ON THE EISENBUD–WIGNER FORMULA FOR TIME-DELAY

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ABSTRACT. It is shown that the Eisenbud–Wigner relation for time-delay holds for potentials $V(r)$ that are $O(r^{-5/2-\epsilon})$ at ∞. This improves previous results in which V was required to be $O(r^{-4-\epsilon})$ and $O(r^{-3-\epsilon})$, respectively.

1. INTRODUCTION

In Martin [1] and Amrein, Jauch, and Sinha [2] the relation between time-delay and the S-matrix (the socalled Eisenbud–Wigner relation) was derived under certain hypotheses which in the context of potential scattering amounted to assuming that the potential V is spherically symmetric and $O(r^{-4-\epsilon})$, $\epsilon > 0$, at $r \to \infty$. In Jauch, Sinha, and Misra [3] and Martin and Misra [4] a time-independent method, utilizing a trace class condition, was employed, and although no spherical symmetry was assumed, the relevant decrease of $V(r)$ at infinity was $O(r^{-3-\epsilon})$. In this note we show by the time-dependent method that it is sufficient that $V(r)$ be $O(r^{-5/2-\epsilon})$ at infinity, thus bridging the gap between the ranges of validity of [1, 2] and [3, 4]. Probably a sharp condition for $V(r)$ is $O(r^{-2-\epsilon})$ if one can avoid a time dependent expression for the derivative of $S(\lambda)$ via Fourier transform.

Recently Tee [5] investigated a sharpening of the approach of [1]. It may be seen that the approach of [4], with some modification, may be pushed through to obtain the Eisenbud–Wigner relation for V that are $O(r^{-3-\epsilon})$. Our approach is simpler and follows that of [2].

For earlier work on time-delay see Jauch and Marchand [6] and Smith [7]. For the existence of a weighted time-delay operator for V which are, roughly, $O(r^{-3-\epsilon})$ in R^3 but without connection to the S-matrix see Lavine [8]. For recent work on a time-delay operator as a dressed limit in the context of hyperbolic equations, see Lax and Phillips [9] and Amrein and Wollenberg [10]. For original papers on the Eisenbud–Wigner relation, which states that

$$T(\lambda) = -iS^*(\lambda)\, dS(\lambda)/d\lambda$$

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where $T(\lambda)$ is the mean time-delay (sometimes called the 'sojourn time') of a particle under interaction as compared to a free particle, and where $S(\lambda)$ is the S-matrix at energy λ, see Eisenbud [11] and Wigner [12].

2. THE MAIN RESULT

Let (H, H_0) be a simple scattering system for which $\Omega_\pm = s - \lim e^{iHt} e^{-iH_0 t}$ exist as $t \to \pm\infty$ and are asymptotically complete (see [2]). Then $S \equiv \Omega_+^\dagger \Omega_-$ is unitary and $[S, H_0] = 0$ so that in the spectral representation of H_0, $S \equiv \{S(\lambda)\}$. Time-delay in such a setting is defined for a particle initially in a scattered state f to be

$$\Delta T(f) = \lim_{r \to \infty} \int_{-\infty}^{\infty} \{\|F_r V_t \Omega_- f\|^2 - \|F_r U_t f\|^2\} \, dt \tag{1}$$

whenever the limit exists. In (1) we have written $V_t \equiv e^{-iHt}$, $U_t = e^{-iH_0 t}$, $F_r \equiv$ multiplication by the characteristic function of the r-ball in R^3, and $f \in \mathcal{M}_\infty(H_0)$, the latter usually comprising all of $\mathcal{H} = L^2(R^3)$. For convenience let us utilize the conditions of Propositions (7.11) and (7.14) of [2] which are abstract versions of the time-delay approach of [1]. From these one may assert that if (α) $\|F_r U_t f\|$ and $\|F_r U_t S f\|$ are integrable on $0 < t < \infty$ for each $0 < \epsilon < \infty$ and (β) $\|(V_t \Omega_- - U_t) f\|$ is integrable on $-\infty < t < 0$ and $\|(V_t \Omega_- - U_t S) f\|$ is integrable on $0 < t < \infty$, then the limit in (1) exists and moreover

$$\Delta T(f) = \lim_{r \to \infty} \int_0^{\infty} \langle S U_t f, [F_r, S] U_t f \rangle \, dt. \tag{2}$$

We now assume furthermore that H_0 has spectrum $[0, \infty)$, S is a function of H_0 and that $f \in m_\infty(H_0) \subseteq \mathcal{H}_{ac}(H_0)$ with compact support in $[0, \infty)$. For a $\rho \in C_0^\infty(0, \infty)$ such that $\rho(H_0) f = f$, we set $S_\rho(\lambda) = S(\lambda)\rho(\lambda)$ and denote by its S_ρ Fourier transform. If $\tilde{S}$ is such that $\int |\tilde{S}_\rho(\tau)|(1 + |\tau|) \, d\tau < \infty$, then one obtains, as in [2], the Eisenbud–Wigner relation:

$$\Delta T(f) = -i\langle f, S^* \, dS/dH_0 f \rangle. \tag{3}$$

In [2], the assumption that S is C^3 was used to verify all the integrability conditions. In Theorem 1 we show that some of the integrability conditions are consequences of the rest. First we recall a lemma, the proof of which is a simple application of functional calculus and Fubini's theorem.

LEMMA. *Let φ be such that its Fourier transform is integrable. Then $\varphi(H_0) = (2\pi)^{-1/2} \int \tilde{\varphi}(\tau) U_\tau^* \, dt$.* □

THEOREM 1. *Let S be a function of H_0, f of compact support in $(0, \infty)$ in the spectral representation of H_0. Assume moreover that $\int_{-\infty}^{\infty} \|E_r U_t f\| \, dt < \infty$ for all $0 < r < \infty$,*

$\int_0^\infty \|V_t\Omega_t f - U_t f\|\ dt| < \infty$*, and that* $\int \|\tilde{S}_\rho(\tau)\|(1 + |\tau|)\, d\tau < \infty$*. Then* (α) *and* (β) *above are satisfied and one obtains the Eisenbud–Wigner formula* (3). □

Proof. We need to verify that the other half of (α) and (β) follows from the first half. For (α) we have

$$\|F_r U_t S(H_0) f\| = \|F_r U_t S_\rho(H_0) f\|$$

$$= (2\pi)^{-1/2} \|F_r U_t \int \tilde{S}_\rho(\tau) U_\tau^* f\, d\tau\|$$

$$\leqq (2\pi)^{-1/2} \int |\tilde{S}_\rho(\tau)|\ \|F_r U_{t-\tau} f\|\ d\tau.$$

Thus

$$(2\pi)^{1/2} \int_0^\infty \|F_r U_t S(H_0) f\|\ dt \leqq \int |\tilde{S}_\rho(\tau)| \int_0^\infty \|F_r U_{t-\tau} f\|\ d\tau\ dt$$

$$\leqq \left(\int_{-\infty}^\infty \|F_r U_t f\|\, dt \right) \left(\int |\tilde{S}_\rho(\tau)|\, d\tau \right) < \infty.$$

Similarly, for (β), we have, using intertwining,

$$(2\pi)^{1/2} \int_0^\infty \|(V_t\Omega_- - U_t S) f\|\, dt = (2\pi)^{1/2} \int_0^\infty \|(V_t\Omega_+ - U_t) S_\rho(H_0) f\|\, dt$$

$$= \int_0^\infty \|(V_t\Omega_+ - U_t) \int \tilde{S}_\rho(\tau) U_\tau^* f\, d\tau\|\, dt$$

$$\leqq \int_0^\infty \int_{-\infty}^\infty |\tilde{S}_\rho(\tau)|\, d\tau \|(V_{t-\tau}\Omega_+ - U_{t-\tau}) f\|\, dt$$

$$= \int_{-\infty}^\infty |\tilde{S}_\rho(\tau)|\, d\tau \left[\int_0^\infty \|(V_s\Omega_+ - U_s) f\|\, ds + \int_{-\tau}^0 \|(V_s\Omega_+ - U_s) f\|\, ds \right]$$

$$\leqq \left(\int_0^\infty \|(V_s\Omega_+ - U_s) f\|\, ds \right) \left(\int |S_\rho(\tau)|\, d\tau \right) + 2\|f\| \int |\tau \tilde{S}_\rho(\tau)|\, d\tau < \infty.$$

In potential scattering, $H_0 = -\Delta$, $H = H_0 + V$, V is a multiplication operator $V(\underline{x})$, and under well known suitable conditions H is selfadjoint with $D(H) = D(H_0)$. Denote by $\mathscr{D}_n = \{f \in L^2(R^3) | \tilde{f} \in C_0^n(R^3 - \{0\})\}$. Then (e.g. see [2, section 13.1]) one has for all $f \in \mathscr{D}_{n+1}$ the known decay estimate $\|(1 + |\underline{Q}|)^{-n-\epsilon} U_t f\| \leqq c_1 (1 + |t|)^{-n-\epsilon/2} \cdot \Sigma_{|m|=0}^{n+1} \|D^m \tilde{f}\|$. Clearly then $\|F_r U_t f\| \in L^1(\bar{R}, \mathrm{d}t)$ since $\|F_r (1 + |\underline{Q}|)^{1+\epsilon}\| = (1 + r)^{1+\epsilon}$. Thus the first part of (α) is satisfied by these f. Moreover

$$\|(V_t \Omega_\pm - U_t) f\| = \|(\Omega_\pm - V_t^* U_t) f\|$$

$$= \left\| \int_t^{\pm\infty} \frac{\mathrm{d}}{\mathrm{d}s} (V_s^* U_s) f \, \mathrm{d}s \right\| \leqq \left| \int_t^{\pm\infty} \|V U_s f\| \, \mathrm{d}s \right|,$$

as in the Jauch–Cook criteria, and for potentials $V = V_1 + V_2$ with $(1 + |x|)^3 V_1(\underline{x}) \in L^2(R^3)$ and $V_2(\underline{x}) \leqq c_2 (1 + |\underline{x}|)^{-2-\eta}$, $\eta > 0$, it is known (see, e.g., Section 13.1 of [2]) that $\|V U_s f\| \leqq c_3 (1 + |s|)^{-2-\eta'} \Sigma_{|m|=0}^{3} \|D^m \tilde{f}\|$, $\eta' < \eta$. Thus $f \in \mathscr{D}_3$ satisfy $|\int_0^{\pm\infty} \|(V_t \Omega_\pm - U_t) f\| \, \mathrm{d}t| < \infty$ and the (α), (β) conditions of Theorem 1.

Turning then to the crucial condition of Theorem 1, namely, that $\int |\tilde{S}_\rho(\tau)| \, (1 + |\tau|) \, \mathrm{d}\tau < \infty$, we restrict attention to the case V spherically symmetric so that S is a function of H_0 in each partial wave expansion component. By partial wave analysis (eqn. (11.45) and problem 13.3(c) of [2]) it follows for all positive integers n that

$$\|\tilde{S}_\rho(\tau) (1 + |\tau|^n)\|_2 = \|S_\rho(\lambda) + (i)^n S_\rho^{(n)}(\lambda)\|_2 \leqq c_4 \|(1 + r^n) V(r)\|_1 .$$

Interpolating between $n = 1$ and $n = 2$ we get

$$\|\tilde{S}_\rho(\tau) (1 + |\tau|)^{3/2+\epsilon}\|_2 \leqq c_5 \|(1 + r^{3/2+\epsilon'}) V(r)\|_1 .$$

Because by Schwarz's inequality,

$$\left| \int \tilde{S}_\rho(\tau) (1 + |\tau| \right| \leqq \left(\int (1 + |\tau|)^{-1-\epsilon} \, \mathrm{d}\tau \right)^{1/2} \cdot \|\tilde{S}_\rho(\tau) (1 + |\tau|)^{3/2+\epsilon}\|_2$$

we have the following:

THEOREM 2. *Let* $H_0 = -\Delta$, $H = H_0 + V$, *V spherically symmetric,* $\int_0^1 r |V(r)| \, \mathrm{d}r < \infty$, $V(r) = O(r^{-5/2-\epsilon})$ *as* $r \to \infty$. *Then the hypotheses of Theorem 1 are satisfied and we have Eisenbud–Wigner relation* (3) *for all* $f \in \mathscr{D}_3$. □

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