A REMARK ON THE INTEGRATION OF SCHRÖDINGER EQUATION USING QUANTUM ITÔ'S FORMULA

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ABSTRACT. When the potential is the Fourier transform of a totally finite complex-valued measure, a formula for the one-parameter unitary group generated by the Schrödinger operator in $L^2(\mathbb{R}^n)$ is obtained entirely in terms of the basic field operators in a suitable Fock space by means of quantum stochastic calculus.

1. INTRODUCTION

In [2], a simple theory of stochastic integrals and differentials with respect to the basic field operator processes was constructed and a quantum Itô's formula for products of differentials was obtained. Just as the classical theory of stochastic differentials with respect to Brownian motion leads to an expression for the contraction semigroups generated by a class of Hamiltonians, we demonstrate here that the theory of quantum stochastic differentials leads to a formula for the unitary group generated by the Hamiltonian when the potential is the Fourier transform of a totally-finite signed measure. The explicit expression that we obtain is entirely in terms of basic field operators in a suitable Fock space. There is a similar attempt by Combe *et al.* in [1] using the classical Poisson processes but involving signed measures.

2. PRELIMINARIES

For any complex separable Hilbert space H let $\Gamma(H)$ denote the associated symmetric Fock space over H and for any $f \in H$, let

$$\psi(f) = (1, f, 2!^{-1/2} f \otimes f, \ldots, n!^{-1/2} f \otimes \cdots \otimes f, \ldots) \tag{2.1}$$

by the corresponding coherent vector. For any $f \in H$ and $U \in \mathcal{U}(H)$, the unitary group of H, define the unitary operator $W(f, U)$ on $\Gamma(H)$ by

$$W(f, U)\psi(g) = \exp[-\tfrac{1}{2}\|f\|^2 - \langle f, Ug\rangle]\, \psi(Ug + f). \tag{2.2}$$

Then

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$$W(f_1, U_1)W(f_2, U_2) = \exp[i \operatorname{Im}\langle f_1, U_1 f_2\rangle]\, W(f_1 + U_1 f_2, U_1 U_2). \tag{2.3}$$

Thus, the map $(f, U) \to W(f, U)$ defines a projective unitary representation of the Euclidean group $H \circledS \mathscr{U}(H)$ which is the semidirect product of H with the unitary group $\mathscr{U}(H)$. This is called the *Weyl representation* of the Euclidean group over H.

For any $f \in H$, we introduce the *annihilation* and *creation* operators $a(f), a^\dagger(f)$, respectively, which satisfy

$$a(f)\psi(g) = \langle f, g\rangle \psi(g), \qquad a^\dagger(f)\psi(g) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\psi(g + \epsilon f)|_{\epsilon = 0}. \tag{2.4}$$

For any bounded operator T on H, we introduce the *gauge* operator $\lambda(T)$ on $\Gamma(H)$ satisfying

$$\lambda(T)\psi(g) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\psi(e^{\epsilon T} g)|_{\epsilon = 0}. \tag{2.5}$$

We now consider the Hilbert space

$$\mathfrak{H} = L^2[0, \infty) \otimes \mathfrak{K} \tag{2.6}$$

where $\mathfrak{K}$ is a fixed complex separable Hilbert space. Let $\chi_{[0,t]}$ denote the indicator function of the interval $[0, t]$ and let P_t^0 be the projection operator of multiplication by $\chi_{[0,t]}$ in $L^2[0, \infty)$. For any $f \in \mathfrak{H}$, $U \in \mathscr{U}(\mathfrak{K})$ define

$$\begin{aligned} f_t &= (P_t^0 \otimes 1)f, \qquad \widetilde{U}(t) = P_t^0 \otimes U + (1 - P_t^0) \otimes 1, \\ W_{f,U}(t) &= W(f_t, \widetilde{U}(t)) \end{aligned} \tag{2.7}$$

where the right-hand side of the third equation is the Weyl operator in $\Gamma(\mathfrak{H})$ defined by (2.1) and (2.2). In the terminology of [2], the family $\{W_{f,U}(t)\}$ is a unitary operator-valued adapted process obeying the stochastic differential equation

$$W_{f,U}(0) = 1; \quad \mathrm{d}W_{f,U} = W_{f,U}(\mathrm{d}\Lambda_{U-1} + \mathrm{d}A_{-U^{-1}f} + \mathrm{d}A_f^\dagger - \tfrac{1}{2}\|f(t)\|_{\mathfrak{K}}^2\, \mathrm{d}t) \tag{2.8}$$

where $\mathfrak{H}$ is viewed as the space of $\mathfrak{K}$-valued square integrable functions on $[0, \infty)$ and

$$\Lambda_{U-1}(t) = \lambda(P_t^0 \otimes (U - 1)), \qquad A_f(t) = a(f_t), \qquad A_f^\dagger(t) = a^\dagger(f_t)$$

are the gauge, annihilation and creation processes respectively.

Let $\mathrm{x} \to U_\mathrm{x}$ be a strongly continuous unitary representation of $\mathbb{R}^n$ in $\mathfrak{K}$ and let

$$\delta(\mathrm{x}) = U_\mathrm{x} u - u \tag{2.9}$$

where $u \in \mathfrak{K}$ is fixed. Let $\mathrm{m} \in \mathbb{R}^n$ be fixed. Define the unitary operators

$$W_{\mathbf{x}}(t) = \exp[it(\mathbf{m}\cdot\mathbf{x} + \mathrm{Im}\langle u, U_{\mathbf{x}}u\rangle_{\mathfrak{R}})]\, W(\chi_{[0,t]}\otimes\delta(\mathbf{x}), \widetilde{U}_{\mathbf{x}}(t)) \tag{2.10}$$

where $\widetilde{U}_{\mathbf{x}}(t)$ is defined by the second equation in (2.7). Since $U_{\mathbf{x}}$ is a representation and (2.9) holds, it follows from (2.3) that $\{W_{\mathbf{x}}(t), t \geq 0, \mathbf{x} \in \mathbb{R}^n\}$ is a commuting family of unitary operators such that for every fixed t, the map $\mathbf{x} \to W_{\mathbf{x}}(t)$ is a unitary representation of $\mathbb{R}^n$. Hence, there exists a commuting family of self-adjoint operators $\{X_j(t), 1 \leq j \leq n, t \geq 0\}$ satisfying

$$W_{\mathbf{x}}(t) = \exp\left[-i\sum_{j=1}^{n} x_j X_j(t)\right], \tag{2.11}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$. Viewed as observables, $\mathbf{X}(t) = (X_1(t), \ldots, X_n(t))$ is a classical mixed Poisson process with independent increments shifted by the function $t\mathbf{m}$. It is to be noted that $W_{\mathbf{x}}$ and $\mathbf{X}$ are constructed by starting from a representation $\mathbf{x} \to U_{\mathbf{x}}$ of $\mathbb{R}^n$ and a 1-cocycle $\delta(\mathbf{x})$ which is a coboundary for the representation.

From (2.8) we conclude

$$\mathrm{d}W_{\mathbf{x}} = W_{\mathbf{x}}\{\mathrm{d}\Lambda_{U_{\mathbf{x}}-1} + \mathrm{d}A_{\delta(-\mathbf{x})} + \mathrm{d}A^{\dagger}_{\delta(\mathbf{x})} + (i\mathbf{m}\cdot\mathbf{x} + \langle u, X_{\mathbf{x}}u - u\rangle_{\mathfrak{R}})\,\mathrm{d}t\}. \tag{2.12}$$

3. THE MAIN RESULT

We introduce three unitary operator-valued adapted processes $\{J_i(t), t \geq 0, i = 1, 2, 3\}$ in the Hilbert space

$$\widetilde{\mathcal{H}} = L^2(\mathbb{R}^n) \otimes \Gamma(L^2[0,\infty) \otimes \mathfrak{R}). \tag{3.1}$$

Let $v \in \mathfrak{R}$ be fixed. Let $\{Q_v(t)\}$ be the 'Brownian motion' defined by

$$Q_v(t) = 1 \otimes (a(\chi_{[0,t]} \otimes v) + a^{\dagger}(\chi_{[0,t]} \otimes v)) \tag{3.2}$$

in $\widetilde{\mathcal{H}}$. Then the operators $Q_v(t)$ can be extended canonically to a commuting family of self-adjoint operators which are denoted by the same symbol. Then we define the process J_1 by

$$J_1(t) = \exp[-iQ_v(t)], \tag{3.3}$$

which obeys the stochastic differential equation

$$J_1(0) = 1; \quad \mathrm{d}J_1 = J_1(-i\,\mathrm{d}A^{\dagger}_v - i\,\mathrm{d}A_v - \tfrac{1}{2}\|v\|^2_{\mathfrak{R}}\,\mathrm{d}t). \tag{3.4}$$

where $A_v(t)$ and $A^{\dagger}_v(t)$ are to be interpreted as $1 \otimes a(\chi_{[0,t]} \otimes v)$ and $1 \otimes a^{\dagger}(\chi_{[0,t]} \otimes v)$, respectively.

Let $H_0(\mathbf{x})$ be a real-valued continuous function on $\mathbb{R}^n$. Using the momentum operators

$\mathbf{p} = (p_1, p_2, \ldots, p_n)$ in $L^2(\mathbb{R}^n)$ and the self-adjoint operator processes $\mathbf{X}(t) = (X_1(t), \ldots, X_n(t))$ defined in $\Gamma(L^2[0,\infty)\otimes\mathfrak{K})$ by (2.10) and (2.11) we construct a process J_2 by putting

$$J_2(t) = \exp\left[-i\int_0^t H_0(\mathbf{p}\otimes 1 + 1\otimes \mathbf{X}(s))\,\mathrm{d}s\right]. \tag{3.5}$$

Then J_2 is a unitary operator-valued process satisfying the equation

$$J_2(0) = 1; \qquad \mathrm{d}J_2 = -iJ_2 H_0(\mathbf{p}\otimes 1 + 1\otimes \mathbf{X}(t))\,\mathrm{d}t. \tag{3.6}$$

Let $\{P(E), E\subseteq \mathbb{R}^n$ is a Borel set$\}$ be the canonical spectral measure of multiplication by the indicator function χ_E in $L^2(\mathbb{R}^n)$. Define the unitary operator-valued process J_3 by

$$J_3(t) = \int_{\mathbb{R}^n} P(\mathrm{d}\mathbf{x})\otimes W_{\mathbf{x}}(t) \tag{3.7}$$

where $W_{\mathbf{x}}(t)$ is defined by (2.10). If we look upon $\widetilde{\mathcal{H}}$ given by (3.1) as $\Gamma(L^2[0,\infty)\otimes\mathfrak{K}$-valued square integrable functions on $\mathbb{R}^n$, $J_3(t)$ is nothing but operator multiplication by $W_{\mathbf{x}}(t)$. It follows from the canonical relations in $L^2(\mathbb{R}^n)$ that

$$H_0(\mathbf{p}\otimes 1 + 1\otimes \mathbf{X}(t))J_3(t) = J_3(t)H_0(\mathbf{p}\otimes 1). \tag{3.8}$$

Further, (2.12) implies that

$$\mathrm{d}J_3 = J_3\int_{\mathbb{R}^n} P(\mathrm{d}\mathbf{x})\otimes(\mathrm{d}\Lambda_{U_{\mathbf{x}}-1} + \mathrm{d}A_{\delta(-\mathbf{x})} + \mathrm{d}A^\dagger_{\delta(\mathbf{x})}) +$$

$$+ J_3 F(\mathbf{q}\otimes 1)\,\mathrm{d}t,$$

where $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ are the position operators in $L^2(\mathbb{R}^n)$ and

$$F(\mathbf{x}) = i\mathbf{m}\cdot\mathbf{x} + \langle u, U_{\mathbf{x}}u - u\rangle_{\mathfrak{K}} \tag{3.9}$$

Define the process J by

$$J(t) = \exp\left[\frac{t\|v\|^2}{2}\right]J_1(t)J_2(t)J_3(t) \tag{3.10}$$

where $J_i(t)$, $i = 1, 2, 3$ are defined by (3.3), (3.5) and (3.7) respectively. Then we have

THEOREM 3.1. *If* $\mathbb{E}_0: \mathscr{B}(L^2(\mathbb{R}^n)\otimes\Gamma(L^2[0,\infty)\otimes\mathfrak{K}))\to\mathscr{B}(L^2(\mathbb{R}^n))$ *is the vacuum conditional expectation map defined by*

$$\langle\varphi_1, \mathbb{E}_0(T)\varphi_2\rangle = \langle\varphi_1\otimes\Omega, T(\varphi_2\otimes\Omega)\rangle$$

for all $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^n)$ *and* Ω *is the Fock vacuum in* $\Gamma(L^2[0,\infty)\otimes \mathfrak{K})$, *then*

$$\mathbb{E}_0 J(t) = \exp[-it(H_0(\mathrm{p}) + V(\mathrm{q}))] \tag{3.11}$$

where

$$V(\mathrm{x}) = \langle v - iu, U_{\mathrm{x}}u - u\rangle_{\mathfrak{K}} - \mathrm{m}\cdot \mathrm{x}, \tag{3.12}$$

and $J(t)$ *is defined by* (3.10).

Proof. We employ the stochastic calculus developed in [2]. Putting $K(t) = \exp[t\|v\|^2/2]J_1(t)J_2(t)$ and using (3.4) and (3.6) we get

$$\mathrm{d}K = -iK\{\mathrm{d}A_v + \mathrm{d}A_v^\dagger + H_0(\mathrm{p}\otimes 1 + 1\otimes \mathrm{X}(t))\,\mathrm{d}t\}.$$

By quantum Itô's formula and (3.8) we have

$$\mathrm{d}J = \mathrm{d}(KJ_3) = (\mathrm{d}K)J_3 + K\,\mathrm{d}J_3 + \mathrm{d}K\cdot \mathrm{d}J_3$$

where

$$(\mathrm{d}K)J_3 = -iJ(\mathrm{d}A_v + \mathrm{d}A_v^\dagger + H_0(\mathrm{p}\otimes 1)\,\mathrm{d}t),$$

$$K\,\mathrm{d}J_3 = J\int_{\mathbb{R}^n} P(\mathrm{dx})\otimes(\mathrm{d}\Lambda_{U_{\mathrm{x}}-1} + \mathrm{d}A_{\delta(-\mathrm{x})} + \mathrm{d}A_{\delta(\mathrm{x})}^\dagger) +$$

$$+ JF(\mathrm{q}\otimes 1)\,\mathrm{d}t,$$

$$\mathrm{d}K\cdot \mathrm{d}J_3 = -iJ\int_{\mathbb{R}^n} P(\mathrm{dx})\otimes \mathrm{d}A_{(U_{-\mathrm{x}}-1)v} - iJG(\mathrm{g}\otimes 1)\,\mathrm{d}t.$$

with

$$G(\mathrm{x}) = \langle v, \delta(\mathrm{x})\rangle_{\mathfrak{K}} = \langle v, U_{\mathrm{x}}u - u\rangle_{\mathfrak{K}}.$$

Thus, in the differential of J the only term in which the differentials of gauge, annihilation and creation processes do not appear is

$$-iJ(H_0(\mathrm{p}\otimes 1) + iF(\mathrm{q}\otimes 1) + G(\mathrm{q}\otimes 1))\,\mathrm{d}t,$$

where

$$G(\mathrm{x}) + iF(\mathrm{x}) = \langle v, U_{\mathrm{x}}u - u\rangle_{\mathfrak{K}} + i\langle u, U_{\mathrm{x}}u - u\rangle_{\mathfrak{K}} - \mathrm{m}\cdot \mathrm{x}$$

$$= \langle v - iu, U_{\mathrm{x}}u - u\rangle_{\mathfrak{K}} - \mathrm{m}\cdot \mathrm{x} = V(\mathrm{x}).$$

Hence, by the discussion in Section 8 of [2] we have

$$d(\mathbb{E}_0 J(t)) = -i(\mathbb{E}_0 J(t))(H_0(\mathbf{p}) + V(\mathbf{q}))\, dt$$

which proves (3.11). □

REMARK. By choosing $\mathbf{m} = 0$ and the vectors u, v and the representation $\mathbf{x} \to U_{\mathbf{x}}$ suitably we can realise all potentials V which are Fourier transforms of totally finite signed measures.

REFERENCES

1. Combe, P., Hoegh-Krohn, R., Sirigue, M., and Sirigue-Collins, M., 'Generalised Poisson Processes in Quantum Mechanics and Field Theory', *Phys. Rep.* 77, No. 3, 1981 (*New Stochastic Methods in Physics* (C. Dewitte-Morette and K.D. Elworthy, eds.), North-Holland, Amsterdam).
2. Hudson, R.L. and Parthasarathy, K.R., 'Quantum Itô's Formula and Stochastic Evolution', submitted to *Comm. Math. Phys.*

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