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A Unitary Analogue of Kato's Theorem on Variation of Discrete Spectra

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Abstract. We obtain an estimate of the distance between extended enumerations of the discrete eigenvalues of two unitary operators whose difference is compact.

In a recent paper [4], T. Kato has obtained an infinite-dimensional version of a well-known estimate of the eigenvalue variation of a Hermitian matrix. The purpose of this Letter is to point out that an analogous known result for unitary matrices can also be extended on the same lines.

Let U be a unitary operator in a separable Hilbert space $\mathscr{H}$. The spectrum of U is a subset of the unit circle $\mathscr{T}$. As in Kato [4], we define an *extended enumeration of the discrete spectrum* of U to be a sequence $\{\alpha_j\}$ whose terms consist of eigenvalues of U with finite multiplicity and boundary points (in $\mathscr{T}$) of the essential spectrum of U, where an eigenvalue of finite multiplicity m appears exactly m times in $\{\alpha_j\}$.

The following is a unitary analogue of Kato's result.

THEOREM. *Let U, V be unitary operators in $\mathscr{H}$ such that their difference $C = U - V$ is a compact operator. Let $\{s_j\}$ be an enumeration of the singular values of C. Then there exist extended enumerations $\{\alpha_j\}$, $\{\beta_j\}$ of discrete eigenvalues of U, V, respectively, such that for every symmetric gauge function ϕ defined on the space of real sequences, we have*

$$\Phi(\{|\alpha_j - \beta_j|\}) \leqslant \frac{\pi}{2}\Phi(\{s_j\}). \tag{1}$$

In particular,

$$\left(\sum_j |\alpha_j - \beta_j|^p\right)^{1/p} \leqslant \frac{\pi}{2}\|C\|_p, \quad 1 \leqslant p < \infty, \tag{2}$$

where

$$\|C\|_p \equiv \left(\sum_j s_j^p\right)^{1/p}$$

The constant $\pi/2$ occurring in the above inequalities cannot be replaced by a smaller constant: for $p = 1$ the estimate (2) is sharp.

REMARKS. 1. For the theory of symmetric gauge functions, the reader is referred to [3]. Recall that the singular values of C are the eigenvalues of the positive compact operator $(C^*C)^{1/2}$, each counted as often as its multiplicity.

2. For finite dimensions, this result was proved in [1] and [2].

An Outline of the Proof. The proof closely follows the arguments in [4]. We will adopt the same notations as there, emphasizing only those points where modifications are required.

Since $U - V$ is compact, we can write $U^{-1}V = I + K$, where K is compact and normal. Hence, we can find a compact self-adjoint operator H such that $U^{-1}V = \exp iH$ and $-\pi I < H \leqslant \pi I$. Let $U(t)$ be the real entire family of unitary operators defined as

$$U(t) = U \exp itH, \quad t \in \mathbb{R}.$$

Then

$$U(0) = U, \qquad U(1) = V, \qquad U'(t) = iU(t)H.$$

Replace $A(t)$ in [4] by $U(t)$ and let $\{\lambda_j(t)\}$, $\{E_j(t)\}$, Δ_j, m_j be defined correspondingly. The only change from Section 2 in [4] is that now we have (see Chapter VII of [5] and Theorem 4.13 of [6]):

$$\frac{d\lambda_j(t)}{dt} = \frac{1}{m_j} \operatorname{tr} U'(t)E_j(t), \quad t \in \Delta_j.$$

Let $\tilde{\lambda}_j(t)$, $\tilde{E}_j(t)$ be the piecewise analytic extensions of $\lambda_j(t)$ and $E_j(t)$ defined on all of $\mathbb{R}$ as in [4]. Then

$$\tilde{\lambda}_j(1) - \tilde{\lambda}_j(0) = \frac{1}{m_j} \int_0^1 \operatorname{tr}(U'(t)\tilde{E}_j(t))\, dt. \tag{3}$$

Let $\gamma_k (|\gamma_k| \leqslant \pi)$ be an enumeration of the eigenvalues of the compact self-adjoint operator H and let ϕ_k be the corresponding eigenvectors. Then

$$\begin{aligned} \operatorname{tr} U'(t)\tilde{E}_j(t) &= i \operatorname{tr} U(t)H\tilde{E}_j(t) \\ &= i \sum_k \langle \tilde{E}_j(t)U(t)H\phi_k, \phi_k \rangle \\ &= i \sum_k \gamma_k \langle \tilde{E}_j(t)U(t)\phi_k, \phi_k \rangle . \end{aligned}$$

But $\tilde{E}_j(t)$ is the spectral projection of $U(t)$ corresponding to the eigenvalue $\tilde{\lambda}_j(t)$. Hence,

$$\tilde{E}_j(t)U(t) = U(t)\tilde{E}_j(t) = \exp(i\tilde{\lambda}_j(t))\tilde{E}_j(t),$$

and

$$\operatorname{tr} U'(t)\tilde{E}_j(t) = i \exp(i\tilde{\lambda}_j(t)) \sum_k \gamma_k \langle \tilde{E}_j(t)\phi_k, \phi_k \rangle . \tag{4}$$

The above two equations replace equations (3) and (4) in [4]. Equation (5) there is replaced by the inequality

$$|\tilde{\lambda}_j(1) - \tilde{\lambda}_j(0)| \leqslant \sum_k \sigma_{jk} |\gamma_k| , \tag{5}$$

where σ_{jk} satisfies the properties (6) and (7) in [4]. Now, if Φ is any symmetric gauge function, then it is monotone and convex. Hence, using the same reasoning as in [4], (see also Theorem 1.16(e) in [7]), we get from (5)

$$\Phi(\{|\alpha_j - \beta_j|\} \leqslant \Phi(\{|\gamma_j|\}) . \tag{6}$$

Now we want to relate γ_j with the singular values s_j of the operator $C = U - V$. Since the singular values of an operator are invariant under multiplication by a unitary operator, these s_j are also the singular values of the operator $I - U^{-1}V = I - \exp iH$. Hence, the s_j can be enumerated as

$$s_j = |1 - \exp i\gamma_j| , \quad j = 1, 2, \dots .$$

Hence, since $|\gamma_j| \leqslant \pi$

$$|\gamma_j| \leqslant \frac{\pi}{2} s_j , \quad j = 1, 2, \dots . \tag{7}$$

Using the monotonicity of Φ, we obtain (1) from (6) and (7). □

REMARKS. 1. The following example showing that the estimate (2) cannot be improved is taken from [2]. Let U, V be operators in $\mathbb{C}^n$ defined as

$$Ue_j = Ve_j = e_{j+1} , \quad j = 1, 2, \dots, n-1 ,$$

$$Ue_n = e_1 , \qquad Ve_n = -e_1 .$$

Then $\|U - V\|_1 = 2$, independent of n. The eigenvalues of U, V are, respectively, the nth roots of 1 and -1. So the minimal value of $\Sigma |\alpha_j - \beta_j|$ over all possible enumerations approaches π as $n \to \infty$.

2. Our inequality (1) can be applied to obtain an estimate for self-adjoint operators. Let A be a self-adjoint operator which is bounded below and let B be a self-adjoint operator such that $B - A$ is compact relative to A. Writing

$$U = (A + i)(A - i)^{-1} \quad \text{and} \quad V = (B + i)(B - i)^{-1}$$

we find that the operator

$$T = \frac{1}{2i}(U - V) = (A - i)^{-1}(B - A)(B - i)^{-1}$$

is compact. Then (1) and a simple calculation leads to an inequality

$$\Phi\left(\{|a_j - b_j|\} \leqslant \frac{\pi}{2} [(a^2 + 1)(b^2 + 1)]^{1/2} \Phi(\{c_j\})\right),$$

where $\{a_j\}$, $\{b_j\}$ are extended enumerations of the discrete eigenvalues of A and B, $\{c_j\}$

is an enumeration of the singular values of T, $a = \max|a_j|$, $b = \max|b_j|$. In a typical case of interest in physics, say when A is the Schrödinger operator and B a compact perturbation of it, a and b are both finite.

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