5. Acknowledgment. The author wishes to express his indebtedness to Professor G. E. Albert for many helpful suggestions made in the pursuance of this research.

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ON A PROBLEM IN MEASURE-SPACES

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Summary. Let $\mathfrak{F}$ be the family of all random variables on a probability space Ω taking values from a separable and complete metric space X. In this paper we prove that $\mathfrak{F}$ is in a certain sense a closed family. More precisely, if $\{\xi_n\}$ is a sequence of X-valued random variables such that their probability distributions converge weakly to a probability distribution P on X, then there exists an X-valued random variable on Ω with distribution P. An example is also given which shows that the assumption of completeness of X cannot in general be dropped.

1. Preliminary remarks. In what follows $(\Omega, \mathcal{S}, \mu)$ is a probability space and X a separable metric space. We denote by $\mathcal{B}$ the class of Borel subsets of X defined as the minimal σ-field containing all open subsets of X.

A map φ of Ω into X is called a random variable if it is measurable i.e., $\varphi^{-1}(A) \in \mathcal{S}$ for each $A \in \mathcal{B}$. If φ is a random variable we define as its distribution the measure μ_φ on $\mathcal{B}$ given by

$$\mu_\varphi(A) = \mu\{\varphi^{-1}(A)\}$$

for all $A \in \mathcal{B}$. A given probability measure P on $\mathcal{B}$ is said to be induced from Ω if there exists a random variable φ such that $P = \mu_\varphi$.

Suppose we are given a sequence $\{P_n\}$ of probability measures on $\mathcal{B}$. We say that $\{P_n\}$ converges weakly to a probability measure P on $\mathcal{B}$ ($P_n \Rightarrow P$ in symbols) if

$$\lim_{n\to\infty} \int_X g\, dP_n = \int_X g\, dP$$

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for every bounded continuous function g on X. In terms of subsets of X this is equivalent to

$$\limsup_{n\to\infty} P_n(C) \leqq P(C)$$

for every closed set $C \subset X$ ([1]). When X is the real line with the usual topology, this convergence is equivalent to the usual convergence of distributions.

2. The main theorem. In this section we state and prove the main theorem. Before doing it we prove a lemma.

LEMMA. *Let X be a separable and complete metric space and $(\Omega, \mathcal{S}, \mu)$ a nonatomic probability space* ([2] p. 168). *Then any probability measure on $\mathcal{B}$ can be induced from Ω.*

PROOF. Since X is a separable metric space, it can be imbedded homeomorphically into a countable product of unit intervals by a celebrated theorem of Urysohn ([3] p. 125). Since it is also complete, the image of X will be a G_δ by a theorem of Larentieff ([3] p. 207). X can thus be regarded as a Borel subset of a countable product of unit intervals. This implies however that X can be regarded as a Borel subset of the unit interval since the unit interval and the countable product of such intervals can be connected by an one-one map which is measurable both ways. It is thus sufficient to show that any probability measure on the unit interval can be induced from Ω. This however is a well-known result.

We now prove the main theorem.

THEOREM. Let X be a separable and complete metric space and $(\Omega, \mathcal{S}, \mu)$ an arbitrary probability space. If $\{\xi_n\}$ is a sequence of X-valued random variables such that $\mu_{\xi_n} \Rightarrow P$ as $n \to \infty$ where P is a probability measure on $\mathcal{B}$, there exists an X-valued random variable ξ such that $P = \mu_\xi$.

PROOF. Any measure space can be decomposed into its atomic and nonatomic components and in view of the previous lemma we can assume that there is no nonatomic component in Ω. We can thus write $\Omega = A_1 \cup A_2 \cup \cdots$ where (i) $A_i \cap A_j = \phi$ for $i \neq j$, (ii) each A_i is an atom of $(\Omega, \mathcal{S}, \mu)$, and (iii) $\mu(A_i) = c_i > 0$ for each i. The distribution $P_n(\equiv \mu_{\xi_n})$ is then atomic and (since X is separable) has mass concentrated in a countable set of points, say $\{a_{n1}, a_{n2}, \cdots\}$. $P_n[a_{ni}] \geqq c_i$ for $i = 1, 2, \cdots$. $\xi(\omega) = a_{ni}$ if $\omega \in A_i$

We first assert that for each i, the set $D_i = \{a_{1i}, a_{2i}, \cdots\}$ has compact closure. If not, then for some i_0, D_{i_0} has a subset which has no limit point and which is infinite. We can assume without losing generality that this subset is D_{i_0} itself and that all the a_{ni_0} are distinct. If then $D \subset D_{i_0}$ is any subset, then D is closed and from $P_n \Rightarrow P$ it follows that $P(D) \geqq \limsup_{n\to\infty} P_n(D)$. If D is infinite then, $\limsup_{n\to\infty} P_n(D) \geqq c_{i_0}$. Thus for any infinite subset $D \subset D_{i_0}$, $P(D) \geqq c_{i_0} > 0$ which is a contradiction.

Thus each D_i has compact closure. We can then, by the diagonal procedure choose a sequence $\{n_k\}$ of integers and points $a_1, a_2, \cdots$ of X such that

$$\lim_{k\to\infty} a_{n_k, i} = a_i$$

for $i = 1, 2, \cdots$. Let ξ be the random variable with values $a_1, a_2, \cdots$ on the sets $A_1, A_2, \cdots$. We complete the proof by showing that $P = \mu_\xi$. It is enough to show that $P_{n_k} \Rightarrow \mu_\xi$. In fact for any bounded continuous g on X,

$$\int_X g\, dP_{n_k} = \sum_i c_i\, g(a_{n_k,i}) \rightarrow \sum_i c_i\, g(a_i) = \int_X g\, d\mu_\xi ,$$

the passage to the limit being justified as $\sum_i c_i g(a_{n_k,i})$ converges uniformly in k. This completes the proof of the theorem.

REMARKS. (1) Suppose X is any separable metric space and X^* its completion. The above theorem will still be true not for X but for X^* and ξ will now be X^*-valued. If then X has the property that as a subset of X^* it is measurable with respect to the completion of *every* measure on X^*, ξ can be reduced to an X-valued random variable and the main theorem is true for such X. This is the case for instance when X is itself a Borel set in X^*. It is interesting to note that there are separable metric spaces X which have the above mentioned property in relation to X^* but which are not complete under any metrization, for example, the set of rationals with the relative real line topology.

(2) It is to be noted that when $(\Omega, \mathcal{S}, \mu)$ is purely atomic the theorem is true with any separable X.

(3) Suppose now $A_1, A_2, \cdots$ is a sequence of sets in $\mathcal{S}$ such that $\mu(A_n) \rightarrow \alpha$. Setting $\xi_n = \chi_{A_n}$, the characteristic function of A_n, we find that $\mu_{\xi_n} \Rightarrow P$ where P is the measure with masses α and $1 - \alpha$ at the points 1 and 0. The above theorem then ensures the existence of $A \in \mathcal{S}$ such that $\mu(A) = \alpha$; in other words that the range of μ is a closed subset of $[0, 1]$.

3. An example. We construct an example to show that the theorem proved in Section 2 requires some such condition on X. We take for X a subset of $[0, 1]$ such that (i) $\mu^*(X) = 1$, $\mu_*(X) = 0$ where μ is Lebesgue measure and (ii) X contains all points of the form $m/2^n$. For $(\Omega, \mathcal{S}, \mu)$ we take the unit interval with Lebesgue measure. The Borel sets of X are precisely the intersections with X of Borel subsets of $[0, 1]$. Lebesgue outer measure on $\mathcal{B}$ is now actually a measure over it, denoted by λ.

Suppose now P_n is the measure on $\mathcal{B}$ with equal masses $1/2^n$ at the points $m/2^n$ $(m = 1, 2, \cdots 2^n)$. It is easy to verify that $P_n \Rightarrow \lambda$. Further each P_n is trivially induced from Ω. We will now show that λ cannot be induced from Ω.

Suppose λ is induced by the map ξ. ξ is obviously a Borel measurable function on $[0, 1]$ and hence by Lusin's theorem ([2]) p.243) we can find for each $\epsilon > 0$ a compact $K_\epsilon \subset [0, 1]$ such that (i) $\mu(K_\epsilon) > 1 - \epsilon$ and (ii) ξ restricted to K_ϵ is continuous. If $M_\epsilon = \xi[K_\epsilon]$, then $M_\epsilon \subset X$ and is a compact subset of the real line. Since λ is induced by ξ, $\lambda(M_\epsilon) > 1 - \epsilon$. But M_ϵ is a Borel set of the real line and this shows that $\mu(M_\epsilon) > 1 - \epsilon$, contradicting the assumption that $\mu_*(X) = 0$.

Thus λ cannot be induced from Ω. This completes the discussion of the example.

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CORRECTION TO "PROBABILITIES OF HYPOTHESES AND INFORMATION-STATISTICS IN SAMPLING FROM EXPONENTIAL-CLASS POPULATIONS"

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In the paper cited in the title (*Ann. Math. Stat.*, Vol. 29 (1958), pp. 571–575): p. 572, line 5. For $\sum \mathbf{x} p(\mathbf{x}, \theta_m)$ read $\sum_{\mathbf{x}} p(\mathbf{x}, \theta_m)$.

CORRECTION TO "POWER FUNCTIONS OF THE GAMMA DISTRIBUTION"

G. D. Berndt

Professor I. R. Savage has called to my attention, through the Editor, the fact that I have overlooked reference to previous work appearing in Eisenhart, Haystay, and Wallis, *Techniques of Statistical Analysis*, and bearing on results reported by me in the *Annals*, Vol. 29, No. 1, March 1958, pages 302–306.

On pages 274–275 of Eisenhart, Haystay, and Wallis, in Figures 8.1 and 8.2, there are given operating characteristic curves for the chi-squared distribution for eight selected degrees of freedom when the significance level is 0.01 and 0.05. Inasmuch as the chi-squared distribution is a gamma distribution with $\frac{1}{2}$ (degrees of freedom) = the parameter gamma in my paper and with 2 = the parameter beta in my paper, and since their rho is equivalent to my delta, there is a similarity in the reported results. This similarity has resulted in some overlap in the results of the two papers in that ten of my forty-eight power curves have an equivalent in the operating characteristic curves in the previous work.

I should like to acknowledge this previous work, and also that of Ferris, Grubbs, and Weaver, by having the following two references added to the two which already appear at the end of my paper:

[3] *Selected Techniques of Statistical Analysis*, Churchill Eisenhart, Millard W. Haystay, and W. Allen Wallis, editors, McGraw-Hill, New York, 1947, pp. 270–278.

[4] Charles D. Ferris, Frank E. Grubbs, and Chalmers L. Weaver, "Operating characteristics for some common statistical tests of significance," *Annals of Mathematical Statistics*, Vol. 17 (1946), pp. 178–197.