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NORMALISATION OF STATISTICAL VARIATES AND THE USE OF RECTANGULAR CO-ORDINATES IN THE THEORY OF SAMPLING DISTRIBUTIONS.

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CONTENTS.

Introduction.

Statistical variates are in general not independent, but show various degrees of intercorrelations. This introduces great algebraic and analytical difficulties in many statistical investigations. In the first section of the present paper* it is shown with the help of matrix algebra how any given set of correlated variates may be transformed into a set of statistically independent variates. In the second section a new type of statistical co-ordinates (called rectangular co-ordinates) is introduced, and the same trasformation is obtained by vector geometrical methods. It is also shown that the matrix of rectangular co-ordinates is identical with the matrix of transformation coefficients used in the first section. This transformation has in practice to be carried out on the sample, and hence these coefficients are subject to sampling fluctuations. The distributions of the coefficients are obtained in the third section with the help of certain auxiliaries which we call normal co-ordinates. In the fourth section we show that many distributions of statistics related to the multivariate normal population can be obtained easily by using the rectangular co-ordinates.

Section I. Normalisation of Variates.

1. *The Observational Matrix.* Let $[x'_{i1}, x'_{i2}, \ldots\ldots x'_{in}]$ be the observed values of the i-th character for the 1st, 2nd,...n-th individual. The complete set of observations

$$[x'_{i\lambda}]_p^n = \left| \begin{array}{cccc} x'_{11}, & x'_{12}, & \ldots\ldots\ldots\ldots & x'_{1n} \\ x'_{21}, & x'_{22}, & \ldots\ldots\ldots\ldots & x'_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \\ x'_{i1}, & x'_{i2}, & \ldots\ldots\ldots\ldots & x'_{in} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \\ x'_{p1} & x'_{p2} & \ldots\ldots\ldots\ldots & x'_{pn} \end{array} \right| \qquad \ldots \quad (1\cdot0)$$

is represented by the matrix (1·0), where p is the number of characters, and is less than n. The elements $x'_{i\lambda}$ may be directly measured quantities (like stature temperature, scores in tests of abilities etc.), or indices (like cephalic index, relative humidity, I. Q., mental age), or other quantities directly derived from the measured quantities.

The mean value a_i and the standard deviation s_i for the i-th character are defined[1] by

$$a_i = \frac{1}{n} S_\lambda [(x'_{i\lambda})] \qquad \ldots \quad (1\cdot1)$$

$$s_i^2 = \frac{1}{n} S_\lambda [(x'_{i\lambda} - a_i)^2] \qquad \ldots \quad (1\cdot2)$$

where S_λ denotes a summation for all values of λ from 1 to n.

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* The first section of this paper together with other matter (which is reproduced in an appendix) was communicated to the Indian Science Congress in 1930 by one of us (P. C. M.).

[1] We shall follow as far as possible the convention that Roman letters (a, s, r, etc) will represent sample statistics, and Greek letters (α, σ, ρ etc) the corresponding population parameters.

Let us now define a set of new quantities

$$x_{i\lambda} = (x'_{i\lambda} - a_i) \quad \dots \quad (1{\cdot}3)$$

We shall call

$$[x]_p^n = \begin{vmatrix} x_{11} & x_{12} & \dots\dots\dots & x_{1n} \\ x_{21} & x_{22} & \dots\dots\dots & x_{2n} \\ \dots & \dots & \dots & \dots \\ x_{i1} & x_{i2} & \dots\dots\dots & x_{in} \\ \dots & \dots & \dots & \dots \\ x_{p1} & x_{p2} & \dots\dots\dots & x_{pn} \end{vmatrix} \quad \dots \quad (1{\cdot}4)$$

the reduced matrix of observations.

Let $\overline{\underline{x}}{}_n^p$ represent the matrix conjugate to $[x]_p^n$, so that

$$\overline{\underline{x}}{}_n^p = \begin{vmatrix} x_{11} & x_{21} & \dots & x_{i1} & \dots & x_{p1} \\ x_{12} & x_{22} & \dots & x_{i2} & \dots & x_{p2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{1n} & x_{2n} & \dots & x_{in} & \dots & x_{pn} \end{vmatrix} \quad \dots \quad (1{\cdot}5)$$

Consider the product of $[x]_p^n$ and $\overline{\underline{x}}{}_n^p$, and let

$$[x]_p^n \cdot \overline{\underline{x}}{}_n^p = [b]_p^p \quad \dots \quad (1{\cdot}6)$$

Then by definition we have $b_{ij} = S_\lambda\,[x_{i\lambda} \cdot x_{j\lambda}]$ ... (1·7)

From this it is clear that if $a_{ij} = s_i \cdot s_j \cdot r_{ij}$ ... (1·8)

where s_i and s_j are the standard deviations for the i-th and j-th characters, and r_{ij} is the correlation between the i-th and the j-th character, then we have

$$b_{ij} = n \cdot a_{ij} \quad \dots \quad (1{\cdot}9)$$

2. *The General Orthogonal Transformation*—Let $[y]_p^n$ be a semi-unit matrix defined by

$$[y]_{p'}^n = \begin{vmatrix} y_{11} & y_{12} & \dots\dots\dots & y_{1n} \\ y_{21} & y_{22} & \dots\dots\dots & y_{2n} \\ \dots & \dots & \dots & \dots \\ y_{p'1} & y_{p'2} & \dots\dots\dots & y_{p'n} \end{vmatrix} \quad \dots \quad (2{\cdot}1)$$

such that

$$[y]^{n}_{p'} \cdot \underline{\overline{y}}^{p'}_{n} = [1]^{p'}_{p'} \qquad \ldots \quad (2\cdot2)$$

where $[1]^{p'}_{p'}$ is a unit matrix defined by

$$[1]^{p'}_{p'} = \left|\begin{array}{cccc} 1 & 0 & 0 \; \ldots\ldots\ldots\ldots & 0 \\ 0 & 1 & 0 \; \ldots\ldots\ldots\ldots & 0 \\ \ldots\ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ 0 & 0 & 0 \; \ldots\ldots\ldots\ldots & 1 \end{array}\right| \qquad \ldots \quad (2\cdot3)$$

Then we have

$$\left.\begin{array}{lll} \mathbf{S}_\lambda[(y_{i\lambda} \cdot y_{j\lambda})] = 1 & \text{when} & i = j \\ \phantom{\mathbf{S}_\lambda[(y_{i\lambda} \cdot y_{j\lambda})]} = 0 & \text{when} & i \neq j \end{array}\right\} \qquad \ldots \quad (2\cdot4)$$

If we consider $y_1, y_2, \ldots\ldots y_{p'}$ to be a set of statistical variates (with zero mean values), then we notice from (2·4) that $[y_1, y_2, \ldots\ldots y_{p'}]$ have zero correlations and are thus statistically independent.

Let us now consider a matrix

$$\underline{\overline{c}}^{p'}_{p} = \left|\begin{array}{cccc} c_{11} & c_{21} & \ldots\ldots\ldots\ldots & c_{p'1} \\ c_{12} & c_{22} & \ldots\ldots \;\; \ldots\ldots & c_{p'2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots \\ c_{1p} & c_{2p} & \ldots\ldots\ldots \;\ldots & c_{p'p} \end{array}\right| \qquad \ldots \quad (2\cdot5)$$

such that

$$\underline{\overline{c}}^{p'}_{p} \cdot [y]^{n}_{p'} = [x]^{n}_{p} \qquad \ldots \quad (2\cdot6)$$

But the conjugate of the product of two matrices taken in a given order is identical with the product of the conjugates of the two factor matrices taken in the reverse order.

Therefore, $$\underline{\overline{x}}^{p}_{n} = \underline{\overline{y}}^{p'}_{n} \cdot [c]^{p}_{p'} \qquad \ldots \quad (2\cdot7)$$

and $$[x]^{n}_{p} \cdot \underline{\overline{x}}^{p}_{n} = \underline{\overline{c}}^{p'}_{p} \cdot [y]^{n}_{p'} \cdot \underline{\overline{y}}^{p'}_{n} \cdot [c]^{p}_{p'}$$

$$= \underline{\overline{c}}^{p'}_{p} \cdot [1]^{p'}_{p'} \cdot [c]^{p}_{p'} = 1 \cdot \underline{\overline{c}}^{p'}_{p} \cdot [c]^{p}_{p} \qquad \ldots \quad (2\cdot8)$$

because matrix products are associative, and in a product a unit matrix is equivalent to a scalar multiplier unity.

Hence by (1·6) and (1·9) we have

$$\underset{\llcorner p}{\overset{\ulcorner p'}{c}} \cdot [c]^{p}_{p'} = n \cdot [a]^{p}_{p} \qquad \ldots \quad (2{\cdot}9)$$

This is a second degree matrix equation, the solution of which is known. Thus $[c]^{p'}_{p}$ can be determined. Going back to

$$\underset{\llcorner p}{\overset{\ulcorner p'}{c}} \quad [y]^{n}_{p'} = [x]^{n}_{p} \qquad \ldots \quad (2{\cdot}95)$$

we can solve for $[y]^{n}_{p'}$, subject to the condition that $[y]^{n}_{p'}$ is a semi-unit matrix.

3. *A Particular Solution.* Let us consider equation (2·9)

$$\underset{\llcorner p}{\overset{\ulcorner p'}{c}} \cdot [c]^{p'}_{p} = n \cdot [a]^{p}_{p} \qquad \ldots \quad (2{\cdot}9)$$

For convenience of reference we shall call the matrix $[a]$ the dispersion matrix (for the sample).

Cullis has shown[2] that the above equation always admits of a solution, when the rank of the matrix $[a]^{p}_{p}$ is p', and that every solution has the same rank p'. It should be noticed that $[a]^{p}_{p}$ being the dispersion matrix of p statistical variates is positive definite. Hence its rank is necessarily p. If we take $p'=p$, the solution given by Cullis will hold. Further let

$$[c]^{p}_{p} = [c']^{p}_{p} \qquad \ldots \quad (3{\cdot}1)$$

be a particular solution, so that

$$\underset{\llcorner p}{\overset{\ulcorner p}{c'}} \cdot [c']^{p}_{p} = n \cdot [a]^{p}_{p} \qquad \ldots \quad (3{\cdot}2)$$

Then the general solution will be given by

$$[c]^{p}_{p} = [z]^{p}_{p} \cdot [c']^{p}_{p} \qquad \ldots \quad (3{\cdot}3)$$

where $[z]^{p}_{p}$ is any semi-unit matrix.

The method of finding a particular solution is laborious but straightforward. Let us represent the successive leading diagonal minor determniants of $[a]^{p}_{p}$ in the following way.

$$A_0 = 1, \qquad A_1 = a_{11} \qquad \ldots \quad (3{\cdot}41)$$

$$A_2 = (a)^2_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \ldots \quad (3{\cdot}42)$$

[2] C. E. Cullis : *Matrices and Determinoids* (Camb. Univ. Press) Vol. II, § 160, p. 356.

$$A_3 = (a)_3^3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \ldots \quad (3\cdot43)$$

$$A_p = (a)_p^p = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ a_{p1} & a_{p2} & \ldots & a_{pp} \end{vmatrix} \quad \ldots \quad (3\cdot44)$$

We now evaluate a set of quantities $s_v^{(1)}, s_v^{(2)}, \ldots\ldots s_v^{(p)}$ defined by the following equations

$$s_v^{(1)} = a_{1,v} \quad \ldots \quad (3\cdot51)$$

$$\Delta_1 s_v^{(2)} = \begin{vmatrix} a_{11} & a_{1,\ 1+v} \\ a_{21} & a_{2,\ 1+v} \end{vmatrix} \quad \ldots \quad (3\cdot52)$$

$$\Delta_2 s_v^{(3)} = \begin{vmatrix} a_{11} & a_{12} & a_{1,\ 2+v} \\ a_{21} & a_{22} & a_{2,\ 2+v} \\ a_{31} & a_{32} & a_{3,\ 2+v} \end{vmatrix} \quad \ldots \quad (3.53)$$

etc., and in general

$$\Delta_{i-1} s_v^{(i)} = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1,\ i-1} & a_{1,\ i-1+v} \\ a_{21} & a_{22} & \ldots & a_{2,\ i-1} & a_{2,\ i-1+v} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{i,\ i-1} & a_{i,\ i-1+v} \end{vmatrix} \quad \ldots \quad (3\cdot54)$$

where i can take any value from 1 to p.

We next obtain a second set of quantities defined by

$$d_v^{(1)} = \frac{s_v^{(1)}}{s_1^{(1)}}, \quad d_v^{(2)} = \frac{s_v^{(2)}}{s_1^{(2)}}, \ \ldots\ldots \ d_v^{(p)} = \frac{s_v^{(p)}}{s_1^{(p)}} \quad \ldots \quad (3\cdot6)$$

This now enables us to form a quasi-scalar square matrix defined by

$$[k]_p^p = \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \sqrt{s_1^{(2)}} & 0 & \ldots & 0 \\ 0 & 0 & \sqrt{s_1^{(3)}} & \ldots & 0 \\ \ldots & \ldots & & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \sqrt{s_1^{(p)}} \end{vmatrix} \quad \ldots \quad (3\cdot7)$$

and a subsidiary matrix

$$[h]_p^p = \left| \begin{array}{cccccc} 0 & d_2^{(1)} & d_3^{(1)} & \cdots & \cdots & d_p^{(1)} \\ 0 & 1 & d_2^{(2)} & d_3^{(2)} & \cdots & d_{p-1}^{(2)} \\ 0 & 0 & 1 & d_2^{(3)} & \cdots & d_{p-2}^{(3)} \\ \cdots & & \cdots & \cdots & \cdot\ \cdot & \cdots \\ 0 & 0 & 0 & 0 & \cdots & d_1^{(p)} \end{array} \right| \qquad \ldots \quad (3{\cdot}8)$$

We can now write down a particular solution

$$[c]_p^p = \sqrt{n} \,.\, [k]_p^p \; [h]_p^p = [c']_p^p \quad \text{say} \qquad \ldots \quad (3{\cdot}91)$$

In numerical calculations it is not necessary to find the quantities defined by equations (2·6), (3·7), (3·8). We can directly write the values for $[c'_{ij}]$ in terms of the quantities defined in the equation (3·5)

$$c'_{ij} = \sqrt{n} \,.\, s^{(i)}{}_{j-i+1} \;/\; \sqrt{s_1^{(i)}} \qquad \ldots \quad (3{\cdot}92)$$

for all values of i less than or equal to j.

Also $\quad c'_{ij} = 0 \quad$ for all values of i greater than j. $\qquad \ldots \quad (3{\cdot}93)$

4. *The General Solution.* We may now consider the equation

$$\ulcorner c \lrcorner_p^p \,.\, [y]_p^n = [x]_p^n \qquad \ldots \quad (4{\cdot}1)$$

Let $\ulcorner C \lrcorner_p^p$ be the reciprocal, and hence $[C]_p^p$ be the conjugate reciprocal of the matrix $\ulcorner c \lrcorner_p^p$, so that

$$[C]_p^p \,.\, \ulcorner c \lrcorner_p^p = \ulcorner C \lrcorner_p^p \,.\, [c]_p^p = \Delta \; [1]_p^p \qquad \ldots \quad (4{\cdot}2)$$

where Δ is determinant of the matrix $\ulcorner c \lrcorner_p^p$.

Prefixing $[C]_p^p$ on both sides of the equation (4·1) we have

$$[C]_p^p \,.\, \ulcorner c \lrcorner_p^p \,.\, [y]_p^n = [C]_p^p \,.\, [x]_p^n = \Delta \,.\, [l]_p^n \qquad \ldots \quad (4{\cdot}3)$$

where $[l]_p^n$ is now a known matrix.

But the left hand side reduces to $\Delta \,.\, [y]_p^n$ where Δ is, as before, the determinant of the matrix $\ulcorner c \lrcorner_p^p$

Hence, $$[y]_p^n = [l]_p^n \qquad \ldots \qquad (4\cdot4)$$

and $$[c]_p^p \cdot [x]_p^n = \Delta \cdot [y]_p^n \qquad \ldots \qquad (4\cdot5)$$

$[c]_p^p$ is therefore the matrix of a general transformation, which converts the correlated variates $[x_1, x_2, \ldots\ldots x_p]$ to the statistically independent variates $[y_1, y_2, \ldots\ldots y_p]$. One such particular transformation is $[c']_p^p$.

SECTION II. RECTANGULAR CO-ORDINATES.

5. *Rectangular Co-ordinates for the Sample.* Let us consider a space of n dimensions. The n measurements $x'_{i\lambda}$ for the i-th character, can now be represented by a single point X'_i in this space, with co-ordinates $x'_{i\lambda}$ ($\lambda = 1, 2, \ldots n$). For the p characters we then get p points $X'_1, X'_2, \ldots X'_p$. Let the line OT be the equiangular line, *i.e.* the line making equal angles with all the axes. Let $X'_i M_i$ be the perpendicular from X'_i on OT ($i = 1, 2, \ldots p$). Then it is known that

$$M_i X'^2_i = n.a_{ii}, \qquad OM_i = n.a_i \qquad \ldots \qquad (5\cdot0)$$

Through O draw OX_i equal and parallel to $M_i X'_i$. Let θ_{ij} be the angle between OX_i and OX_j. Then we know that

$$\cos\theta_{ij} = r_{ij}, \qquad OX_i \cdot OX_j \cdot \cos\theta_{ij} = n.a_{ij} \qquad \ldots \qquad (5\cdot1)$$

Consider the figure formed by $OX_1, OX_2, \ldots OX_p$. If we reduce the whole figure in the ratio $1 : \sqrt{n}$, we get the figure $OZ_1 Z_2 \ldots Z_i \ldots Z_p$, where Z_i ($i = 1, 2, \ldots p$) is the point on OX_i such that

$$\frac{OZ_i}{OX_i} = \frac{1}{\sqrt{n}} \qquad \ldots \qquad (5\cdot2)$$

We then have $$OZ_i^2 = a_{ii}, \qquad OZ_i \cdot OZ_j \cdot \cos Z_i OZ_j = a_{ij} \qquad \ldots \qquad (5\cdot3)$$

The figure $OZ_1 Z_2 \ldots Z_p$ is fundamental to our investigations. We may call this figure the fundamental polyhedron for the sample.

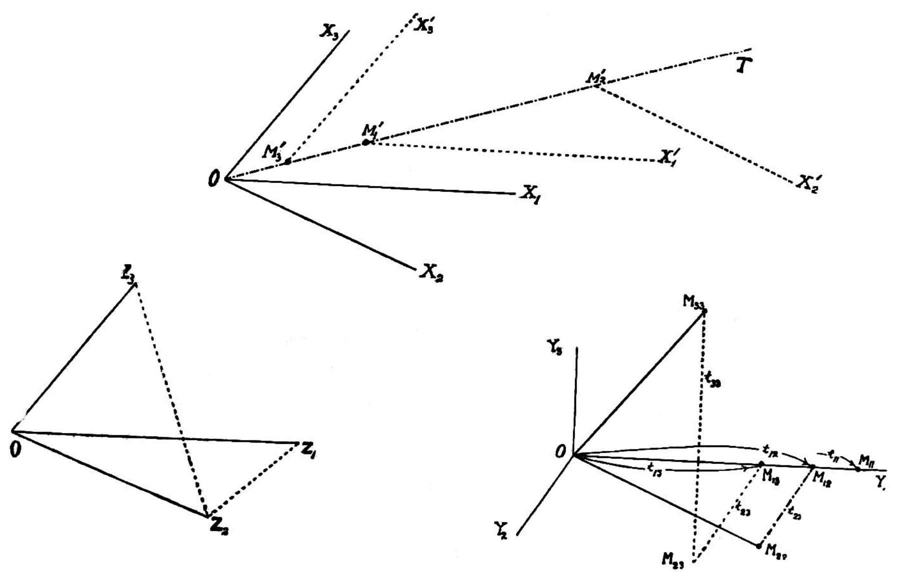
The lines $OZ_1, OZ_2 \ldots OZ_p$ all lie in a linear subspace of p dimensions. Different samples have different fundamental polyhedra, but they are all immersed in a space of $n-1$ dimensions, orthogonal to the equiangular line OT through O.

Let M_{ij} denote the foot of the perpendicular from the point Z_j to the subspace $OZ_1 Z_2 \ldots\ldots Z_i$. It should be noticed that in M_{ij}, the value of i may be taken 1, 2, 3, ... up to j, but not greater than j; also $M_{jj} = Z_j$. Then any two links of the broken chain $OM_{1j} M_{2j} \ldots M_{jj}$ are perpendicular.

We now take a new system of rectangular axes OY_1, OY_2,...OY_p immersed in the space $OZ_1\, Z_2 \ldots Z_p$ such that OY_1 is identical with OZ_1 ; OY_2 lies in the plane $OZ_1\, Z_2$, and is perpendicular to OY_1, and in general OY_j $(j \leqq p)$ is taken to lie in the subspace $OZ_1\, Z_2 \ldots Z_j$ and is perpendicular to OY_1, OY_2, ... OY_{j-1}. Then if t_{1j}, t_{2j} ... t_{ij} be the co-ordinates of Z_j with reference to this system of co-ordinates,

$$t_{ij} = M_{i-1,j}\, M_{ij} \qquad (i \leqq j)$$
$$t_{ij} = 0 \qquad (i>j) \qquad \ldots \quad (5\cdot 4)$$

where it is to be remembered that the point $M_{o,j}$ is the origin O, for $j = 1, 2, 3, \ldots p$. The diagram for three variates $(p=3)$ is given below.



We now write out the matrix

$$[t]_p^p \equiv \begin{vmatrix} t_{11}, & t_{12}, & t_{13} & \ldots & t_{1p} \\ 0, & t_{22}, & t_{23} & \ldots & t_{2p} \\ 0, & 0, & t_{33} & \ldots & t_{3p} \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & t_{pp} \end{vmatrix} \qquad \ldots \quad (5\cdot 5)$$

Thus the elements in the q-th column of the matrix $[t]_p^p$ are the co-ordinates of the point Z_q, with reference to the new system. We shall call such a system of $p\,(p+1)/2$ co-ordinates, the rectangular co-ordinates of the sample.

Before proceeding to the problem of their distribution we shall investigate the connection in which they stand to the quantities c'_{ij} introduced in the first section.

6. *Rectangular Co-ordinates and the Dispersion Matrix.* Now from (5·5) we have

$$t_{11} = M_{01}\,M_{11} = OZ_1 = s_1 = |\,a_{11}\,|^{1/2} \qquad \ldots \qquad (6\cdot10)$$

$$t_{22} = \text{(parallelogram formed by } OZ_1,\ OZ_2)/OZ_1$$

$$= \frac{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{1/2}}{|\,a_{11}\,|^{1/2}} \qquad \ldots \qquad (6\cdot11)$$

$$t_{ii} = \frac{\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1i} \\ a_{21} & a_{22} & \ldots & a_{2i} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i1} & a_{i2} & \ldots & a_{ii} \end{vmatrix}^{1/2}}{\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1,\,i-1} \\ a_{21} & a_{22} & \ldots & a_{2,\,i-1} \\ \ldots & \ldots & \ldots & \ldots \\ a_{i-1,1} & a_{i-1,2} & \ldots & a_{i-1,\,i-1} \end{vmatrix}^{1/2}} \qquad \ldots \qquad (6\cdot12)$$

where $i \leqq p$; as $t_{ii} = M_{i-1,i}\,M_{ii}$ is the ratio of the volumesof the parallelopipeds formed by (OZ_1, OZ_2, ... OZ_i) and (OZ_1, OZ_2, ... OZ_{i-1}) and these volumes are given by the numerator and denominator respectively in (6·12).

Again, $$t_{12} = M_{02}\,M_{12} = OM_{12} = OZ_2\cos\theta_{12} = s_2\,r_{12}$$

$$= \frac{s_1 s_2 r_{12}}{s_1} = \frac{a_{12}}{(a_{11})^{1/2}} \qquad \ldots \qquad (6\cdot21)$$

Similarly we can prove that

$$t_{1k} = s_k\,r_{1k} = a_{1k}/(a_{11})^{1/2} \qquad \ldots \qquad (6\cdot22)$$

where $k \gtreqless 1$, and is of course less than or equal to p, the number of characters.

Again, $$s_2\,s_3\,r_{23} = OZ_2\,OZ_3\cos\theta_{23} = t_{12}\,t_{13} + t_{22}\,t_{23} + 0\,.\,t_{33}$$

therefore, $t_{23} = (s_2\,s_3\,r_{23} - t_{12}\,t_{13})/t_{22} = (s_2\,s_3\,r_{23} - s_2\,s_3\,r_{12}\,r_{23})/t_{22}$ from (6·22)

$$= \frac{s_2\,s_3\begin{vmatrix} 1 & r_{13} \\ r_{12} & r_{23} \end{vmatrix}}{t_{22}} = \frac{\dfrac{s_2\,s_3}{s_1^2\,s_2\,s_3}\begin{vmatrix} s_1^2 & s_1\,s_3\,r_{13} \\ s_1\,s_2\,r_{12} & s_2\,s_3\,r_{23} \end{vmatrix}}{t_{22}}$$

Thus from (6·12), we have

$$t_{23} = \frac{\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}}{(a_{11})^{1/2} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{1/2}} \quad \ldots \quad (6·30)$$

In a similar manner we can prove the general result

$$t_{2k} = \frac{\begin{vmatrix} a_{11} & a_{1k} \\ a_{21} & a_{2k} \end{vmatrix}}{(a_{11})^{1/2} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{1/2}} \quad \ldots \quad (6·31)$$

where k ranges from 2 to p.

Again, $s_3\, s_4\, r_{34} = OZ_3\, OZ_4 \cos \theta_{34} = t_{13}\, t_{14} + t_{23}\, t_{24} + t_{33}\, t_{34} + 0\,.\, t_{44}$

that is $t_{34} = (s_3\, s_4\, r_{34} - t_{13}\, t_{14} - t_{23}\, t_{24})/t_{33}$

which after substitution from (6·12), (6·22), (6·31) and some reduction gives us

$$t_{34} = \frac{\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{1/2} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{1/2}} \quad \ldots \quad (6·40)$$

In the same way we show that

$$t_{3k} = \frac{\begin{vmatrix} a_{11} & a_{12} & a_{1k} \\ a_{21} & a_{22} & a_{2k} \\ a_{31} & a_{32} & a_{3k} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^{1/2} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{1/2}} \quad \ldots \quad (6·41)$$

where k ranges from 3 to p.

The form of (6·31) and (6·41) suggests the following general result.

$$t_{ik} = \frac{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1}; & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2,i-1}, & a_{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{i,i-1}; & a_{ik} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,i-1} \\ a_{21} & a_{22} & \cdots & a_{2,i-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,i-1} \end{vmatrix}^{1/2} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,i} \\ a_{21} & a_{22} & \cdots & a_{2,i} \\ \cdots & \cdots & \cdots & \cdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,i} \end{vmatrix}^{1/2}} \qquad (6·5)$$

where i does not exceed k, and k ranges from 1 to p.

Assuming the above result to be true up to $t_{i-1,k}$ we can prove it to be true for $t_{i,k}$ by arguments similar to those used before. As we have shown it to be true for $i = 1, 2, 3$, the result in question is thus rigorously established by induction.

Now from (3·44) and (3·54) it follows that

$$t_{ik} = \frac{\Delta_{i-1}\ s^{(i)}{}_{k-i+1}}{\Delta_{i-1}{}^{1/2}\ \Delta_i{}^{1/2}} = \left(\frac{\Delta_{i-1}}{\Delta_i}\right).s^{(i)}{}_{k-i+1} \qquad \cdots \qquad (6·6)$$

If in (3·54) we put $v = 1$, we have $\Delta_{p-1}\ s_1{}^{(p)} = \Delta_p$ ... (6·7)

From (6·6) and (6·7) we have $t_{ik} = s^{(i)}{}_{k-i+1} / \sqrt{s_1{}^{(i)}}$ ... (6·8)

where i does not exceed p and k ranges from i to p, and it has been already noticed that $t_{ik} = 0$ when $k < i$.

Comparing with (3·9) we have $c'_{ij} = t_{ij} \sqrt{n}$ ... (6·9)

for all values of i and j from 1 to p.

We have thus established the identity of the matrix of rectangular co-ordinates and the matrix $[c']_p^p$ which we obtained as a particular solution of the fundamental matrix equation (2·9).

7. *Vectorial Interpretation.* The connexion between the investigations of section I and the present section can be still better seen from the following considerations.

Consider a system of p mutually perpendicular unit vectors $[\boldsymbol{i}_1, \boldsymbol{i}_2, \ldots \boldsymbol{i}_p,]$ immersed in the space $OX_1\ X_2 \ldots X_p$. Let OX_1, OX_2, ... OX_p, when considered as vectors be $[\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots \boldsymbol{u}_p]$. Then each of these may be expressed as a linear combination of the unit vectors $\boldsymbol{i}$ as follows

$$\boldsymbol{u}_k = u_{1k}\,\boldsymbol{i}_1 + u_{2k}\,\boldsymbol{i}_2 + \ldots + u_{pk}\,\boldsymbol{i}_p \qquad \text{for} \quad k = 1, 2, \ldots p. \qquad \ldots \quad (7{\cdot}0)$$

Then the linear transformation with matrix $\lceil u \rfloor_p^p$ converts the vectors $\boldsymbol{i}$ to $\boldsymbol{u}$ where

$$\lceil u \rfloor_p^p = \begin{vmatrix} u_{11} & u_{12} & \cdot\cdot & u_{1p} \\ u_{21} & u_{22} & \ldots & u_{2p} \\ \ldots & \ldots & \cdot\cdot & \ldots \\ u_{p1} & u_{p2} & \ldots & u_{pp} \end{vmatrix} \qquad \ldots \quad (7{\cdot}2)$$

If $[U]_p^p$ denotes the conjugate reciprocal of $\lceil u \rfloor_p^p$ and $\Delta_u = \left| u_{ij} \right|$ is the determinant of $\lceil u \rfloor_p^p$, then the linear transformation with the matrix

$$\frac{1}{\Delta_u} \cdot [U]_p^p \qquad \ldots \quad (7{\cdot}3)$$

will convert the vectors $\boldsymbol{u}$ to $\boldsymbol{i}$, and so this linear transformation acting on the correlated variates $[x_1, x_2, \ldots x_p]$ will convert them into statistically independent variates with unit standard deviations.

If instead of the vectors $\boldsymbol{i}$ we had started with any other set of mutually perpendicular vectors $[\,\boldsymbol{j}_1, \boldsymbol{j}_2, \ldots \boldsymbol{j}_p\,]$ and expressed

$$\boldsymbol{u}_k = v_{1k}\,\boldsymbol{j}_1 + v_{2k}\,\boldsymbol{j}_2 + \ldots + v_{pk}\,\boldsymbol{j}_p \qquad \text{and set}$$

$$\lceil v \rfloor_p^p = \begin{vmatrix} v_{11} & v_{12} & \ldots & v_{12} \\ v_{21} & v_{22} & \ldots & v_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ v_{p1} & v_{p2} & \ldots & v_{pp} \end{vmatrix} \qquad \ldots \quad (7{\cdot}4)$$

then the matrix $\lceil v \rfloor_p^p$ has the same property as $\lceil u \rfloor_p^p$ namely that if $[V]_p^p$ is its conjugate reciprocal then the linear transformation with matrix

$$\frac{1}{\Delta_v} \cdot [V]_p^p \qquad \ldots \quad (7{\cdot}5)$$

will convert the variates $[\,x_1, x_2, \ldots x_p\,]$ into a set of statistically independent variates with unit standard deviations, Δ_v being the determinant of the matrix $\lceil v \rfloor_p^p$.

It is geometrically evident that we can pass from $\overline{\underline{u}}_p^p$ to $\overline{\underline{v}}_p^p$ by multiplying the former with a semi-unit matrix.

If now in particular we take our p orthogonal unit vectors $[i_1, i_2, \ldots i_p,]$ to lie along OY_1, OY_2, ... OY_p, then

$$u_{ij} = t_{ij} \sqrt{n} \qquad \ldots \qquad (7{\cdot}6)$$

where t_{ij}'s are rectangular co-ordinates for the sample. Consequently from (6·9) we have

$$u_{ij} = c'_{ij} \qquad \ldots \qquad (7{\cdot}7)$$

where c'_{ij} 's are the quantities occuring in the investigations of section 1.

We thus arrive by another route at the proposition proved already, that if $[C']_p^p$ is the conjugate reciprocal of $\overline{\underline{c'}}_p^p$, and Δ' the determinant of $\overline{\underline{c'}}_p^p$, then the transformation with the matrix

$$\frac{1}{\Delta'} \cdot [C']_p^p$$

is one of the linear transformations which change the variates $[x_1, x_2, \ldots x_p]$ into a set of statistically independent variates with unit standard deviations. It also appears that the most general transformation with this property is one with matrix

$$\ldots \quad \frac{1}{\Delta} \cdot [\,C\,]_p^p \qquad \text{where} \qquad \overline{\underline{c}}_p^p = [z]_p^p \cdot \overline{\underline{c'}}_p^p$$

$[z]_p^p$ being any arbitrary semi-unit matrix; $[C]_p^p$ is the conjugate reciprocal of $\overline{\underline{c}}_p^p$, and Δ is the determinant of the matrix $\overline{\underline{c}}_p^p$.

8. *Rectangular Co-ordinates for the Population.* Before proceeding to the question of the distribution of the rectangular co-ordinates for the sample, we proceed to define the rectangular co-ordinates for the population.

In a space of p dimensions, let us take a set of lines OZ'_1, OZ'_2, ... OZ'_p, such that

$$\left.\begin{aligned} OZ'_i &= \alpha_{ii}, \\ \cos Z'_i\, OZ'_j &= \rho_{ij}, \end{aligned}\right\} \qquad \ldots \qquad (8{\cdot}0)$$

where α_{ii} is the population variance for the i-th character and ρ_{ij} is the population correlation between the i-th and j-th characters.

We now take $OZ'_1\, Z'_2 \ldots Z'_p$ as the fundamental polyhedron and denote as before by M'_{ij} the foot of the perpendicular from the point Z'_j to the subspace $OZ'_1\, Z'_2 \ldots Z'_i$ where in M'_{ij} i can be taken equal to 1, 2, ... j.

Then setting

$$\left.\begin{aligned} \tau_{ij} &= M_{i-1.j}\, M_{ij}, \qquad (i \leq j) \\ \tau_{ij} &= 0, \qquad (i > j) \end{aligned}\right\} \qquad \ldots \qquad (8{\cdot}1)$$

we may call τ_{ij} the rectangular co-ordinates of the population.

SECTION III. JOINT DISTRIBUTION OF RECTANGULAR AND OF NORMAL CO-ORDINATES.

9. *Density Factor.* On the hypothesis of normal distribution, the distribution of the reduced observations $(x_{i\lambda})$ [cf. 1·3] $(i = 1, 2, \ldots p$, and $\lambda = 1, 2, \ldots n)$ can be written, after integrating out for the means $(a_1, a_2 \ldots\ldots a_p)$, as

$$df = \frac{1}{n^{p/2}.(2\pi)^{(n-1)p/2}.\,|\alpha|^{(n-1)/2}}.e^{-\frac{1}{2}n\{\alpha^{11}a_{11} + \alpha^{22}a_{22} + \ldots 2\alpha^{12}a_{12} + \ldots\}}.[dx_{i\lambda}] \quad (9{\cdot}0)$$

where $[dx_{i\lambda}]$ stands for the product of all differentials $dx_{i\lambda}$, for values of $i = 1, 2, \ldots p$; and $\lambda = 1, 2, \ldots n$; also $|\alpha|$ is the determinant of the matrix of population dispersions (α_{ij}), that is.

$$|\alpha| = |\alpha_{ij}|, \text{ and } \alpha^{ij} = \frac{A_{ij}}{|\alpha|} \quad (9{\cdot}1)$$

where A_{ij} is the minor of α_{ij} in $|\alpha|$.

Now we take, following R.A. Fisher, a space of np dimensions, and subdivide it into p n-dimensional orthogonal subspaces. In the i-th subspace we can take n orthogonal axes $OX_{i\lambda}$, $(\lambda = 1, 2, \ldots n)$, and can represent the i-th reduced character by a point Q_i with co-ordinates, $x_{i\lambda}$ $(\lambda = 1, 2, \ldots n)$, with respect to the axes just chosen. If we rotate all the subspaces so as to make them fall on the first subspace, then the points $Q_1, Q_2, \ldots Q_p$ may be considered to take the positions $X_1, X_2, \ldots X_p$, considered in paragraph 6. Let Q be the point of the complete np space whose projections on the p subspaces are $Q_1, Q_2, \ldots Q_p$. Then we can call Q, the representative point of the reduced observations, and the complete space may be considered as being populated by the representative points Q, with density

$$\frac{1}{n^{p/2}.(2\pi)^{(n-1)p/2}.\,|\alpha|^{(n-1)/2}}.e^{-\frac{n}{2}\{\alpha^{11}a_{11} + \alpha^{22}a_{22} + \ldots 2\alpha^{12}a_{12} + \ldots\}} \quad \ldots \quad (9{\cdot}2)$$

Let t_{ij} be the rectangular co-ordinates for the sample, and τ_{ij} the similar rectangular co-ordinates for the population, defined before. Then we shall first show that the density (9·2) is expressible in terms of the rectangular co-ordinates.

$$\text{Let} \quad [\tau]_p^p = \begin{vmatrix} \tau_{11} & \tau_{12} & \ldots & \tau_{1p} \\ \tau_{21} & \tau_{22} & \ldots & \tau_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ \tau_{p1} & \tau_{p2} & \ldots & \tau_{pp} \end{vmatrix} \quad \ldots \quad (9{\cdot}3)$$

be the matrix of rectangular co-ordinates for the population; and let $\overset{\frown p}{\underset{\smile p}{\tau}}$ be the conjugate matrix.

Suppose $\overline{\underline{\tau}}^{p}_{p} \cdot [\tau]^{p}_{p} = [T]^{p}_{p}$

$$= \begin{vmatrix} T_{11} & T_{12} & \cdots & T_{1p} \\ T_{21} & T_{22} & \cdots & T_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ T_{p1} & T_{p2} & \cdots & T_{pp} \end{vmatrix} \quad \ldots \quad (9{\cdot}4)$$

then we shall show that $T_{ij} = \alpha_{ij}$.

Now from the multiplication rule for matrices it follows easily that

$$T_{ij} = S_{\lambda}\ (\tau_{\lambda i} \cdot \tau_{\lambda j}) \quad \ldots \quad (9{\cdot}5)$$

where it is to be remembered that $\tau_{ij} = 0$ for $i > j$.

But $S_{\lambda}\ (\tau_{\lambda i} \cdot \tau_{\lambda j})$ is from geometry the scalar product of the vector OZ'_i and OZ'_j and is therefore equal to $\sigma_i . \sigma_j . \rho_{ij} = \alpha_{ij}$.

$$\text{Hence} \quad [T]^{p}_{p} \equiv [\alpha]^{p}_{p} \quad \ldots \quad (9{\cdot}6)$$

$$\text{Consequently also} \quad \alpha^{ij} = T^{ij} \quad \ldots \quad (9{\cdot}7)$$

where T^{ij} is the minor of T_{ij} in the determinant of $[T]^{p}_{p}$, divided by this determinant.

$$\text{Again} \quad a_{ij} = S_{\lambda}\ (t_{\lambda i} \cdot t_{\lambda j}) \quad \ldots \quad (9{\cdot}8)$$

where as before $t_{ij} = 0$ for $i > j$.

$$\text{Also} \quad (\alpha^{11} a_{11} + \alpha^{22} a_{22} + \ldots\ldots 2\alpha^{12} a_{12} + \ldots\ldots)$$
$$= \{T^{11} t_{11}^2 + T^{22} (t_{12}^2 + t_{22}^2) + \ldots\ldots T^{pp} (t_{1p}^2 + t_{2p}^2 + \ldots\ t_{pp}^2) + \}$$
$$+ \{2\,T^{12}\, t_{11}\, t_{12} + 2\,T^{13}\, t_{11}\, t_{13} + 2\,T^{23}\, (t_{12}\, t_{13} + t_{22}\, t_{23}) + \ldots$$
$$\ldots + 2\,T^{ij} (t_{11}\, t_{1j} + t_{21}\, t_{2j} + \ldots\ t_{ii}\, t_{ij}) + \ldots\} \quad (9{\cdot}9)$$

We shall find it convenient to denote the left hand side in equation (9·9) by $F(T, t)$. We can then write the density factor (9·2) as

$$\frac{1}{n^{p/2} . (2\pi)^{(n-1)p/2} . \mid T \mid^{(n-1)/2}}\ e^{-\frac{1}{2}n\ .\ F(T,t)} \quad \ldots \quad (9{\cdot}95)$$

where $\mid T \mid$ is the determinant of the matrix $[T]^{p}_{p}$.

10. *Joint Distribution of Rectangular Co-ordinates.* We now want the total density for those samples for which the rectangular coordinates lie between t_{ij} and $t_{ij} + dt_{ij}$ where i will take all values from 1 to p, and for any given value of i, j will take all values from that given value of i to p.

When t_{11} is constant, the point X_1 describes an $(n-1)$-dimensional hypersphere of radius $t_{11}\sqrt{n}$, because OX_1 is always perpendicular to the equiangular line in the first subspace. When t_{11}, t_{12}, t_{22} are constants, the point X_2 describes an $(n-2)$-dimensional hypersphere of radius $t_{22}\sqrt{n}$, and so on. Finally X_p will describe a hypersphere of radius $t_{pp}\sqrt{n}$. Now the surface of a k-dimensional hypersphere is

$$2 . \frac{\pi^{k/2}}{\Gamma(k/2)} . r^{k-1} \qquad \ldots \qquad (10{\cdot}1)$$

where r is the radius of the hypersphere. Hence consistently with the restrictions on the representative point Q, it describes a hypervolume

$$\frac{2^p . \pi^{p(2n-p-1)/4} . n^{np/2}}{\Gamma\frac{1}{2}(n-1).\Gamma\frac{1}{2}(n-2). \ldots \Gamma\frac{1}{2}(n-p)} \times t_{11}^{n-2} . t_{22}^{n-3} . \ldots t_{pp}^{n-p-1} [dt_{ij}] \qquad \ldots \qquad (10{\cdot}2)$$

where $[dt_{ij}]$ stands for the product of all differential elements like dt_{ij} where i will take all values from 1 to p, and for any given value of i, j will take all values from that value of i to p.

Hence finally the joint distribution of the rectangular co-ordinates takes the form :—

$$\frac{n^{p(n-1)/2} . |T^{ij}|^{(n-1)/2}}{2^{p(n-1)/2} \pi^{p(p+1)/4}} e^{-\frac{1}{2}n . F(T,\, t)} . t_{11}^{n-2} t_{22}^{n-3} \ldots\ldots t_{pp}^{n-p-1} . [dt_{ij}] \qquad \ldots \qquad (10{\cdot}3)$$

We shall show later that J. Wishart's joint distribution of the sample dispersions is directly deducible from this.

11. *Normal Form and Normal Co-ordinates.* We have denoted by F (T, t) the expression on the right hand side of (9·9). We can regard this expression as a quadratic form in the $p(p+1)/2$ variables t_{ij}. We shall now construct a linear transformation which leaves unchanged the p variables $[t_{11}, t_{22}, \ldots t_{pp}]$ except for constant factors, and at the same time reduces the quadratic form under consideration to a sum of squares.

$$\begin{aligned} F(T,t) = \; & \{T^{11} t_{11}^2 + T^{22}(t_{12}^2 + t_{22}^2) + \ldots T^{pp}(t_{1p}^2 + t_{2p}^2 + \ldots\ldots + t_{pp}^2)\} \\ & + \{2T^{12} t_{11} t_{12} + 2T^{13} t_{11} t_{13} + \ldots 2T^{1p} t_{11} t_{1p} \\ & + 2T^{23}(t_{12} t_{13} + t_{22} t_{23}) + \ldots 2T^{2p}(t_{12} t_{1p} + t_{22} t_{2p}) + \quad \ldots \quad \ldots \\ & + 2T^{p-1,p}(t_{1,p-1} t_{1,p} + t_{2,p-1} t_{2,p} + \ldots t_{p-1,p-1} t_{p-1,p})\} \\ = \; & \mathop{S^p}_{i,j=1} [T^{ij} . \mathop{S^k}_{\lambda=1} (t_{\lambda i} . t_{\lambda j})] \end{aligned}$$

where k is the lesser of i and j, and $T^{ij} = T^{ji}$. ... (11·1)

Let $|T|$ denote the determinant of the matrix $[T]_p^p$ defined by

$$|T| = \begin{vmatrix} T^{11} & T^{12} & \dots & T^{1,p-1} & T^{1,p} \\ T^{21} & T^{22} & \dots & T^{2,p-1} & T^{2,p} \\ \dots & \dots & \dots & \dots & \dots \\ T^{p-1,1} & T^{p-1,2} & \dots & T^{p-1,p-1} & T^{p-1,p} \\ T^{p,1} & T^{p,2} & \dots & T^{p,p-1} & T^{p,p} \end{vmatrix} \quad \dots \quad (11{\cdot}2)$$

If we take the last r constituents of the diagonal elements of $|T|$, where r is of course less than p, and consider that minor of T, which has these terms for its leading diagonal terms, then we can call this minor $|T_{(r)}|$.

Thus

$$|T_{(1)}| = |T^{pp}| = T^{pp} \quad \dots \quad (11{\cdot}21)$$

$$|T_{(2)}| = \begin{vmatrix} T^{p-1,p-1} & T^{p-1,p} \\ T^{p,p-1} & T^{pp} \end{vmatrix} \quad \dots \quad (11{\cdot}22)$$

$$= T^{p-1,p-1} \,.\, T^{p,p} - (T^{p-1,p})^2 \quad \dots \quad (11{\cdot}23)$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$|T_{(p)}| = |T| \quad \text{itself} \quad \dots \quad (11{\cdot}24)$$

Let T^{ij} be any element of $|T|$ such that $i<p-r$, $j<p-r$. We shall define the determinant $|T_{(r)}^{ij}|$ with the help of the following equation (11·25).

$$|T_{(r)}|\;|T_{(r)}^{ij}| \equiv \begin{vmatrix} T^{ij} & T^{i,p-r+1} & T^{i,p-r+2} & \dots & T^{i,p} \\ T^{p-r+1,j} & T^{p-r+1,p-r+1} & T^{p-r+1,p-r+2} & \dots & T^{p-r+1,p} \\ T^{p-r+2,j} & T^{p-r+2,p-r+1} & T^{p-r+2,p-r+2} & \dots & T^{p-r+2,p} \\ \dots & \dots & \dots & \dots & \dots \\ T^{p,j} & T^{p,p-r+1} & T^{p,p-r+2} & \dots & T^{pp} \end{vmatrix} \quad \dots \quad (11{\cdot}25)$$

It is to be noticed that

$$|T_{(r)}^{p-r,p-r}| \equiv |T_{(r+1)}| \,/\, |T_{(r)}| \quad \dots \quad (11{\cdot}26)$$

We have, therefore,

$$T_{(p-1)}^{11}\; T_{(p-2)}^{22} \dots T_0^{pp}$$

$$= \frac{|T_{(p)}|}{|T_{(p-1)}|}\;\frac{|T_{(p-1)}|}{|T_{(p-2)}|} \;\dots\dots\; \frac{|T_{(1)}|}{|T_{(0)}|} = |T_{(p)}| = |T| = |T^{ij}| \quad (11{\cdot}30)$$

Let $$l_{1p}\sqrt{T^{pp}} = T^{1p}\, t_{11} + T^{2p}\, t_{12} + \dots T^{pp}\, t_{1p} \quad \dots \quad (11\cdot31)$$

$$= \mathop{S}_{\lambda=1}^{p} (T^{\lambda p}\, t_{1\lambda}) \quad \dots \quad (11\cdot32)$$

$$l_{2p}\sqrt{T^{pp}} = \mathop{S}_{\lambda=1}^{p} (T^{\lambda p}\, t_{2\lambda}) \quad \dots \quad (11\cdot33)$$

where it is to be remembered that $t_{ij} = 0$, if $i > j$, and in general

$$l_{ip}\sqrt{T^{pp}} = \mathop{S}_{\lambda=1}^{p} (T^{\lambda p}\, t_{i\lambda}) \qquad (i = 1, 2, \dots p\,.) \quad \dots \quad (11\cdot34)$$

Then we can write

$$F(T,t) = \mathop{S}_{i,j=1}^{i,j=p} [T^{ij} \,.\, \mathop{S^{k}}_{\lambda=1} (t_{\lambda i}\, t_{\lambda j})] \text{ where } k \text{ is the lesser of } i \text{ and } j$$

$$= l_{1p}^{\,2} + l_{2p}^{\,2} + l_{3p}^{\,2} + \dots\dots l_{pp}^{\,2}$$

$$+ \mathop{S}_{i,j=1}^{i,j=p-1} [T_{(1)}^{ij} \,.\, S^{k}_{\lambda=1} (t_{\lambda i}\, t_{\lambda j})] \quad \dots \quad (11\cdot35)$$

where, as before, k is the lesser of i and j, and

$$T_{(1)}^{ij} = \{T^{ij}\, T^{pp} - T^{i,p}\, T^{j,p}\} \,/\, T^{pp} = |\, T_{(1)}^{ij} \,| \quad \dots \quad (11\cdot36)$$

We now set

$$l_{i,\,p-1} \sqrt{(T_{(1)}^{p-1,\,p-1})} = \mathop{S}_{\lambda=1}^{p-1} (T_{(1)}^{\lambda,\,p-1}\, t_{i\lambda}) \;\; (i = 1, 2, \dots\; p-1) \quad (11\cdot37)$$

We also write

$$\mathop{S}_{i,j=1}^{i,j=p-1} [T_{(1)}^{ij} \,.\, \mathop{S}_{\lambda=1}^{k} (t_{\lambda i} t_{\lambda j})] = l^{2}_{1,p-1} + l_{2,p-1} \dots\dots l^{2}_{p-1,\,p-1}$$

$$+ \mathop{S}_{i,j=1}^{i,j=p-2} [T_{(2)}^{ij} \,.\, \mathop{S}_{\lambda=1}^{k} (t_{\lambda i} t_{\lambda j})] \quad \dots \quad (11\cdot38)$$

where $$T_{(2)}^{ij} = \{T_{(1)}^{ij}\, T_{(1)}^{p-1,\,p-1} - T^{i,\,p-1}\, T^{j,p-1}\} / T^{p-1,\,p-1} \quad \dots \quad (11\cdot39)$$

Now, $$T_{(1)}^{ij} = \{T^{ij}\, T^{pp} - T^{i,p}\, T^{j,p}\} \,/ T^{pp} \quad \dots \quad (11\cdot40)$$

$$T_{(1)}^{p-1,\,p-1} = \{T^{p-1,\,p-1}\, T^{p,p} - (T^{p-1,p})^{2}\} \,/ T^{pp} \quad \dots \quad (11\cdot41)$$

$$T_{(1)}^{i,p-1} = \{T^{i,p-1}\, T^{pp} - T^{i,p}\, T^{p-1,p}\} \,/ T^{pp} \quad \dots \quad (11\cdot42)$$

$$T_{(1)}^{p-1,j} = \{T^{p-1,j}\, T^{pp} - T^{p-1,p}\, T^{j,p}\} \,/ T^{pp} \quad \dots \quad (11\cdot43)$$

Now consider

$$| T_{(2)} | \,.\, | T_{(2)}{}^{ij} | = \begin{vmatrix} T^{ij} & T^{i,p-1} & T^{i,p} \\ T^{p-1,j} & T^{p-1,\,p-1} & T^{p-1,p} \\ T^{p,j} & T^{p,p-1} & T^{p,p} \end{vmatrix} \quad \ldots \quad (11{\cdot}50)$$

Denoting the minor of any term in this determinant by placing a bracket ( ) round it, we have $T_{(1)}{}^{ij} = (T^{p-1,\,p-1})/T^{pp}$ ... (11·51)

$$T_{(1)}{}^{p-1,\,p-1} = (T^{ij})/T^{pp} \quad \ldots \quad (11{\cdot}52)$$

$$T_{(1)}{}^{i,p-i} = (T^{p-1,j})/T^{pp} \quad \ldots \quad (11{\cdot}53)$$

$$T_{(1)}{}^{j,p-1} = (T^{p-1,i})/T^{pp} \quad \ldots \quad (11{\cdot}54)$$

Accordingly $T_{(2)}{}^{ij} = \{(T^{p-1,\,p-1})\ (T^{ij}) - (T^{p-1,i})(T^{p-1,j})\}\ /T^{pp}(T^{ij})$ (11·55)

$$= | T_{(2)} | \quad | T_{(2)}{}^{ij} | \ /(T^{ij}) \quad \ldots \quad (11{\cdot}56)$$

$$= | T_{(2)}{}^{ij} | \quad \ldots \quad (11{\cdot}57)$$

Proceeding in this way and setting

$$l_{i,\,p-r} \sqrt{\,| T_{(r)}{}^{p-r,\,p-r} |} = \mathop{S}_{\lambda=1}^{p-r} [\, T_{(r)}^{\lambda,p-r} \,.\, t_{i\lambda} \,] \qquad (i=1,\, 2,\, \ldots\, p-r)\ldots \quad (11{\cdot}6)$$

we have $F(T,t) = l^2_{1p} + l^2_{2p} + \ldots\, l_{pp}{}^2 + l^2_{1,p-1} + l^2_{2,p-1} + \ldots\, l^2_{p-1,\,p-1}$
$+ \ldots\ldots + l_{13}{}^2 + l_{23}{}^2 + l_{33}{}^2 + l_{12}{}^2 + l_{22}{}^2 + l_{11}{}^2$ ... (11·7)

$$= \mathop{S}_{i,j=1}^{i,j=p} [l^2_{ij}] \qquad \text{where } l_{ij} = 0, \text{ when } i > j \quad \ldots \quad (11{\cdot}8)$$

It is to be noticed that $l_{ii} = t_{ii} \sqrt{\,| T_{(p-i)}^{ii} |}\ \ (i=1,\, 2,\, \ldots\, p)$ ... (11·9)

12. *Rectangular Co-ordinates in terms of Normal Co-ordinates.* We shall now express t_{ij} as a linear combination of the l's. In what follows we shall write $T_{(r)}^{ij}$ for $| T_{(r)}^{ij} |$, the two having been identified earlier.

$$l_{ii} \sqrt{T_{(p-i)}{}^{i,i}} = T_{(p-i)}{}^{i,i} \,.\, t_{ii} \quad \ldots \quad (12{\cdot}11)$$

$$l_{i,\,i+1} \sqrt{T^{i+1,\,i+1}_{(p-i-1)}} = T^{i,i+1}_{(p-i-1)} \,.\, t_{ii} + T^{i+1,\,i+1}_{(p-i-1)} \,.\, t_{i,i+1} \quad \ldots \quad (12{\cdot}12)$$

$$l_{i,j} \sqrt{\,T^{i,j}_{(p-j)}} = \mathop{S}_{\lambda=1}^{j} [T^{\lambda,j}_{(p-j)}\ t_{i,\lambda}] \qquad (j \geqq i) \quad \ldots \quad (12{\cdot}13)$$

$$l_{i,p} \sqrt{\,T^{pp}} = \mathop{S}_{\lambda=1}^{p} [T^{\lambda,p}\ t_{i,\lambda}] \quad \ldots \quad (12{\cdot}14)$$

We have here in hand $(p-i+1)$ linear equations from which the $(p-i+1)$ quantities $t_{i,j}$ $(j = i, i+1,...p)$ can be obtained as linear functions of l_{ij}'s $(j = i, i+1. ... p)$. As we shall presently show, we shall get t_{ij} in the form

$$t_{ij} = k_j^{i,i} l_{ii} + k_j^{i,i+1} l_{i,i+1} + \dots k_j^{i,j} l_{ij} = \mathop{S}_{\lambda=i}^{j} [k_j^{i\lambda} . l_{i\lambda}] \quad \dots \quad (12\cdot2)$$

Consider the determinant of the $(p-i+1)$th order

$$\Delta_{(i)} \equiv \begin{vmatrix} T_{(p-i)}^{i,i} & 0 & 0 & 0 & \dots & 0 \\ T_{(p-i-1)}^{i,i+1} & T_{(p-i-1)}^{i+1,\,i+1} & 0 & 0 & \dots & 0 \\ T_{(p-i-2)}^{i,i+2} & T_{(p-i-2)}^{i+1,\,i+2} & T_{(p-i-2)}^{i+2,i+2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & & \\ T_{(0)}^{i,p} & T_{(0)}^{i+1,p} & T_{(0)}^{i+2,p} & \dots & \dots & T_{(0)}^{p,p} \end{vmatrix} \quad \dots \quad (12\cdot3)$$

We shall use the notation $\Delta_{(i)}^{l,m}$ to denote the minor of the element in the l-th row and m-th column of $\Delta_{(i)}$ divided by $\Delta_{(i)}$ itself. ... (12·31)

It is readily seen that $\Delta_{(i)}^{l,m} = 0$ when $m < l$. ... (12·32)

We now have from the theory of linear equations,

$$t_{ij} = \sqrt{T_{(p-i)}^{i,i}}\, \Delta_{(i)}^{1,j-i+1}\, l_{ii} + \sqrt{T_{(p-i-1)}^{i+1,\,i+1}}\, \Delta_{(i)}^{2,j-i+1}\, l_{i,i+1} + \dots + \sqrt{T_{(p-j)}^{j,j}}\, \Delta_{(i)}^{j-i+1,\,j-i+1}\, l_{i,j} \quad (12\cdot4)$$

It is thus easily seen that the relation (12·2) holds when we take

$$k_j^{i,q} = \sqrt{T_{(p-q)}^{q,q}} \,.\, \Delta_{(i)}^{q-i+1,\,j-i+1} \quad \dots \quad (12\cdot5)$$

13. *Joint Distribution of Normal Co-ordinates.* The joint distribution of the t_{ij}'s has already been obtained in (10·3) in the form

$$\frac{n^{p(n-1)/2} \,|\, T^{ij} \,|\, ^{(n-1)/2}}{2^{p(n-3)/2}\, \pi^{p(p-1)/4}} \cdot \frac{1}{\mathop{P^p}\limits_{k=1} \{\Gamma(n-k)/2\}} \cdot e^{-\frac{1}{2} n\, F\,(T,t)}$$

$$\times \quad (t_{11})^{n-2} (t_{22})^{n-3} \dots\dots (t_{pp})^{n-p-1} . \; [dt_{ij}] \quad \dots \quad (13\cdot1)$$

where $[dt_{ij}]$ denotes the product of all differential elements dt_{ij} $(i=1, 2,...p,$ and $j=1, 2, ...p$; but $t_{ij} = 0$ when $i > j)$, and $F\,(T,t)$ is given by the left hand side of (9·9).

If we denote by $[dl_{ij}]$ the product of all differential elements dl_{ij} ($i=1, 2 \ldots p$, and $j = 1, 2 \ldots\ldots p$; but $l_{ij} = 0$, when $i > j$), then

$$[dl_{ij}] \text{ transforms to } \frac{\partial\,(l_{11}, l_{12}, l_{22}, \ldots l_{pp})}{\partial\,(t_{11}, t_{12}, t_{22}, \ldots t_{pp})} \times [dt_{ij}] \quad \ldots \quad (13\cdot21)$$

Equations (12·11) to (12·14) show that $l_{ii}, l_{i,i+1} \ldots l_{i,p}$ are functions of $t_{ii}, t_{i,i+1}, \ldots t_{i,p}$, and of no other t's. Hence the Jacobian reduces to the product of a number of factors.

Having regard to the form of these equations we can write

$$[dl_{ij}] = \mathbf{P}^{p}_{i=1} (\sqrt{\Delta_{(i)}}) \cdot [dt_{ij}] \quad \ldots \quad (13\cdot22)$$

where $\mathbf{P}^{p}_{i=1}$ denotes the product of the terms for all values of i from 1 to p, and

where $\Delta_{(i)}$, which is given by (12·3), can be written as

$$\Delta_{(i)} = \mathbf{P}^{p}_{k=i} (T^{k,k}_{(p-k)}) \quad \ldots \quad (13\cdot23)$$

$$\text{Therefore,} \quad [dl_{ij}] = \left\{ \left(T^{11}_{(p-1)}\right) \left(T^{22}_{(p-2)}\right)^{2} \left(T^{33}_{(p-3)}\right)^{3} \ldots\ldots \left(T^{pp}\right)^{p} \right\}^{\frac{1}{2}} \times [dt_{ij}] \quad \ldots \quad (13\cdot3)$$

Using (13·3), (11·9), (11·7), and (11·30), the expression (13·1) can be written as

$$\frac{n^{p(n-1)/2}}{2^{p(n-3)/2}\,\pi^{p(p-1)/4}} \cdot \frac{1}{\mathbf{P}^{p}_{k=1} \{\Gamma(n-k)/2\}} \times e^{-\frac{1}{2}n \mathbf{S}^{p}_{j=1} \mathbf{S}^{j}_{i=1} (l_{ij}^2)} \times (l_{11})^{n-2} (l_{22})^{n-3} \ldots (l_{pp})^{n-p-1} [dl_{ij}] \quad \ldots \quad (13\cdot4)$$

where according to the notation already used $[dl_{ij}]$ stands for the product of all differentials like dl_{ij} ($i=1, 2, \ldots\ldots p$, and $j = 1, 2 \ldots\ldots p$; where l_{ij} and $dl_{ij} = 0$ for $i > j$.)

14. *Distributions connected with Normal Co-ordinates.* From (13·4) we now integrate out for all variables l_{ij} ($i \neq j$), and get

$$\frac{n^{p(2n-p-1)/4}}{2^{p(2n-p-5)/4} \mathbf{P}^{p}_{k=1} \{\Gamma(n-k)/2\}} \times e^{-\frac{1}{2}n \mathbf{S}^{p}_{i=1} (l_{ii}^2)} \times (l_{11})^{n-2} (l_{22})^{n-3} \ldots\ldots (l_{pp})^{n-p-1} [dl_{ii}] \quad \ldots \quad (14\cdot1)$$

It is easily seen that the p-fold integral of the above over all values of the variates $l_{11}, l_{22} \ldots\ldots l_{pp}$ from 0 to ∞ reduces, as it should, to unity.

To obtain the distribution of l_{ii} only, we have to integrate out for the remaining $(p-1)$ variables in (14·1). We then get,

$$\frac{n^{(n-i)/2}}{2^{(n-i-2)/2}\,\Gamma(n-i)/2} \cdot e^{-\frac{1}{2}n l_{ii}^2} \cdot (l_{ii})^{n-i-1} \cdot dl_{ii} \text{ as the required distribution} \ldots\ldots \quad (14\cdot2)$$

This shows that the distribution of l_{ii}^2 is of Pearsonian Type III.

If from (13·4) we integrate out for every other variable except l_{ij} $(i \neq j)$, we get the distribution of l_{ij} as

$$\sqrt{(n/2\pi)} \, . \, e^{-\frac{1}{2}n l_{ij}^2} \, dl_{ij} \qquad \ldots \quad (14·3)$$

Thus the l_{ij}'s $(i \neq j)$ are distributed normally.

15. *Distributions connected with Rectangular Co-ordinates.* Distribution of t_{ii} and t_{ij}

Now $$l^2_{ii} = t^2_{ii} \, . \, T^{i,i}_{(p-i)} = t^2_{ii} \, . \, | T_{(p-i+1)} | \, / \, | T_{(p-i)} | \qquad \ldots \quad (15·1)$$

Substituting in (14·2), we get as the distribution of t_{ii}

$$\frac{n^{(n-i)/2} \, | T_{(p-i+1)} |^{(n-i)/2}}{2^{(n-i-2)/2} . \Gamma(n-i)/2 . | T_{(p-i)} |^{(n-i)/2}} \, . \, e^{-\frac{1}{2}n t_{ii}^2 . | T_{(p-i+1)} | / | T_{(p-i)} |} . (t_{ii})^{n-i-1} . dt_{ii} \qquad (15·2)$$

To obtain the distribution of t_{ij} we first note that

$$t_{ij} = k_j^{ii} l_{ii} + k_j^{i,i+1} l_{i,i+1} + \ldots\ldots k_j^{i,j} l_{ij} \quad \text{where } k_j^{iq} = \sqrt{(T_{(p-q)}{}^{q,q}) . \Delta_{(i)}{}^{q-i+1, j-i+1}} \ldots \quad (15·3)$$

We first write down the joint distribution of $l_{ii}, l_{i,i+1}, l_{i,i+2} \ldots\ldots l_{ij}$, which is

$$\frac{n^{(j+n-2i)/2}}{2^{(j+n-2i-2)/2} \, \pi^{(j-i)/2}} \times e^{-\frac{1}{2}n\{(l_{ii})^2 + (l_{i,i+1})^2 \ldots (l_{ij})^2\}} . (l_{ii})^{n-i-1} . dl_{ii} dl_{i,i+1} dl_{i,i+2} \ldots dl_{ij} \ldots \quad (15·4)$$

We now take a new variate

$$y_{ij} = \Big[\mathop{S}_{\lambda=i+1}^{j} (k_j^{i,\lambda} . l_{i\lambda}) \Big] \Big/ \Big[\mathop{S}_{\lambda=i+1}^{j} (k_j^{i,\lambda})^2 \Big]^{\frac{1}{2}} \qquad \ldots \quad (15·45)$$

Then the distribution of y_{ij} will be a normal distribution with a variance equal to that of the l's.

Hence the distribution of y_{ij} is $$\sqrt{(n/2\pi)} \, . \, e^{-\frac{1}{2}n y^2_{ij}} \, dy_{ij} \qquad \ldots \quad (15·51)$$

We have now

$$t_{ij} = k_j^{ii} l_{ii} + \Big[\mathop{S}_{\lambda=i+1}^{j} (k_j^{i,\lambda})^2 \Big]^{\frac{1}{2}} . \, y_{ij} = \alpha \, . \, l_{ii} + \beta . y_{ij} \quad \text{(provisionally).} \ \ldots \quad (15·52)$$

Hence the joint distribution of l_{ii} and y_{ij} becomes

$$\frac{n^{(n-i+1)/2}}{2^{(n-i-1)/2} . \sqrt{\pi} . \Gamma(n-i)/2} \, . \, e^{-\frac{1}{2}n\{(l_{ii})^2 + (y_{ij})^2\}} . (l_{ii})^{n-i-1} . \, dl_{ii} \, dy_{ij} \qquad \ldots \quad (15·7)$$

It should be remembered that l_{ii} varies from 0 to ∞, and y_{ij} from $-\infty$ to $+\infty$.

Let us set $t_{1j} = \eta = \alpha l_{11} + \beta y_{1j}, \quad \xi = \alpha l_{11}$ ... (15·71)

Then $$\frac{\partial(\xi, \eta)}{\partial(l_{11}, y_{1j})} = \begin{vmatrix} \alpha & 0 \\ \alpha & \beta \end{vmatrix} = \alpha\beta \quad \text{... (15·72)}$$

Therefore transforming (15·7) to the new variables ξ and η, the joint distribution of ξ and η is obtained as

$$C \cdot e^{-\frac{1}{2}n\{(\alpha^2+\beta^2)\xi^2 - 2\alpha^2\xi\eta + \alpha^2\eta^2\} / (\alpha^2 \cdot \beta^2)} (\xi)^{n-i-1} \cdot d\xi \cdot d\eta \quad \text{... (15·73)}$$

where $$C = \frac{n^{(n-i+1)/2}}{2^{(n-i-1)/2} \cdot \sqrt{\pi}\ \Gamma(n-i)/2} \cdot \frac{1}{\alpha^{n-i}\beta} \quad \text{... (15·74)}$$

Putting $\alpha = r\cos\theta$, and $\beta = r\sin\theta$ this integral reduces to

$$C.e^{-\frac{1}{2}n\{\xi^2 - 2\xi\eta\cos^2\theta + \eta^2\cos^2\theta\}/r^2\sin^2\theta\cos^2\theta} (\xi)^{n-i-1}\ d\xi \cdot d\eta \quad \text{... (15·8)}$$

or $$C\, e^{-\frac{1}{2}n\{(\xi-\eta\cos^2\theta)^2 + \eta^2\sin^2\theta\cos^2\theta\}/r^2\sin^2\theta\cos^2\theta} \cdot (\xi)^{n-i-1} \cdot d\xi \cdot d\eta \quad \text{... (15·81)}$$

Remembering that ξ varies from 0 to ∞ and η from $-\infty$ to $+\infty$, **we get the distribution of η in the form :—**

$$C.e^{-(n\eta^2/2r^2)}\left[\int_0^\infty e^{-\frac{1}{2}n(\xi-\eta\cos^2\theta)^2/r^2\sin^2\theta\cos^2\theta} \cdot (\xi)^{n-i-1} \cdot d\xi\right].d\eta \quad \text{... (15·82)}$$

Take a new variable u such that $\xi - \eta\cos^2\theta = u\cos^2\theta$ ... (15·9)

Then the integral within the square brackets is transformed and the distribution of η reduces to

$$C.e^{-(n\eta^2/2r^2)}\left[\int_{-\eta}^\infty e^{-(n\cot^2\theta.u^2/2r^2)} .(u+\eta)^{n-i-1} \cdot du\right] \times (\cos^2\theta)^{n-i} \cdot d\eta \quad \text{... (15·91)}$$

Finally substituting the value of C given in equation (15·74), we have the distribution of η or t_{1j} in the form :—

$$\frac{(\cos\theta)^{n-i}}{r^{n-i+1}\sin\theta} \cdot \frac{n^{(n-i+1)/2}}{\sqrt{\pi}.\Gamma(n-i)/2} \cdot e^{-(n\eta^2/2r^2)} \cdot F_{n-i}(\eta, n\cot^2\theta/r^2).\ d\eta \quad \text{... (15·92)}$$

where $$F_m(\eta,\alpha) = \int_{-\eta}^\infty e^{-\alpha u^2/2} \cdot (u+\eta)^{m-1} \cdot du \quad \text{... (15·93)}$$

SECTION IV. SPECIAL DISTRIBUTIONS.

16. *Joint Distribution of Dispersions.* The joint distribution of the rectangular co-ordinates t_{ij} ($i=1, 2......p$; $j=1, 2......p$, but $t_{ij} = 0$, when $i > j$) has already been obtained in the form :—

$$\frac{n^{\frac{1}{2}p(n-1)} \mid T^{ij} \mid ^{\frac{1}{2}(n-1)}}{2^{p(n-3)/2} \cdot \pi^{p(p-1)/4}} \cdot \frac{1}{\underset{k=1}{\mathrm{P}^p}\{\Gamma(n-k)/2\}} \cdot e^{-\frac{1}{2}nF(T,t)} \cdot (t_{11})^{n-2} (t_{22})^{n-3} \ldots (t_{pp})^{n-p-1} \cdot [dt_{ij}] \quad (16\cdot 1)$$

where $F(T,t) = \alpha^{11}a_{11} + \alpha^{22}a_{22} + ...\alpha^{pp} a_{pp} + 2(\alpha^{12} a_{12} + \alpha^{13} a_{13} + ...\alpha^{p,p-1} a_{p,p-1})$

and $$a_{ij} = (t_{1i}\, t_{1j} + t_{2i}\, t_{2j} + \ldots \ldots t_{ii}\, t_{ij}) , \quad i \leqq j. \quad \ldots (16\cdot 11)$$

If now in (16·1) we change the variables t_{ij} ($i=1,2......p$; $j=1, 2......p$; but $t_{ij} = 0$, when $i > j$), to a_{ij} ($i=1, 2......p$; $j = 1, 2...p$; and a_{ij} is same as a_{ji}), then

$$\frac{\partial(a_{11}, a_{12}, a_{22}, ...a_{pp})}{\partial(t_{11}, t_{12}, t_{22}, ... t_{pp})} = 2^p (t_{11})^p (t_{22})^{p-1} ...... t_{pp}. \quad \ldots (16\cdot 12)$$

Substituting in (16·1) we get

$$\frac{1}{2^p} \cdot \frac{n^{\frac{1}{2}p(n-1)} \mid T^{ij} \mid ^{\frac{1}{2}(n-1)}}{2^{p(n-3)/2} \cdot \pi^{p(p-1)/4}} \cdot \frac{1}{\underset{k=1}{\mathrm{P}^p}\{\Gamma(n-k)/2\}} \cdot e^{-\frac{1}{2}nF(T,t)} \cdot (t_{11}.t_{22}......t_{pp})^{n-p-2} . [da_{ij}] \quad \ldots (16\cdot 2)$$

But from (6·12) it is easily seen that

$$t_{11}t_{22}\ldots t_{pp} = \begin{vmatrix} a_{11} & a_{12} & \ldots \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots \ldots & a_{2p} \\ \ldots & \ldots & \ldots \ldots & \ldots \\ a_{p1} & a_{p2} & \ldots \ldots & a_{pp} \end{vmatrix}^{1/2} = \mid a_{ij} \mid ^{\frac{1}{2}} \quad \ldots (16\cdot 3)$$

Also we have shown that $T^{ij} = \alpha^{ij}$, therefore (16·2) reduces to

$$\frac{(\frac{1}{2}n)^{p(n-1)/2} \mid \alpha^{ij} \mid ^{(n-1)/2}}{\pi^{p(p-1)/4}} \cdot \frac{1}{\underset{k=1}{\mathrm{P}^p}\{\Gamma(n-k)/2\}} \quad e^{-\frac{1}{2}n\{\alpha^{11}a_{11} + \alpha^{22}a_{22} + 2\alpha^{12}a_{12} + ...\}}$$

$$\times \quad \mid a^{ij} \mid ^{(n-p-2)/2} . [da_{ij}] \quad \ldots (16\cdot 4)$$

which apart from minor notational differences is identical with Wishart's joint distribution of the variances.[3]

[3] J. Wishart: "The Generalised Product Moment Distribution in Samples drawn from a Normal Multivariate Population". *Biometrika*, Vol. XXA (1928), pp. 38—40.

17. *Distribution of the Dispersion Determinant.* S. S. Wilks[4] has defined the generalised variance as the determinant $| a_{ij} |$ $(i=1,2......p;\ j=1,2......p,$ and $a_{ji}=a_{ij})$.

From (6·12) it follows that $$| a_{ij} | = (t_{11}t_{22}t_{33}......t_{pp})^2 \quad ... \quad (17·1)$$

Now from (14·1) the joint distribution of l_{ii}'s $(i=1,2......p)$ is given by

$$\frac{n^{p(2n-p-1)/4}}{2^{p(2n-p-3)/4}\ \underset{k=1}{\overset{p}{P}}\ \{\Gamma(n-k)/2\}} \cdot e^{-\frac{1}{2}n \underset{i=1}{\overset{p}{S}} (l_{ii})^2} \cdot (l_{11})^{n-2}(l_{22})^{n-3}...(l_{pp})^{n-p-1} \cdot [dl_{ii}] \quad ..(17·2)$$

where $[dl_{ii}]$ stands, of course, for $dl_{11}dl_{22}......dl_{pp}$.

Let us now make the substitutions

$$u_1=l_{11},\quad u_2=l_{11}l_{22},\quad u_3=l_{11}l_{22}l_{33},\ ....\ u_p=l_{11}l_{22}......l_{pp}. \quad ... \quad (17·3)$$

It is now readily seen that

$$\frac{\partial(u_1,\ u_2,\ ......\ u_p)}{\partial(l_{11},\ l_{22},\ ......\ u_{pp})} = (l_{11})^{p-1}(l_{22})^{p-2}......l_{p-1,\ p-1}. \quad ... \quad (17·31)$$

Hence the distribution of $(u_1,\ u_2,......u_p)$ is easily seen to be proportional to

$$e^{-\frac{1}{2}n\{u_1^2+(u_2^2/u_1^2)+(u_3^2/u_2^2)+......(u_p^2/u_{p-1}^2)\}}\ u_p^{n-p-1} \cdot [du_i] \quad ... \quad (17·4)$$

where $[du_i]$ stands for $du_1du_2.........du_p$.

It is also evident that all the u's vary from 0 to ∞. The distribution of u_p is proportional to

$$\left[\int_0^\infty ......\int_0^\infty e^{-\frac{1}{2}n\{u_1^2+(u_2^2/u_1^2)+(u_3^2/u_2^2)+...(u_p^2/u_{p-1}^2)\}} du_1du_2...du_{p-1}\right].u_p^{n-p-1}du_p.$$
$$... \quad (17·41)$$

Substituting h for u_p^2, the distribution of h is proportional to

$$\left[\int_0^\infty e^{\frac{1}{2}n\{u_1^2+(u_2^2/u_1^2)+....(h/u_{p-1}^2)\}} [du_i]\right] \cdot h^{(n-p-2)/2}dh \quad ... \quad (17·42)$$

where $[du_i]$ stands for $du_1du_2......du_{p-1}$; the single integral sign stands for the multiple integral; and h varies from 0 to ∞.

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[4] S. S. Wilks: "Certain Generalisations in the Analysis of Variance". *Biometrika*, Vol. XXIV (1932) pp. 476—477.

If we now set $g = |a_{ij}| = (t_{11}\, t_{22} \ldots\ldots t_{pp})^2$ ... (17·5)

then we see from (11·9) that

$$h = (l_{11}\, l_{22} \ldots\ldots l_{pp})^2 = (\overset{11}{T}_{(p-1)})\, \overset{22}{T}_{(p+2)} \ldots\ldots \overset{pp}{T}_{(0)} \,.\, (t_{11}\, t_{22} \ldots\ldots t_{pp})^2 = |T^{ij}| \,.\, g \quad \ldots (17·51)$$

Thus, $\partial h / \partial g = |T^{ij}| = |\alpha^{ij}|$ ... (17·52)

Therefore the distribution of g is given by

$$\frac{|\alpha^{ij}|^{(n-p)/2}\, n^{p(2n-p-1)/4}}{2^{p(2n-p-5)/4} \mathbf{P}^{p}_{k=1} \{\Gamma(n-k)/2\}} \cdot \left[\int_0^{\infty} e^{-\frac{1}{2}n\{u_1^2 + (u_2^2/u_1^2) + \ldots (g\,|\alpha^{ij}|)/u_{p-1}^2\}} [du_1] \right] . g^{(n-p-2)/2} . dg$$

... (17·53)

Putting $\frac{1}{2}n.u_1^2 = w_1^2$, $(\frac{1}{2}n)^2.u_2^2 = w_2^2$, $\ldots\ldots (\frac{1}{2}n)^{p-1}.(u_{p-1})^2 = (w_{p-1})^2$ ... (17·54)
the distribution reduces to

$$\frac{(n/2)^{p(n-p)/2}\, |\alpha^{ij}|^{(n-p)/2}\, 2^{p-1}}{\mathbf{P}^{p}_{k=1} \{\Gamma(n-k)/2\}}$$

$$\times \left[\int_0^{\infty} e^{-\{w_1^2 + (w_2^2/w_1^2) + \ldots |\alpha^{ij}|\, (\frac{1}{2}n)^p g/(w_{p-1})^2\}} [dw_1] \right] . g^{(n-p-2)/2} dg \quad \ldots (17·55)$$

This can be easily identified with the form given by Wilks[5] on setting

$w_1^2 = v_1$, $w_2^2 = v_2$, ... $(w_{p-1})^2 = v_{p-1}$, and noting that $(n/2)^p . |\alpha^{ij}| = A$ ... (17·56)

In particular when $p = 1$, the distribution of g reduces to

$$\frac{(n/2)^{(n-1)/2}}{(\sigma)^{n-1}\, \Gamma\, \frac{1}{2}(n-1)} \cdot e^{-(ng/2\sigma^2)} . dg \quad \ldots (17·61)$$

where g is really s^2, and σ is the standard deviation for the population.

When $p = 2$, the distribution of g is given by

$$\frac{(n/2)^{n-2} . 2}{\{\sigma_1^2 \sigma_2^2 (1-\rho^2)\}^{(n-2)/2}\, \Gamma\frac{1}{2}(n-1)\, \Gamma\, \frac{1}{2}(n-2)} \left\{ \int_0^{\infty} e^{-[w_1^2 + (Ag/w_1^2)]} dw_1 \right\} . g^{(n-4)/2} . dg$$

$$= \frac{(\frac{1}{2}n)^{n-2} \sqrt{\pi}}{\{\sigma_1^2 \sigma_2^2 (1-\rho^2)\}^{\frac{1}{2}(n-2)}\, \Gamma\{\frac{1}{2}(n-1)\}\, \Gamma\{\frac{1}{2}(n-2)\}} \quad e^{-2\sqrt{(Ag)}}\, g^{(n-4)/2} dg \quad \ldots (17·62)$$

[5] S. S. Wilks, *Biom.* XXIV (1932), 476-477.

Consider now the expression (16·10). It is easily seen from the theory of definite integrals that within the multiple integral sign we can integrate out for the odd-suffixed variables w_1, w_3, ... etc. and thus reduce the order of multiple integration.

When p is odd, we finally get the distribution in the form :—

$$\frac{(\tfrac{1}{2}n)^{p(n-p)/2} . \mid \alpha^{ij} \mid^{(n-p)/2} . 2^{p-1}}{\mathop{P}\limits_{k=1}^{p} \{\Gamma \tfrac{1}{2}(n-k)\}} \quad g^{(n-p-2)/2} . dg \times$$

$$\left[\int_0^\infty e^{-2\{w_2 + (w_4/w_2) + (w_6/w_4) + ..(w_{p-1}/w_{p-3}) + (\Lambda g/2w_{p-1}^2)\}} . 2w_2 w_4 .. w_{p-3} . dw_2 dw_4 .. dw_{p-1} \right] \quad \ldots (17{\cdot}71)$$

When p is even we finally get the distribution in the form

$$\frac{(\tfrac{1}{2}n)^{p(n-p)/2} . \mid \alpha^{ij} \mid^{(n-p)/2} . 2^{p-1}}{\mathop{P}\limits_{k=1}^{p} \{\Gamma \tfrac{1}{2}(n-k)\}} \;.\; g^{(n-p-2)/2} . dg \times \quad \ldots \quad (17{\cdot}72)$$

$$\left[\int_0^\infty e^{-2\{w_2 + (w_4/w_2) + ...(w_{p-2}/w_{p-4}) + (\sqrt{(\Lambda g)}/w_{p-2})\}} . w_2 w_4 ... w_{p-2} . dw_2 . dw_4 .. dw_{p-2} \right]$$

In particular putting $p=3$ in (17·71), we get for the trivariate case the distribution of the generalised variance in the form

$$\frac{4 . (n/2)^{3(n-3)/2} . \mid \alpha^{ij} \mid^{(n-3)/2}}{\mathop{P}\limits_{k=1}^{3} \{\Gamma \tfrac{1}{2}(n-k)\}} \;.\; \left[\int_0^\infty e^{-2\{w_2 + \Lambda g/2w_2^2\}} . w_2 \, dw_2 \right] . g^{(n-5)/2} . dg \qquad (17{\cdot}73)$$

Putting $p=4$ in (17·72) we get for the four-variate case the distribution of the genaralised variance in the form

$$\frac{8 . (n/2)^{2(n-4)} . \mid \alpha^{ij} \mid^{(n-4)/2}}{\mathop{P}\limits_{k=1}^{4} \{\Gamma \tfrac{1}{2}(n-k)\}} \quad \left[\int_0^\infty e^{-2\{w_2 + \sqrt{(\Lambda g)}/w_2\}} \, w_2 \, dw_2 \right] . g^{(n-6)/2} . dg \qquad (17{\cdot}74)$$

Noting that $\quad K_m(z) = \tfrac{1}{2} . (\tfrac{1}{2} z)^{-m} \int_0^\infty e^{-(x + z^2/4x)} . x^{m-1} . dx$

and putting $2w_2 = x$, $\quad z = 4 \, (\Lambda . g)^{1/4}$, $\quad$ and $m = 2$, $\quad$ the distribution reduces to

$$\frac{16 . (n/2)^{2(n-4)} . \mid \alpha^{ij} \mid^{(n-4)/2}}{\mathop{P}\limits_{k=1}^{4} \{\Gamma \tfrac{1}{2}(n-k)\}} \;.\; \Lambda^{\frac{1}{2}} . g^{(n-5)/2} . K_2\{4 \, (\Lambda \, g)^{1/4}\} . dg \qquad \ldots (17{\cdot}8)$$

18. *Ratio of independent Dispersion Determinants.* Consider two p-variate normal populations. The parameters for the first population, and statistics for samples drawn from the first populaton, will be denoted as before. The parameters or the second population, and statistics for samples drawn from it, will be denoted by placing a dash on each letter corresponding to a parameter for the first population or statistic for a sample drawn from it.

We want to study the distribution of

$$z = \frac{g}{g'} = \frac{|a_{ij}|}{|a'_{ij}|} = \frac{(t_{11}\, t_{22} \ldots\ldots t_{pp})^2}{(t'_{11}\, t'_{22} \ldots\ldots t'_{pp})^2} \qquad \ldots \quad (18{\cdot}1)$$

Take $u_1, u_2 \ldots\ldots u_p$ as in the last paragraph for the first population, and $u'_1, u'_2 \ldots u'_p$ for the corresponding parameters of the second population. We have (cf. 17·4) the joint distribution of $[u_1, u_2, \ldots\ldots u_p]$ and $[u'_1, u'_2, \ldots\ldots u'_p]$ given by

$$k \times e^{-\frac{1}{2}[n\{u_1^2 + (u_2/u_1)^2 \ldots (u_p/u_{p-1})^2\} + n'\{u'^2_1 + (u'_2/u'_1)^2 + \ldots (u'_p/u'_{p-1})^2\}]}$$
$$\times\ (u_p)^{n-p-1} (u'_p)^{n'-p-1} [du_i][du'_i] \qquad \ldots \quad (18{\cdot}21)$$

where as before $[du_i]$ denotes the product of $du_1, du_2, \ldots\ldots du_p$, and $[du'_i]$ denotes the product of $du'_1, du'_2 \ldots\ldots du'_p$, also

$$k = \frac{(n)^{p(2n-p-1)/4}\,(n')^{p(2n'-p-1)/4}}{(2)^{p(n+n'-p-5)/2}\ \mathbf{P}^p_{k=1}\{\Gamma\tfrac{1}{2}(n-k)\}\ \mathbf{P}^p_{k=1}\{\Gamma\tfrac{1}{2}(n'-k)\}} \qquad \ldots \quad (18{\cdot}22)$$

Now put $\quad u_p u'_p = \phi, \quad$ and $\quad u_p/u'_p = \psi \qquad \ldots \quad (18{\cdot}3)$

Therefore, $\quad u_p^2 = \phi\psi, \quad u'^2_p = \phi/\psi \quad$ and $\quad \dfrac{\partial(\phi,\ \psi)}{\partial(u'_p, u_p)} = 2\,\psi.$

Making the transformation to ϕ and ψ, we can now write the distribution as

$$\tfrac{1}{2}k e^{-\frac{1}{2}n[\{u_1^2 + (u_2/u_1)^2 + \ldots (u_{p-1}/u_{p-2})^2 + \phi\psi/u_{p-1}^2\} + n'\{u'^2_1 + (u'_2/u'_1)^2 + \ \ldots \phi/\psi u'_{p-1}\}^2]}$$
$$\times\ \phi^{\frac{1}{2}(n+n')-p-1}\ \psi^{\frac{1}{2}(n-n'-2)}\ .\ du_1\, du_2 \ldots\ldots du_{p-1}\, du'_1\, du'_2 \ldots\ldots du'_{p-1}\ .\ d\phi\ .\ d\psi \qquad \ldots \quad (18{\cdot}4)$$

If ζ denotes the population value of z, that is, the ratio of the generalised population variances, then we have

$$\psi^2 = \frac{u_p^2}{u'^2_p} = \frac{(l_{11}\ \ l_{22} \ldots .\ l_{pp})^2}{(l'_{11}\, l'_{22} \ldots\ \ldots l'_{pp})^2} = \frac{(t_{11}\, t_{22} \ldots\ldots t_{pp})^2}{(t'_{11} t'_{22} \ldots\ldots t'_{pp})^2} \times \frac{|T^{ij}|}{|T'^{ij}|} = \frac{z}{\zeta} \qquad \ldots \quad (18{\cdot}5)$$

Then the above distribution can be written as

$$e^{-\frac{1}{2}[n\{u_1^2 + (u_2/u_1)^2 + \ldots (u_{p-1}/u_{p-2})^2 + \phi z^{\frac{1}{2}}/u_{p-1}\zeta^{\frac{1}{2}}\} + n'\{u'^2_1 + (u'_2/u'_1)^2 + \ldots (u'_{p-1}/u'_{p-2})^2 + \phi\zeta^{\frac{1}{2}}/u'^2_{p-1} z^{\frac{1}{2}}\}]}$$

$$\times\ (k/2\zeta) \times \phi^{\frac{1}{2}(n+n')-p-1}\ (z/\zeta)^{(n-n'-4)/4} \times du_1 du_2 \ldots du_{p-1}\ du'_1\ du' \ \ldots du'_{p-1}\ d\phi\ dz. \qquad \ldots \quad (18{\cdot}6)$$

Hence the required distribntion[5] of z is

$$\frac{k}{2(\zeta)^{(n-n'+1)/4}} \cdot \Big[\int_0^\infty \ldots\ldots \int_0^\infty e^{-\frac{1}{2}n\{u_1^2 + (u_2/u_1)^2 \ldots (u_{p-1}/u_{p-2})^2 + \phi z^{\frac{1}{2}}/u_{p-1}^2 \zeta^{\frac{1}{2}}\}} \times e^{-\frac{1}{2}n'\{u'^2_1 + (u'_2/u'_1)^2 + \ldots (u'_{p-1}/u'_{p-2})^2 + \phi \zeta^{\frac{1}{2}} u'^2_{p-1} z^{\frac{1}{2}}\}}$$

$$\times \quad (\phi)^{\frac{1}{2}(n+n')-p-1}\, du_1\, du_2 \ldots\ldots du_{p-1} du'_1\, du'_2 \ldots\ldots du'_{p-1} \Big] \quad (z)^{(n-n'-4)/4} \cdot dz \quad \ldots \quad (18{\cdot}7)$$

19. *Ratio of Dispersion Determinant to a Principal Minor.* We can by a suitable renaming of the variates, take the principal minor in question to be the leading principal minor. Let this minor be of the k-th order. Then we have to find the distribution of

$$y_k = | a_{ij} | \,/\, | a_{\lambda\mu} | \quad \ldots \quad (19{\cdot}1)$$

where $i, j = 1, 2, \ldots\ldots p,$ and $\lambda, \mu = 1, 2, \ldots\ldots k.$

Then as before $y_k = (t_{k+1,\,k+1}\, t_{k+2,\,k+2} \ldots\ldots t_{pp})^2$ ... (19·2)

The joint distribution of $(l_{k+1,\,k+1},\, l_{k+2,\,k+2}, \ldots\ldots l_{pp})$ is given by

$$C_k \cdot e^{-\frac{1}{2}n\mathbf{S}^p_{i=k+1}(l_{ii}^2)} \cdot \{(l_{k+1,\,k+1})^{n-k-2} (l_{k+2,\,k+2})^{n-k-3} \ldots\ldots (l_{pp})^{n-p-1}\} \cdot [dl_{ii}] \quad \ldots \quad (19{\cdot}3)$$

where $[dl_{ii}] = dl_{k+1,\,k+1} \cdot dl_{k+2,\,k+2} \ldots\ldots dl_{pp}$ and

$$C_k = \mathbf{P}^p_{i=k+1} \left\{ \frac{n^{\frac{1}{2}(n-i)}}{2^{\frac{1}{2}(n-i-2)} \cdot \Gamma\frac{1}{2}(n-i)} \right\} \quad \ldots \quad (19{\cdot}4)$$

Put

$$\left. \begin{aligned} u_{k,\,k+1} &= l_{k+1,\,k+1} \\ u_{k,\,k+2} &= l_{k+1,\,k+1} \times l_{k+2,\,k+2} \\ \ldots & \quad \ldots \quad \ldots \quad \ldots \\ u_{k,\,p} &= l_{k+1,\,k+1} \times l_{k+2,\,k+2} \ldots \times l_{pp} \end{aligned} \right\} \quad \ldots \quad (19{\cdot}5)$$

Therefore,

$$\frac{\partial(u_{k,\,k+1},\, u_{k,\,k+2} \ldots\ldots u_{kp})}{\partial(l_{k+1,\,k+1},\, l_{k+2,\,k+2}, \ldots\ldots l_{pp})} = l_{k+1,\,k+1}^{p-k-1} \cdot l_{k+2,\,k+2}^{p-k-2} \ldots\ldots l_{pp}^{0} \quad \ldots \quad (19{\cdot}55)$$

Transforming to the new variables, the distribution of $(u_{k,k+1}, u_{k,k+2} \ldots u_{k,p})$ reduces to

$$C_k \cdot e^{-\frac{1}{2}n\{u^2_{k,k+1} + (u_{k,k+2}/u_{k,k+1})^2 + \ldots\ldots (u_{k,p}/u_{k,p-1})^2\}} \cdot u_{k,p}^{n-p-1} \cdot [du_{k,i}] \quad \ldots \quad (19{\cdot}6)$$

where $[du_{k,i}]$ stands for $du_{k,k+1}\, du_{k,k+2} \ldots\ldots du_{k,p}$

[5] S. S. Wilks: "Certain Generalisations in the Analysis of Variance." *Biom.* XXIV (1932), 478-480.

But
$$(u_{k,p})^2 = (l_{k+1,k+1} \cdot l_{k+2,k+2} \ldots\ldots l_{pp})^2$$
$$= \Big(\mathbf{P}_{i=k+1}^{p} \; \mathrm{T}_{(p-i)}^{ii} \Big)\Big(t_{k+1,k+1} \cdot t_{k+2,k+2} \cdot \ldots, t_{pp} \Big)^2$$
$$= | \mathrm{T}_{p-k} | \cdot y_k \quad \text{from} \quad (11\cdot26) \qquad \ldots \quad (19\cdot7)$$

Therefore, the required distribution of y_k is given by[6]

$$\tfrac{1}{2} \left| \mathrm{T}_{(p-k)} \right|^{(n-p)/2} . C_k \left[\int_0^{\infty} e^{-\frac{1}{2}n\{u^2_{k,k+1} + (u_{k,k+2}/u_{k,k+1})^2 + \ldots + \,|\, \mathrm{T}_{p-k} | \cdot y_k / \, u^2_{k,p-1}\}} \cdot [du_{ki}] \right]$$
$$\times \; y_k^{(n-p-2)/2} \, . \, dy_k \qquad \ldots \quad (19.8)$$

where C_k is given by $(19\cdot4)$ and $[du_{ki}] = du_{k,k+1} \cdot du_{k,k+2} \ldots\ldots du_{k,p-1}$

20. *Ratio of Standard Deviations of two Correlated Variates.* The distribution of $w = \sqrt{(a_{11}/a_{22})}$, which is the ratio of the two standard deviations for the sample, can be easily obtained.

The joint distribution of (t_{11}, t_{12}, t_{22}) can be written as

$$k \, . \, e^{-\frac{1}{2}n\{\alpha^{11}t_{11}^2 + \alpha^{22}(t_{12}^2 + t_{22}^2) + 2\alpha^{12} \, . \, t_{11} \, t_{12}\}} \cdot (t_{11})^{n-2} (t_{22})^{n-3} \, . \, dt_{11} \, dt_{22} \, . \, dt_{12} \ldots \quad (20\cdot12)$$

where n is the size of the sample, and

$$k = \frac{n^{n-1} \, | \, \alpha^{ij} \, |^{\frac{1}{2}(n-1)}}{2^{n-3}\sqrt{\pi}} \cdot \frac{1}{\Gamma \frac{1}{2}(n-1) \, \Gamma \frac{1}{2}(n-2)} \qquad \ldots \quad (20\cdot13)$$

Now $\quad a_{11} = t_{11}^2, \quad a_{22} = t_{12}^2 + t_{22}^2, \quad w^2 = t_{11}^2/(t_{12}^2 + t_{22}^2) \quad .. \quad (20\cdot14)$

Introduce two new variables r, θ, such that $t_{12} = r \cos\theta$, $t_{22} = r \sin\theta$ $\quad \ldots \quad (20\cdot21)$

Then the joint distribution can be written in terms of r, θ and t_{11} in the form :—

$$k e^{-\frac{1}{2}n\{\alpha^{11}t_{11}^2 + \alpha^{22}r^2 + 2\alpha^{12}t_{11}r\cos\theta\}} \cdot (t_{11})^{n-2} (r)^{n-2} (\sin\theta)^{n-3} \, . \, dt_{11} \, . \, dr . d\theta \ldots \quad (20\cdot22)$$

Then $\quad w^2 = t_{11}^2/r^2, \quad$ or $\quad w = t_{11}/r \qquad \ldots \quad (20\cdot23)$

Set $\quad b = t_{11} \cdot r \qquad \ldots \quad (20\cdot31)$

Therefore
$$\frac{\partial\,(w, b)}{\partial\,(t_{11}, r)} = 2t_{11}/r = 2w \qquad \ldots \quad (22\cdot32)$$

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[6] S. S. Wilks. *Biom.* XXIV (1932), 480-481.

In terms of the new variables the joint distribution reduces to

$$\tfrac{1}{2}k.e^{-\frac{1}{2}n\{\alpha^{11}wb+\alpha^{22}(b/w)+2\alpha^{12}b\cos\theta\}}.(b^{n-2}/w)(\sin\theta)^{n-3}.db.dw.d\theta \quad \ldots \quad (20\cdot33)$$

Here θ varies from 0 to π, and w and b vary from 0 to ∞. Integrating out for θ, we get

$$(kb^{n-2}/2w).e^{-\frac{1}{2}nb(\alpha^{11}w+\alpha^{22}/w)}.db.dw.\int_0^{\pi}e^{-\frac{1}{2}n\alpha^{12}b\cos\theta}.(\sin\theta)^{n-3}.d\theta \ldots \quad (20\cdot34)$$

But
$$I_m(z) = \frac{z^m}{2^m\Gamma(\frac{1}{2})\Gamma(m+\frac{1}{2})}\int_0^{\pi}e^{\pm z\cos\theta}.(\sin\theta)^{2m}.d\theta \quad \ldots \quad (20\cdot41)$$

Hence (20·34) reduces to

$$\frac{k.(2)^{(n-5)/2}\Gamma(\frac{1}{2})\Gamma\frac{1}{2}(n-2)}{(n)^{(n-3)/2}(\alpha^{12})^{(n-3)/2}}.\frac{(b)^{(n-1)/2}}{w}.e^{-\frac{1}{2}nb(\alpha^{11}w+\alpha^{22}/w)}.\{I_{(n-3)/2}(n\alpha^{12})b\}.db.dw \quad \ldots \quad (20\cdot42)$$

Finally integrating out for b, we get the distribution of w in the form :—

$$\frac{k.(2)^{(n-5)/2}\Gamma(\frac{1}{2})\Gamma\frac{1}{2}(n-2)}{(n)^{(n-3)/2}(\alpha^{12})^{(n-3)/2}}.\frac{dw}{w}.\int_0^{\infty}e^{-\frac{1}{2}nb(\alpha^{11}w+\alpha^{12}/w)}.(b)^{(n-1)/2}.\{I_{(n-3)/2}.(n\alpha^{12}b)\}db \quad \ldots \quad (20\cdot51)$$

But
$$\int_0^{\infty}e^{-at}\,I_{\gamma}.\{(bt)\}.t^{\mu-1}$$

$$=\frac{(\frac{1}{2}b/a)^{\gamma}\,\Gamma(\mu+\gamma)}{a^{\mu}\,\Gamma(\gamma+1)}.(1-b^2/a^2)^{\frac{1}{2}-\mu}.\,{}_2F_1\{(\gamma-\mu+1)/2,\ \frac{\gamma-\mu}{2}+1,\ \gamma+1,\ -b^2/a^2\} \cdots \quad (20\cdot52)$$

Hence (20·51) can be written as

$$\frac{k(2)^{(n-5)/2}\Gamma(\frac{1}{2})\Gamma\frac{1}{2}(n-2)}{(n)^{(n-3)/2}(\alpha^{12})^{(n-3)/2}}.\frac{dw}{w}\ \frac{[\alpha^{12}/(\alpha^{11}w+\alpha^{22}/w)]^{(n-3)/2}\Gamma(n-1)}{(\frac{1}{2}n)^{(n+1)/2}.\Gamma\frac{1}{2}(n-1)[\alpha^{11}w+\alpha^{22}/w]^{(n+1)/2}}$$

$$\times\{1-4\alpha_{12}^2/(\alpha_{11}w+\alpha^{22}/w)^2\}^{-n/2}.\,{}_2F_1\{-\tfrac{1}{2},0,\tfrac{1}{2}(n-1),-4(\alpha^{12})^2w^2/(\alpha^{11}w^2+\alpha^{22})^2\} \ldots (20\cdot6)$$

where the notation followed is the same as given in Watson's *Bessel Functions*, p. 100.

It is clear that
$$F(\alpha,0,\rho,2) = 1 \quad \ldots \quad (20\cdot71)$$

Hence putting in the value of k from (20·13), the distribution reduces to

$$\frac{2(w)^{n-2}\,|\alpha^{ij}|^{(n-1)/2}}{(\alpha^{11}w^2+\alpha^{22})^{n-1}}.\left[1-\frac{4(\alpha^{12})^2w^2}{(\alpha^{11}w^2+\alpha^{22})^2}\right]^{-\frac{1}{2}n}.dw \quad \ldots \quad (20\cdot72)$$

But $|\alpha^{ij}| = 1/|\alpha_{ij}|$. Hence finally the distribution takes the form

$$\frac{1}{|\alpha_{ij}|^{(n-1)/2}} \cdot \frac{2(w)^{n-2}}{(\alpha^{11} w^2 + \alpha^{22})^{n-1}} \left[1 - \frac{4(\alpha^{12})^2 w^2}{(\alpha^{11} w^2 + \alpha^{22})^2}\right]^{-n/2} .dw \quad \ldots \quad (20{\cdot}73)$$

If in particular $\alpha_{11} = \alpha_{22}$, that is $\sigma_1 = \sigma_2$, then remembering that $\alpha_{12} = \rho.\sigma_1.\sigma_2$, we get the distribution in the form

$$\frac{2\Gamma(n-1)}{\{\Gamma \frac{1}{2}(n-1)\}^2} \cdot (1-\rho^2)^{(n-1)/2} \cdot \left[1 - \frac{4\rho^2 w^2}{(1+w^2)^2}\right]^{-n/2} \cdot \{(w^{n-2}/(1+w^2)^{n-1}\}.\ dw \quad \ldots \quad (20{\cdot}8)$$

which is the distribution given by S. S. Bose[7].

21. *Distribution of the Covariance of two Correlated Variates.* We can now find the distribution of $a_{12} = s_1 s_2 r_{12}$, where s_1 and s_2 are the sample standard deviations, and r_{12} is the sample correlation-coefficient of a bivariate normal population.

We have evidently $a_{12} = t_{11} t_{12}$. We have as before the joint distribution of (t_{11}, t_{12}, t_{22}) given by

$$k \cdot e^{-\frac{1}{2}n\{a^{11}t_{11}{}^2 + \alpha^{22}(t_{12}{}^2 + t_{22}{}^2) + 2\alpha^{12}t_{11}t_{12}\}} \cdot (t_{11})^{n-2} \cdot (t_{22})^{n-3} \cdot dt_{11}\, .dt_{22} \cdot dt_{12} \ \ldots \quad (21{\cdot}1)$$

Integrating out for t_{22}, (21·1) reduces to

$$\frac{k \cdot \Gamma(n-2)/2}{2(\frac{1}{2}n\alpha^{22})^{(n-2)/2}} \cdot e^{-\frac{1}{2}n\{\alpha^{11}t_{11}{}^2 + \alpha^{22}t_{12}{}^2 + 2\alpha^{12}t_{11}t_{12}\}} \cdot (t_{11})^{n-2}.dt_{11}.dt_{12} \quad \ldots \quad (21{\cdot}2)$$

Now set $\quad t_{11}\, t_{12} = u, \quad t_{12}/t_{11} = v \quad \ldots \quad (21{\cdot}31)$

Then $$\frac{\partial\,(u, \ v)}{\partial\,(t_{11}, t_{12})} = 2v \quad \ldots \quad (21{\cdot}32)$$

In terms of the new variables the distribution reduces to

$$\frac{k \cdot \Gamma(n-2)/2}{2(\frac{1}{2}n\alpha^{22})^{(n-2)/2}} \cdot e^{-\frac{1}{2}n\{(\alpha^{11}u/v) + \alpha^{22}uv + 2\alpha^{12}u\}} \cdot (u/v)^{(n-2)/2} \cdot (1/2v) \cdot du \cdot dv \quad (21{\cdot}33)$$

It should be noticed that u and v vary in such a way that (u, v) lies either in the first quadrant or in the third quadrant of the (u, v) plane.

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[7] S. S. Bose : "On the Distribution of the Ratio of Variances of two Samples drawn from a given Normal Bivariate Correlated Population." *Sankhyā*, Vol. 2 (1), 1935, 65—72.

To get the required distribution of u we integrate out for v and obtain the expression

$$\frac{k\ .\ \Gamma\ (n-2)/2}{4(\frac{1}{2}n\alpha^{22})^{(n-2)/2}}\ .\ e^{-n\alpha^{12}u}\ .u^{(n-3)/2}\ .\ du\ .\int_0^\beta e^{-\frac{1}{2}nu\{(\alpha^{11}/v)+\alpha^{22}v\}}\ .\ (1/v^{\frac{1}{2}n})\ .\ dv \qquad \ldots (21\cdot4)$$

where it is to be remembered that β is $+\infty$ or $-\infty$ according as u is positive or negative.

To evaluate the integral we put $v = cw$, where $c = \sqrt{(\alpha^{11}/\alpha^{22})}$ ... (21·51)
and the integral reduces to

$$(\alpha^{22}/\alpha^{11})^{(n-2)/4}.\int_0^\beta e^{-\frac{1}{2}n\ u\ \{w\ +\ (1/w)\}\sqrt{(\alpha^{11}\ \alpha^{22})}}.(w)^{-\frac{1}{2}n}\ .\ dw \qquad \ldots (21\cdot52)$$

If β is $+\infty$, then on using the equation (32) on page 51 of Gray, Matthews and MacRobert's *Bessel Functions*, we find that the integral comes out in the form

$$(\alpha^{22}/\alpha^{11})^{(n-2)/4}\ .\ 2\ K_{\frac{1}{2}(n-2)}\ \{nu\sqrt{(\alpha^{11}\alpha^{22})}\} \qquad \ldots (21\cdot53)$$

When β is $-\infty$, we get the same result on using the same equation, and the well known relation between $K_n(z)$ and $K_n(-z)$. Hence finally the distribution of u is given by

$$\tfrac{1}{2}k\ .\ \frac{\Gamma\ \frac{1}{2}(n-2)}{(\frac{1}{2}n\alpha^{22})^{(n-2)/2}}\ .\ (\alpha^{22}/\alpha^{11})^{(n-2)/4}.\ e^{-n\alpha^{12}u}\ .\ u^{(n-2)/2}\ .\ K_{\frac{1}{2}(n-2)}\ \{nu\sqrt{(\alpha^{11}\alpha^{22})}\}du \qquad \ldots(21\cdot61)$$

Putting in the value of k from (20·13), this reduces to

$$\frac{n^{\frac{1}{2}n}}{2^{(n-2)/2}.\sqrt{\pi}\ .\ |\ \alpha_{ij}\ |^{(n-2)/2}.\Gamma\ \frac{1}{2}(n-1).(\alpha^{11}\alpha^{22})^{(n-2)/4}}\ .\ e^{-n\ .\ \alpha^{12}.\ u}$$

$$\times\ u^{\frac{1}{2}(n-2)}\ .\ K_{\frac{1}{2}(n-2)}\ \{n\ u\ \sqrt{(\alpha^{11}\ \alpha^{22})}\}\ .\ du \qquad \ldots (21\cdot62)$$

Putting $n\ u\sqrt{(\alpha^{11}\ \alpha^{22})} = W = \frac{n}{1-\rho^2}\ .\ \frac{a_{12}}{\sigma_1\ \sigma_2}$ ... (21·71)

we have the distribution of W given by

$$\frac{(1-\rho^2)^{\frac{1}{2}(n-1)}}{\sqrt{\pi}\ .\ (2)^{\frac{1}{2}(n-2)}\ .\ \Gamma\frac{1}{2}(n-1)}\ .\ e^{\rho W}.\ W^{\frac{1}{2}(n-2)}\ .\ \{K_{\frac{1}{2}(n-2)}(W)\}\ .dW \qquad \ldots (21\cdot8)$$

which agrees with the distribution given by Pearson, Jeffrey and Elderton [8].

September, 1936.

[8] Karl Pearson, G. B. Jeffery and Ethel M. Elderton: "On the Distribution of the First Product Moment-Coefficient in Samples drawn from an indefinitely large Normal Population," *Biom.* XXI (1929 p. 168, equation (xxiv).

APPENDIX.

[*Note by P. C. Mahalanobis.* I worked out the earlier portion of the present paper in 1930, and sent it (together with certain applications to anthropometric and psychological problems, which are reproduced below in the form of an appendix) to Dr. G. M. Morant of the Biometric Laboratory, London, in October 1930 for publication in the *Biometrika*. Dr. Morant informed me however that the Editor (Karl Pearson) was unable to accept it as in his opinion the normalised variates lacked physical significance. (I may mention here that Karl Pearson had long ago considered the possibilty of correlated variates having arisen as linear functions of statistically independent components* but had not developed this method.) The paper was then communicated to the Indian Science Congress, and was presented before the Physics and Mathematics Section presided over by Dr. C. W. B. Normand, Director General of Observatories, India, in January 1931 at the Nagpur Session of the Congress. The paper however was not published as the question of sampling distributions had not been considered. With the help of my two young colleagues it has now become possible to start systematic work on this aspect of the problem. In the meantime the use of normalised variates has been extensively developed by H. Hotelling and others. I am however reproducing here that portion of my paper which dealt with applied problems in the form in which it was originally communicated in 1930, as it may throw some light on the historical development of the subject.]

A (1). The chief obstacle in the way of measuring the amount of divergence between statistical groups arises from the fact that the observed variates are correlated. Prof. Karl Pearson[1] has discussed these difficulties in detail in regard to C^2, the Pearsonian Coefficient of Racial Likeness. Similar difficulties arise in using certain other Coefficients of Divergence, D_1^2, D_2^2 etc constructed by me and described in a paper "On Tests and Measures of Group-Divergence" [2].

The transformation considered in the present paper suggests a way out of the difficulties. We may transform the observed variates into a system of statistically independent variates, and use these transformed variates for calculating the different coefficients.

(a) Let the intra-group variances and covariances be the same for all groups, i. e, the matrix $[a]_p^p$ is the same for each statistical group. In this case it is obvious that $[y]_n^p$ will be different for different groups, and will transform into different matrices, but the functional relation between $[x_1, x_2, \ldots\ldots x_p]$ and $[y_1, y_2, \ldots\ldots y_p]$ will remain identical.

This method will suffice for all comparative purposes (including the calculation of C^2, D^2 etc) but can be used only when the intra-group correlations and standard deviations are constant.

(b) When the above condition is not fulfilled a different procedure is necessary.

Let $[\bar{X}'_{11}, \bar{X}'_{12}, \ldots\ldots \bar{X}'_{1k}]$ represent the mean values of the different groups (1, 2, ... k) for the i-th character. If $\bar{X}'_i$ is the mean of means for the i-th character, and S_i the corresponding between-group standard deviation, then we may reduce $[\bar{X}']_p^k$ into $[\bar{x}]_p^k$ with the help of the equation

$$\bar{x}_{i\lambda} = (\bar{X}'_{i\lambda} - \bar{X}'_i)/S_i \qquad \ldots \quad A(1\cdot1)$$

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* For example, in *Phil. Trans.* Vol. 186 (1897), 261.

[1] Karl Pearson : "On the Coefficients of Racial Likeness". *Biometrika*, Vol. XVIII (1926), 105-117.

[2] *Jour. Asiat. Soc. Bengal*, Vol. XXVI (1930), 541-588.

It is obvious that

$$[\bar{x}]_p^k \; \underline{\overline{\bar{x}}}_k^p = k \, . \, [r]_p^p \qquad \ldots \text{A}(1\cdot2)$$

will then define the matrix of between-group correlations.

We may then transform $[\bar{x}]_p^k$ into $[\bar{y}]_{p'}^k$ in the usual way, and use $[y]_{p'}^k$ for purposes of comparison.

(c) A third alternative is open to us. We may calculate $[R]_p^k$ from the pooled measurements for all the different groups. Let there be k groups, and let $[x^{(1)}]_k^{n_1}$, $[x^{(2)}]_k^{n_2}$, ... $[x^{(k)}]_k^{n_k}$ be the actual reduced values for the 1st, 2nd .... k-th groups consisting respectively of n_1, n_2, ... n_k individuals. We may then form the pooled matrix :

$$\left[x^{(1)}, \; x^{(2)}, \ldots\ldots\ldots x^{(k)}\right]_p^{n_1,\, n_2, \ldots \ldots n_k}$$

and calculate the pooled correlations defined by

$$\left[x^{(1)}, x^{(2)}, \ldots.. x^{(k)}\right]^{n_1 \; n_2, \ldots\ldots n_k} \cdot \; \underline{\overline{x^{(1)}, x^{(2)}, \ldots.. x^{(k)}}}_{n_1,\, n_2,\, \ldots\ldots n_k}^{p} = N \, . [R]_p^p \qquad \ldots \text{A}(1\cdot3)$$

where $N = n_1 + n_2 + . \; \ldots n_k$ gives the total number of individuals in all the groups combined, and $[R]_p^p$ is the matrix of observed correlations.

We may then use $[R]_p^p$ to obtain a system of variates $[y_1, y_2 \ldots\ldots y_p]$ which will be statistically independent for the pooled data, but may of course show inter-correlations within particular groups.

The choice between (b) and (c) will ultimately depend on the nature of the actual empirical data under consideration.

A (2). I believe that the present transformation will be found useful in another direction. To fix our ideas, let us consider a particular problem, say, measurements on the head in living subjects. Theoretically there is no limit to the number of different characters which may be measured. For example, it is usual to measure the greatest head-length and the greatest head-breadth practically at right angles to each other. We may, if we like, proceed to measure diameters inclined at different angles to the direction of the greatest head length, and in this way obtain a very large number of head diameters, say $[d_1, d_2, \ldots\ldots d_p]$.

There are now two possibilities. Let $[a]_p^p$ represent the variances and covariances between these p measured diameters for any group of n individuals. If the matrix $[a]_p^p$ is undegenerate, then the system $[d_1, d_2, \ldots\ldots d_p]$ will transform into a system of p independent variates $[y_1, y_2, \ldots\ldots y_p]$. Let us now increase p to q. If the matrix $[a]_q^q$ is still undegenerate, then we are likely to obtain different values of $[y_1, y_2, y_p \ldots\ldots y_q]$. Obviously, in this case, we can never be sure that by including fresh measurements our results of comparison will not be completely changed.

The other possibility is that as we go on increasing the number of characters, a stage is reached where the matrix $[a]_q^q$ will become degenerate, and $[d_1, d_2, \ldots\ldots d_q]$ will transform into a set of lower number of say q' independent variates $[y_1, y_2 \ldots\ldots y_{q'}]$. In this case, even if we take different selections of characters we shall get fairly consistent results, prov ded of course our choice of characters is wide enough to lead to reliable values of the transformed independent variates $[y_1, y_2, \ldots\ldots y_{q'}]$

In order that results of comparison may have stability, it is therefore necessary that that this condition should be fulfilled. We may thus enunciate the two following axioms :—

(A) In any given problem there is a finite number of statistically independent transformed variates.

(B) It is immaterial which particular set of observed characters is selected for study provided these characters are sufficiently wide in range to be capable of being transformed into a set of q' independent variates with reasonable reliability.

To deny these axioms is to deny the possibility of stable comparisons.

From this point of view, one of the fundamental problems of comparative anthropometry is to determine what number q of independent characters will be sufficient for purposes of comparison in a given problem. The transformation described in the present paper makes it possible to investigate this question in a systematic manner.

A (3). We may also use the present transformation for studying the factorial theory of human abilities. Let $[x]_p^n$ be the reduced scores of n individuals in p different tests (or the reduced measures for n individuals in p different traits). The observed correlation between the different tests or traits is then given by

$$[x]_p^n \cdot \overline{x}_n^p = n \cdot [r]_p^p \qquad \cdots \text{A(3·01)}$$

The factorial theory of abilities attempts to express the observed variates $[x_1, x_2, \ldots x_p]$ in terms of a number of statistically independent variates $[y_1, y_2, \ldots\ldots y_{p'}]$ such that only particular y-variates are common to particular groups of the observed variates.

That is, $[x]_p^n$ is expressed as

$$[x]_p^n = \underset{n}{\overset{p'}{l}} \cdot [y]_{p'}^n \qquad \ldots \text{ A}(3{\cdot}02)$$

where the matrix $[l]$ has a special form, and $[y]_{p'}^n$ is a semi-unit matrix.

Spearman[3] defines a general factor to be one (say y_0) which occurs in all the observed variates. Hence the l-coefficients corresponding to y_0 will be p in number.

A specific factor, on the other hand, is one which will occur in one observed variate only. If $[y_1, y_2, \ldots\ldots y_i]$ are the specific factors corresponding to $[x_1, x_2, \ldots\ldots x_i]$ then the corresponding l-coefficients are obviously given by $[l_{11}, l_{22}, l_{33}, \ldots\ldots l_{ii}]$. It is clear that in this case all l-coefficients of the type l_{ij} must vanish when i is not equal to j.

A group factor of order q is one which occurs in q of the observed variates. For each y-variate which is a group factor of order q, it is clear that the number of non-vanishing l-coefficients must also be q. (The specific factors may be called group-factors of order l, while the general factor is a group factor of order p).

Equation A(2·2) will give rise to a second degree matrix equation

$$[\,l\,]_p^s \cdot \underset{s}{\overset{p}{l}} = n \cdot [r]_p^p \qquad \ldots \text{ A}(3{\cdot}03)$$

of the same form as equation (3·2)*. Here s will depend on the different kinds of group-factors supposed to be present, and will usually be greater than p. A general solution, therefore, is not possible ; but solutions may be obtained in particular cases.

(1) Group factors of a particular order q only are present. Let $q=1$, then $s=p$, and equation A(3·03) takes the form

$$\begin{bmatrix} l_{11}, & 0, & 0, \ldots\ldots & 0 \\ 0, & l_{22}, & 0, \ldots\ldots & 0 \\ \cdots & \cdots & \cdots\;\cdots\cdots & \cdots \\ 0. & 0, & 0, \ldots\ldots & l_{pp} \end{bmatrix} \begin{pmatrix} l_{11}, & 0, & 0, \ldots\ldots & 0 \\ 0, & l_{22}, & 0, \ldots\ldots & 0 \\ \cdots & \cdots & \cdots\;\cdots\cdots & \cdots \\ 0, & 0, & 0, \ldots\ldots & 0 \end{pmatrix} = n \cdot [r]_p^p \qquad \ldots \text{ A}(3{\cdot}1)$$

It is obvious that that $[r]_p^p$ must be a unit matrix, i.e. all the observed characters must be statistically independent. This is obviously a necessary and sufficient condition for an analysis of this type being possible.

[3] C. Spearman : The Abilities of Man (Macmillan, 1927,) Chapters X and XI.
* Main paper, Section I, p. 5.

(2) Let $q=2$. The total number of vertical rows in equation A(3·03) (i.e., the number of y-variates required) is obviously ${}^{m}C_2$ in order that all possible group-factors of order 2 may be present.

Thus $$s_2 = {}^{m}C_2 = m(m-1)/2\,! \qquad \text{... A(3·21)}$$

Since only 2 l-coefficients are non-vanishing in each vertical row, the total number of non-vanishing l-coefficients is given by

$$2\,.\,{}^{p}C_2 = p(p-1) = f_2 \qquad \text{... A(3·22)}$$

The total number of observed parameters furnished by $[r]_p^p$ is $p(p+1)/2 = e$. It is clear therefore that (f_2-e) gives the number of l-coefficients which are arbitrary. Thus[4] the number of arbitrary l-coefficients is equal to

$$(f_2-e) = p(p-3)/2 = b_2 \qquad \text{... A(3·23)}$$

Assigning arbitrary values to b_2 of the l-coefficients, the remaining coefficients can be easily obtained. The solution is however not unique.

(3) The general case for group-factors of order q may be treated in the same way. The total number of y-variates required is

$$s_q\,.\,{}^{p}C_q = \frac{p\,!}{q\,!\,(p-q)\,!} \qquad \text{... A(3·31)}$$

The total number of non-vanishing l-coefficients is given by

$$f_q = q\,.{}^{p}C_q = \frac{p\,!}{(q-1)\,!\,(p-q)\,!} \qquad \text{... A(3·32)}$$

The number of arbitrary l-coefficients is given by

$$b_q = f_q - e = \frac{p\,!}{(q-1)\,!\,(p-q)\,!} - \frac{p(p+1)}{2} \qquad \text{... A(3·33)}$$

(4) Let all group factors of order 1, 2, ... q inclusive be present. The total number of y-variates required will then be

$$s_1+s_2+\ldots\ldots s_q = {}^{p}C_1+{}^{p}C_2+.\ \ldots{}^{p}C_q = \Sigma_q({}^{p}C_q) = g_q \quad \text{say} \qquad \text{... A(3·41)}$$

and the total number of non-vanishing l-coefficients is given by

$$f_1+f_2+\ldots\ldots f_q = 1\,.\,{}^{p}C_1+2\,.\,{}^{p}C_2+\ldots\ldots q\,.\,{}^{p}C_q = \Sigma_q(q.{}^{p}C_q) = h_q \qquad \text{... A(3·42)}$$

where Σ_q represents a summation for all values of $q=1, 2, \ldots\ldots q$.

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[4] It is clear that for $p=2$, $s_2=1$, and the group-factor of order 2 becomes a general factor, and the two observed variates must have a correlation $=\pm 1$. For $p=3$, s_2 is 3, and a general solution of the equation may be obtained in the usual way.

(5) Let all group factors of all orders (1 and p inclusive) be present. Then the **total number of y-variates $= g_p = 2^p - 1$** ... A(3·51)

The number of non-vanishing l-coefficients $= h_p =$ $\sum_{q=1}^{q=p} (g_q . {}^nC_q)$ ... A(3·52)

This will give a complete solution. It is clear however that by far the large number of l-coefficients will be arbitrary, and a large variety of different solutions will be possible.

(3) A general factor and specific factors only are present. The equation now takes the form :

$$\begin{pmatrix} l_{01}, & l_{11}, & 0, & 0 \ldots\ldots 0 \\ l_{02}, & 0, & l_{22}, & 0 \ldots\ldots 0 \\ l_{03}, & 0, & 0, & l_{33}, \ldots. 0 \\ \ldots & \ldots & \ldots & \ldots \ \ldots \ldots \\ l_{0p}, & 6, & 0_1 & 0 \ldots\ldots l_{pp} \end{pmatrix} \begin{pmatrix} l_{01}, & l_{02}, & l_{03} \ldots\ldots l_{0p} \\ l_{11}, & 0, & 0 \ldots\ldots 0 \\ 0, & l_{22}, & 0 \ldots\ldots 0 \\ \ldots & \ldots & \ldots \ldots\ldots \ldots \\ 0, & l_{32}, & 0 \ldots\ldots 0 \end{pmatrix} = n.[r]_p^p \quad \ldots \text{A(3·61)}$$

We notice that for all values of i, j, a, b from 1 to p,

$$\left. \begin{aligned} r_{ii} &= l_{0i}^2 + l_{ii}^2 = +1 \\ r_{ij} &= l_{0i} . l_{0j} \\ r_{ab} &= l_{0a} . l_{0b} \end{aligned} \right\} \quad \ldots \text{A(3·52)}$$

It follows immediately that

$$r_{ab} . r_{ij} = l_{0a} . l_{0b} . l_{0i} . l_{0j} = r_{ai} . r_{bj} = r_{aj} . r_{bi} \quad \ldots \text{A(3·63)}$$

This is the well known tetrad equation of Spearman[5]. Equation A(3.63) **gives the necessary and sufficient condition for a solution of the present type being possible.**

Here $s = p+1, \quad f = 2p$ ... A(3·64)

Thus when p is greater than 3, the number of observed parameters $p(p+1)/2$ **will be greater than $2p$, the number of non-vanishing l-coefficients.** It follows therefore **that at least $p(p-3)/2$ connections must exist between the** different rows of the corre**lation matrix.**

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[5] C. Spearman : Abilities of Man, (1927). Appendix.