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ON SOME SERIES OF EFFICIENCY-BALANCED BLOCK AND ROW-COLUMN DESIGNS

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SUMMARY. Two series of efficiency-balanced block and row-column designs have been constructed using balanced incomplete block designs and the concept of Youden type designs. An upper bound of the efficiency factor has been derived and it is found that the designs constructed here have efficiency factors close to this bound.

1. Introduction

Variance- and efficiency-balanced designs in one-way and two-way elimination of heterogeneity set-up have been studied quite extensively in the literature. We consider a block design with v treatments, b blocks each of size k. For it, let $\boldsymbol{R} = \text{diag}\ (r_1, \ldots, r_v)$, $\boldsymbol{r} = (r_1, \ldots, r_v)'$ and $\boldsymbol{N} = (n_{ij})$ be the $v \times b$ incidence matrix of the design where n_{ij} is the number of times the i-th treatment occurs in the j-th block, and r_i is the replication number of the i-th treatment for $i = 1, \ldots, v$ and $j = 1, \ldots b$, Under the usual fixed effects, additive homoscedastic linear model, the coefficient matrix (C-matrix) of the reduced normal equations for estimating linear functions of treatment effects is given by $\boldsymbol{C} = \boldsymbol{R} - k^{-1}\boldsymbol{NN}'$ which is symmetric, non-negative definite with zero row sums. A design is said to be connected if and only if rank $(\boldsymbol{C}) = v-1$.

The row-column designs for two-way elimination of heterogeneity considered here have bk experimental units arranged in a rectangular array of k rows and b columns such that each unit receives only one of the v treatments being investigated. Under an appropriate model, the C-matrix of a row-column design is given by

$$\boldsymbol{C}^{(RC)} = \boldsymbol{R} - k^{-1}\boldsymbol{NN}' - b^{-1}\boldsymbol{MM}' + (bk)^{-1}\boldsymbol{rr}' \qquad \ldots \quad (1.1)$$

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where $\boldsymbol{R}$, $\boldsymbol{r}$ are as defined earlier, $\boldsymbol{M}$ and $\boldsymbol{N}$ are the $v \times k$ treatment-row and $v \times b$ treatment-column incidence matrices, respectively. As in case of block designs, $\boldsymbol{C}^{(RC)}$ is symmetric, non-negative definite with zero row sums, and a row-column design is said to be connected if and only if rank $(\boldsymbol{C}^{(RC)}) = v-1$.

We now have the following definitions.

Definition 1.1. A connected block (resply. row-column) design is said to be variance-balanced (VB) if and only if it permits the estimation of all normalized treatment contrasts with the same variance.

It is known (cf. Rao (1958)) that a connected block (resply. row-column) design is VB if and only if

$$\boldsymbol{C}(\boldsymbol{C}^{(RC)}) = \theta\{\boldsymbol{I}-v^{-1}\boldsymbol{1}\boldsymbol{1}'\}$$

where $\theta\,(>0)$ is the unique non-zero eigenvalue of $\boldsymbol{C}(\boldsymbol{C}^{(CR)})$, $\boldsymbol{I}$ is the identity matrix (of appropriate order) and $\boldsymbol{1}$ is a column vector of unities.

The positive eigenvalues of $\boldsymbol{R}^{-1/2}\boldsymbol{C}\,\boldsymbol{R}^{-1/2}$ (resply. $\boldsymbol{R}^{-1/2}\boldsymbol{C}^{(RC)}\boldsymbol{R}^{-1/2}$) are called the canonical efficiency factors of designs, see James and Wilkinson (1971), and Pearce, Calinski and Marshall (1974).

Definition 1.2. A connected block (resply. row-column) design is said to be efficiency-balanced (EB) if and only if the canonical efficiency factors are all equal.

It can be shown that a connected block (resply. row-column) design is EB if and only if

$$\boldsymbol{C}(\boldsymbol{C}^{(RC)}) = e\{\boldsymbol{R}-(bk)^{-1}\boldsymbol{r}\boldsymbol{r}'\}$$

where $0 < e \leqslant 1$ is the unique non-zero eigenvalue of $\boldsymbol{R}^{-1/2}\boldsymbol{C}\,\boldsymbol{R}^{-1/2}$ (resply. $\boldsymbol{R}^{-1/2}\boldsymbol{C}^{(RC)}\boldsymbol{R}^{-1/2}$). For such designs, every treatment contrast is estimated with the same efficiency factor e.

It is known that a VB (block or row-column) design with $v > 2$ is EB, and conversely, if and only if the design is equireplicate. Also, it may be noted that in the class of proper (equal block sized) designs, any binary VB or EB block design is necessarily equireplicate.

In the present paper, two series of proper EB block designs, using balanced incomplete block (BIB) designs, are constructed. Further, using the idea of Youden type designs (to be discussed later), several designs belonging to the above two series can be converted to EB row-column designs. An upper bound for the efficiency factor of any row-column design is obtained. This upper bound comes out to be the same as that for a block design. A list of EB designs with replications $\leqslant 30$ is provided, and it is found that these designs have efficiency factors close to its upper bound.

The definition of a BIB design can be found in Raghavarao (1971).

2. Method of construction

Consider a BIB design, having incidence matrix $\boldsymbol{N}$, with parameters v', b', r', k', λ. Throughout this paper $\mathbf{0}_c$ is a $c\times 1$ zero vector $\mathbf{1}_m$ is an $m\times 1$ column vector of all unities, $\boldsymbol{I}_m$ is the identity matrix of order m, and $\otimes$ stands for the Kronecker product of matrices.

The following two theorems are straightforward by Definition 1.2.

Theorem 2.1. *For non-negative integers p, q, s and w if $\{r'pw+sq(k'+w-s)\}/(p\lambda) = \{b'pw+v'q(k'+w-s)\}/(r'p+sq)$, there exists a proper EB block design with parameters $v = v'+1$, $b = b'p+v'q$, $r_1 = r'p+sq$, $r_2 = b'pw+v'q(k'+w-s)$, $k = k'+w$, $e = p\lambda b/r_1^2$, whose inci-dence matrix is given by*

$$\begin{bmatrix} \mathbf{1}_p' \otimes \boldsymbol{N} & \mathbf{1}_q' \otimes s\boldsymbol{I}_{v'} \\ s\mathbf{1}_{pb'}' & (k'+w-s)\mathbf{1}_{qv'}' \end{bmatrix}.$$

Theorem 2.2. *For non-negative integers p, q, s and w, if $\{r'p(v'-k')+sq(v'-s)\}/(p\lambda+w) = \{(v'-k')b'p+(v'-s)v'q\}/(r'p+sq+w)$, there exists a proper EB block design with parameters $v = v'+1$, $b = b'p+v'q+w$ $r_1 = r'p+sq+w$, $r_2 = b'p(v'-k')+v'q(v'-s)$, $k = v'$, $e = (p\lambda+w)b/r_1^2$, whose incidence matrix is given by*

$$\begin{bmatrix} \mathbf{1}_p' \otimes N & \mathbf{1}_q' \otimes s\boldsymbol{I}_{v'} & \mathbf{1}_{v'}\mathbf{1}_w' \\ (v'-k')\mathbf{1}_{pb'}' & (v'-s)\mathbf{1}_{qv'}' & \mathbf{0}_x' \end{bmatrix}$$

Example 2.1. Consider the BIB design with parameters $v' = b' = 6$, $r' = k' = 5, \lambda = 4$. Then Theorem 2.1 with $p = 1$ $q = 1$, $s = 3$ and $w = 0$, yields an EB block design with parameters $v = 7, b = 12, k = 5$, $r_1 = 8, r_2 = 12$ and $e = 0.750$. The incidence matrix of the design is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}.$$

Example 2.2. Consider the BIB design with parameters $v' = b' = 3$, $r' = k' = 2, \lambda = 1$. Then Theorem 2.2 with $p = 2, q = 1, s = 1$ and $w = 1$, yields an EB block design with parameters $v = 4, b = 10, k = 3, r_1 = 6$, $r_2 = 12$ and $e = 0.833$. The incidence matrix of the design is

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}.$$

3. Efficiency-balanced row-column designs

We first quote some results and definitions from Das and Dey (1989).

Definition 3.1. A $k \times b$ array containing entries from a finite set $\Omega = \{1, 2, \ldots, v\}$ of v treatment symbols is called a Youden type (YT) row-column design if the i-th treatment symbols occurs in each row of the array m_i times, for $i = 1, \ldots, v$, where $m_i = r_i/k$ and r_i is the replication of the i-th treatment symbol in the array.

Theorem 3.1. *A necessary and sufficient condition for the existence of a YT design is that r_i/k is an integer for all $i = 1, \ldots, v$.*

With each row-column design d are associated the block desings d_M and d_N with incidence matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ respectively, i.e., $d_M(d_N)$ is the block design obtained by treating the {rows} ({columns}) of d as blocks. Then it follows from (1.1) that the C-matrix of d is

$$\boldsymbol{C}_d^{(RC)} = \boldsymbol{C}_d^N - \boldsymbol{b}^{-1}\boldsymbol{M}(\mathbf{I}_k - k^{-1}\mathbf{1}\mathbf{1}')\boldsymbol{M}' \qquad \ldots \quad (3.1)$$

where $\boldsymbol{C}_d^N = \boldsymbol{R} - k^{-1}\boldsymbol{N}\boldsymbol{N}'$ is the C-matrix of d_N.

Theorem 3.2. *A necessary and sufficient condition for $\boldsymbol{C}_d^{(RC)} = \boldsymbol{C}_d^N$ is that d is a YT design.*

By Definition 3.1, Theorems 3.1 and 3.2, the following results on EB row-column designs can be obtained.

Theorem 3.3. *The block contents of the block design in Theorem* 2.1 *can be rearranged to yield a YT design provided r_i/k is an integer for $i = 1, 2$. In such a case, the YT design is an EB row-column design with parameters v, k, r_1, r_2, e, as in Theorem* 2.1.

Theorem 3.4. *The block contents of the design in Theorem* 2.2 *can be rearranged to yield a YT design provided r_i/k is an integer for $i = 1, 2$. Further in such a case, the YT design is an EB row-column design with parameters v, b, k, r_1, r_2, e, as in Theorem* 2.2.

Example 3.1. The block contents of the EB design in Example 2.2 can be rearranged to yield the following EB row-column design with parameters $v = 4, b = 10, k = 3, r_1 = 6, r_2 = 12$ and $e = 0.833$:

$$\begin{array}{cccccccccc} 4 & 4 & 4 & 1 & 3 & 2 & 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 & 4 & 3 & 4 & 4 & 4 & 1 \\ 1 & 3 & 2 & 4 & 1 & 4 & 4 & 4 & 3 & 2 \end{array}$$

4. Efficiency bounds

Das and Kageyama (1991) showed that for a connected proper block design, the efficiency factor e, i.e. Harmonic mean of the canonical efficiency factors, is bounded above. The following is due to Das and Kageyama (1991).

Theorem 4.1. *In a connected design with v treatments and b blocks of size k each,*

$$e \leqslant v(k-1)/\{k(v-1)\}$$

and the equality holds if and only if the design is a binary EB design (and thus BIB).

A similar result can be obtained for row-column designs as follows :

Theorem 4.2. *In a connected row-column design with v treatments arranged in k rows and b columns,*

$$e \leqslant v(k-1)/\{k(v-1)\}$$

and the equality holds if and only if the design is a Youden design.

Proof. Let $e_i, i = 1, \ldots, v-1$, be the canonical efficiency factors of a connected row-column design d. Then,

$$e = (v-1) \Big/ \sum_{i=1}^{v-1} e_i^{-1} \leqslant (v-1)^{-1} \sum_{i=1}^{v-1} e_i$$

$$= (v-1)^{-1}\mathrm{tr}(\boldsymbol{R}^{-1/2}\boldsymbol{C}_d^{(RC)}\boldsymbol{R}^{-1/2}) \leqslant v(k-1)/\{k(v-1)\},$$

since from (3.1) and because of symmetry and idempotence of $\boldsymbol{I}_k - k^{-1}\mathbf{1}\mathbf{1}'$,

$$\mathrm{tr}(\boldsymbol{R}^{-1/2}\boldsymbol{C}_d^{(RC)}\boldsymbol{R}^{-1/2}) = \mathrm{tr}(\boldsymbol{R}^{-1/2}\boldsymbol{C}_d^{N}\boldsymbol{R}^{-1/2}) - b^{-1}\mathrm{tr}(\boldsymbol{R}^{-1/2}\boldsymbol{M}(\boldsymbol{I}-k^{-1}\mathbf{1}\mathbf{1}')\boldsymbol{M}'\boldsymbol{R}^{-1/2})$$
$$\leqslant v - k^{-1}\textstyle\sum_i \sum_j n_{ij}/r_i$$
$$= v(k-1)/k.$$

It is easy to see that the equality holds if and only if the design is a Youden design.

Remark 4.1. As upper bounds for EB row-column designs, in the same manner as in Das and Kageyama (1991 ; Theorem 2.2 and Corollary 2.1), we can get the following.

(A) In an EB row-column design with v treatments, k rows and b columns, in which $r_1 \leqslant r_2 \leqslant \ldots \leqslant r_v$ are the replication numbers,

$$e \leqslant b(k-1)/(bk-r_1).$$

(B) In an EB row-column design with v treatments, k rows and b columns, the inequalities

$$e \leqslant b(k-1)/(bk-[bk/v]) \leqslant v(k-1)/\{k(v-1)\},$$

hold, the second inequality becoming equality if bk/v is an integer, where $[m]$ means the largest integer $\leqslant m$.

5. Tabulation

Using the results of Sections 2 and 3, we have listed EB designs (other than BIB designs) with replications $\leqslant 30$. The parameters of these designs along with the values of e and the upper bound of e (eb, say), as in Theorem 4.1, are given in Tables 1 and 2. By these tables, it is seen (as revealed by the ratio $R = e/eb$) that several of the designs constructed here have efficiency factors close to eb.

We refer to the table of BIB designs given in Raghavarao (1971 ; pages 91-97) for our search of EB designs. The column under BIB in Tables 1 and 2 gives the serial number of BIB designs listed in Raghavarao's table. The serial number zero stands for the BIB design with parameters $v = b = 3$, $r = k = 2, \lambda = 1$.

The designs marked with asterisk can be converted to a YT design and give rise to EB row-column designs. It is found that 7 of the designs in these tables are VB.

TABLE 1. PARAMETRIC VALUES AND EFFICIENCY FACTORS OF EB BLOCK AND ROW-COLUMN DESIGNS BASED ON THEOREMS 2.1 AND 3.3

v	b	k	r_1	r_2	e	eb	R	BIB	p	q	s	w
5	12	3	8	4	.750	.833	.900	2	2	1	2	0
5	24	3	16	8	.750	.833	.900	2	4	2	2	0
5*	36	3	24	12	.750	.833	.900	2	6	3	2	0
6	10	5	6	20	.833	.960	.868	4	1	1	2	1
6	20	4	15	5	.800	.900	.889	4	3	1	3	0
6	40	4	30	10	.800	.900	.889	4	6	2	3	0
6*	55	3	30	15	.733	.800	.917	5	4	3	2	0
7*	26	3	12	6	.722	.778	.929	7	2	1	2	0
7*	52	3	24	12	.722	.778	.929	7	4	2	2	0
7	12	5	8	12	.750	.933	.804	8	1	1	3	0
7	24	5	16	24	.750	.933	.804	8	2	2	3	0
7	30	5	24	6	.833	.933	.893	8	4	1	4	0
8	35	3	14	7	.714	.762	.938	10	4	1	2	0
8	70	3	28	14	.714	.762	.938	10	8	2	2	0
9*	50	4	24	8	.781	.844	.926	15	3	1	3	0
10*	57	3	18	9	.704	.741	.950	17	4	1	2	0
10	63	4	27	9	.778	.833	.933	19	3	1	3	0
11	70	3	20	10	.700	.733	.955	25	2	1	2	0
13*	100	3	24	12	.694	.722	.962	34	2	1	2	0
13	34	6	15	24	.756	.903	.837	36	1	1	4	0
14	117	3	26	13	.692	.719	.964	38	4	1	2	0
16*	155	3	30	15	.689	.711	.969	42	4	1	2	0
16	60	8	30	30	.800	.933	.857	44	3	1	6	0

TABLE 2. PARAMETRIC VALUES AND EFFICIENCY FACTORS OF EB BLOCK AND ROW-COLUMN DESIGNS BASED ON THEOREMS 2.2 AND 3.4

v	b	k	r_1	r_2	e	eb	R	BIB	p	q	s	w
4*	11	3	9	6	.815	.889	.917	0	1	1	2	5
4*	13	3	12	3	.722	.889	.813	0	1	1	3	7
4	10	3	5	15	.800	.889	.900	0	1	2	1	1
4*	18	3	15	9	.800	.889	.900	0	1	2	2	9
4*	10	3	6	12	.833	.889	.938	0	2	1	1	1
4*	15	3	12	9	.833	.889	.938	0	2	1	2	6
4*	17	3	15	6	.756	.889	.850	0	2	1	3	8
4*	22	3	18	12	.815	.889	.917	0	2	2	2	10
4	20	3	10	30	.800	.889	.900	0	2	4	1	2
4*	19	3	15	12	.844	.889	.950	0	3	1	2	7
4*	21	3	18	9	.778	.889	.875	0	3	1	3	9
4*	23	3	18	15	.852	.889	.958	0	4	1	2	8
4*	25	3	21	12	.794	.889	.893	0	4	1	3	10
4*	20	3	12	24	.833	.889	.938	0	4	2	1	2
4	21	3	14	21	.857	.889	.964	0	5	1	1	3
4*	27	3	21	18	.857	.889	.964	0	5	1	2	9
4*	31	3	24	21	.861	.889	.969	0	6	1	2	10
5*	13	4	8	20	.813	.938	.867	1	1	1	2	3
5*	15	4	12	12	.625	.938	.667	1	1	1	4	5
5*	19	4	12	28	.792	.938	.844	1	1	2	2	5
5*	23	4	20	12	.575	.938	.613	1	1	2	4	9
5*	22	4	16	24	.688	.938	.733	1	2	1	4	6
5*	30	4	24	24	.625	.938	.667	1	2	2	4	10
5*	18	4	16	8	.844	.938	.900	2	1	1	3	10
5	24	4	18	24	.889	.938	.948	2	4	1	2	4
6	10	5	6	20	.833	.960	.868	4	1	1	2	0
6	15	5	12	15	.833	.960	.868	4	1	1	3	5
6	30	5	24	30	.833	.960	.868	4	2	2	3	10
6*	21	5	15	30	.840	.960	.875	5	1	1	3	6
6*	24	5	20	20	.720	.960	.750	5	1	1	5	9
7*	23	6	18	30	.639	.972	.657	7	1	1	6	7
7*	21	6	18	18	.843	.972	.867	8	1	1	4	9
7*	28	6	24	24	.875	.972	.900	8	2	1	4	10
8*	24	7	21	21	.653	.980	.667	11	1	1	7	10
8*	18	7	14	28	.827	.980	.844	13	1	1	4	4

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