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SHARPER SPEED OF CONVERGENCE TO NORMALITY FOR SOME m DEPENDENT PROCESSES

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SUMMARY. For a stationary m dependent process under certain assumptions which ensure that the individual random variables have finite moment generating function but the random variables may not be bounded, the non uniform rates of convergence of standardised sample sum to normality are studied. The results obtained turn out to be quite sharp even for i.i.d random variables. Application of these rates are made in moment type convergences, L_q versions of the Berry-Esseen theorem and to probabilities of deviations. Possible extensions are indicated for non stationary m dependent processes. The conditions assumed are shown to be fulfilled for moving average process.

1. Introduction

Let $\{X_n, n \geqslant 1\}$ be a stationary m dependent process with

$$EX_1 = 0, EX_1^2 + 2 \sum_{i=1}^{m-1} EX_1X_{1+i} \equiv 1 \qquad \dots \quad (1.1)$$

and $Eg(X_1) < \infty$ where $g(x)$ is non negative, even and non decreasing function on $[0, \infty)$. Define $F_n(t) = P\left(n^{-1/2} \sum_{i=1}^{n} X_i \leqslant t\right)$. Then $F_n \rightarrow \Phi$ where Φ is the standard $N(0, 1)$ distribution function. Uniform speed of such convergences are studied by Stein (1972) among others.

To study the moment type convergences of standardised sample sum and to compute probabilities of deviation it is essential that the role of t, the point of convergence be reflected in the rate with which $|F_n(t) - \Phi(t)|$ goes to zero as $n \rightarrow \infty$. Such non uniform rates of convergence under the structural assumption on g which ensures that some finite moment of X of order $\geqslant 2$ exists, or that all the moments of X exist but the moment generating function of X may not exist are studied in a separate paper, Dasgupta (1992b). One of the pleasant features of such non uniform rates is that it produces several deviation and other allied results as by products.

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In this paper we partially cover an extreme spectrum of g which ensures that the moment generating function of the random variables exist but the random variables may not be bounded. We specifically assume that

$$EX^{2m}/(EX^2)^m \leqslant l^{-m}(2m)\,!/m\,! \qquad \dots \quad (1.2)$$

where $1 < l \leqslant 2$ and $m = 2, 3, 4, \dots$

In a similar set up the case of independent random variables in a triangular array is studied in Dasgupta (1992a). Although we shall not use it, under the additional assumption that all the odd order moments of X are zero, (1.2) implies

$$E \exp(h|X|) \leqslant \exp(h^2\sigma^{*2}/l) \text{ where } \sigma^{*2} = EX^2 \qquad \dots \quad (1.3)$$

See after (2.6) of Dasgupta (1992a).

Without such additional assumption (1.2) implies that

$$E \exp(h|X|) \leqslant 2 \exp(h^2\sigma^{*2}/l), \qquad \dots \quad (1.4)$$

To see this observe that

$$E\, e^{h|X|} \leqslant E\, e^{-hG} + Ee^{hX} \leqslant 2 \sum_{m=0}^{\infty} \frac{h^{2m}}{(2m)\,!} EX^{2m}.$$

$$\leqslant 2 \exp(h^2\sigma^{*2}/l) \qquad \text{from (1.2)}$$

Following the lines of (5.1) of Dasgupta (1992a) it can be shown that (1.2), without any additional restriction on odd order moments implies that

$$E \exp(cX^2) < \infty \text{ for } 0 < c < t/(4\sigma^{*2}), \sigma^{*2} = EX^2$$

The non uniform rates of convergence are obtained in this frame work for m dependent processes. The result obtained are nearly optimal even for i.i.d set up and as such it naturally imposes some restriction on the correlations of the variables depending on the value of m.

There are now quite a number of works on nonuniform speed of convergence in CLT under weak dependance set up e.g., see Babu, Ghosh and Singh (1978), Babu and Singh (1978), Dasgupta (1988, 1983, 1992b, 1990). But none of the above results are as sharp as (2.2) on the entire range of t. The contribution of the present paper is to show that it is possible to obtain near optimal rates even for weak dependent set up under appropriate assumptions, see Remark 2.1.

The technique used can be briefly described as follows. Along with usual short block long block technique, the random variables within each long block are grouped in a special way so as to use the results of independent random variables within each block. Then the moment generating function for sum of the random variables in each block is estimated in terms of that of the individual random variables.

Although for simplicity of notations and computations the stationary process is considered, similar technique may be used for non stationary m dependent processes with identical results ; since the results for independent random variables in a triangular array of Dasgupta (1989) are used as basic tools. For moving average process the technical condition (2.1) is shown to be satisfied.

2. Results for m dependent processes

Our main result in this section is the following.

Theorem 2.1. *Let* $\{X_n, \ n \geqslant 1\}$ *be a stationery m dependent process satisfying* (1.1) *and* (1.2). *If*

$$\bar{\rho} > (m-l)/(2(m-1)l) \quad \ldots \quad (2.1)$$

Where

$$\bar{\rho} = (m-1)^{-1} \sum_{i=1}^{m-1} \rho_i, \quad \rho_i = corr\,(X_1, X_{1+i})$$

then the following holds

$$|F_n(t)-\Phi(t)| \leqslant bn^{-1/2} \, exp\,(-\gamma\, t^2), \, -\infty < t < \infty \quad \ldots \quad (2.2)$$

where $b > 0$ *and* $0 < \gamma(< 1/2)$ *depends on* m, l *and* $\bar{\rho}$.

Remark 2.1. (2.2) cannot be substantially improved even for bounded random variables since $|F_n(t)-\Phi(t)| = 1-\Phi(t) \sim (2\pi)^{-1/2} \exp(-t^2/2)$ for $t > an^{1/2}$, 'a' sufficiently large for bounded random variables. The values of $\bar{\rho}$ becomes more restrictive for large m. As $m \to \infty$ the r.h.s. of (2.1) tends to $1/2l$. For $m = 2$, (2.1) asserts $\rho > 0$; for $m = 3, l = 2$ this asserts $\bar{\rho} > 1/8$.

Example. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with mean zero and variance σ^2. Then for the moving average process of order m

$$X_i = \frac{1}{m} \sum_{j=0}^{m-1} Y_{i+j}, \quad i \geqslant 1,$$

we get

$$\rho(X_1, X_2) = (m-1)/m, \rho(X_1, X_3) = (m-2)/m, \ldots, \rho(X_1, X_m) = 1/m.$$

Hence

$$\bar{\rho} = (m-1)^{-1} \sum_{i=1}^{m-1} \rho(X_1, X_{1+i}) = \frac{1}{2}$$

and this is always greater than $(m-l)/(2(m-1)l)$ as $l > 1$.

In what follows b represents a generic constant whose value may differ in different appearances.

Proof of the theorem. For $p \geqslant m$ with $k = [n/(p+m)]$ where $[x]$ denotes the integer part of x and $l' = n-k(p+m)$ if $n > k(p+m)$ define

$$\epsilon_i = \sum_{j=1}^{p} X_{(i-1)(p+m)+j}, \qquad i = 1, 2, \ldots, k$$

$$\eta_i = \sum_{j=1}^{m} X_{ip+(i-1)m+j} \quad \ldots \quad (2.3)$$

$$\epsilon_{k+1} = \sum_{j=1}^{l'} X_{k(p+m)+j} \text{ or 0 according as } l' \geqslant 1 \text{ or not.}$$

Now write ϵ_1 in the following form

$$\begin{aligned}\epsilon_1 &= X_1+X_{m+1}+X_{2m+1}+\dots\\ &+X_2+X_{m+2}+X_{2m+2}+\dots\\ &+\dots+\dots+\dots\\ &+X_m+X_{2m}+\dots\\ &= z_1+z_2+\dots+z_m \qquad \dots \quad (2.4)\end{aligned}$$

where each z_i is a sum of $[p/m]$ or $[p/m]+1$ independent components.

Take $t > 0$, let $t_n = t \pm f(n)$. For proper choice of $f(n) = f(n, t)$, we shall complete the proof by showing

$$|\Phi(t_n)-\Phi(t)| \leqslant bn^{-1/2}\exp(-\gamma t^2) \qquad \dots \quad (2.5)$$

$$P\left(\left|\sum_{i=1}^{k}\eta_i\right| > tn^{1/2}f(n)\right) \leqslant bn^{-1/2}\exp(-\gamma t^2) \qquad \dots \quad (2.6)$$

$$\left|P\left(\sum_{1}^{k+1}\epsilon_i > t_n n^{1/2}\right)-\Phi(-t_n)\right| \leqslant bn^{-1/2}\exp(-\gamma t^2). \qquad \dots \quad (2.7)$$

For $g(x) = \exp(cx^2)$ and $g_1(x) = g(x/m)$, observe that since g is nonnegative, even and non decreasing function

$$\begin{aligned}Eg_1(\eta_1) &= Eg\left(\frac{1}{m}(X_1+\dots+X_m)\right) \leqslant Eg\left(\sup_{1\leqslant i\leqslant m}|X_i|\right)\\ &\leqslant E\left(\sum_{i=1}^{m} g(X_i)\right) = m\,E\,g(X_1) < \infty \text{ from (1.5).} \qquad \dots \quad (2.8)\end{aligned}$$

Again η_i's are i.i.d random variables and therefore by Markov inequality and (2.8)

$$P\left(\left|\sum_{i=1}^{k}\eta_i\right| > tn^{1/2}f(n)\right) = o(g_1(t(n/k)^{1/2}f(n)))^{-1} \leqslant bn^{-1/2}\exp(-\gamma t^2) \dots \quad (2.9)$$

for some $\gamma > 0$ if

$$t^2(n/k)f^2(n) > \log n. \qquad \dots \quad (2.10)$$

Next note that

$$\begin{aligned}E\exp(|\epsilon_1|h/m) &= E\exp\left(\left|\sum_{i=1}^{m} z_i\right|h/m\right) \leqslant E\exp\left(h\sup_{1\leqslant i\leqslant m}|z_i|\right)\\ &\leqslant E\left(\sum_{i=1}^{m}\exp(h|z_i|)\right) = m\,E(\exp h|z_1|)\\ &\leqslant m\prod_{i=1}^{p/m} E\exp(h|X_i|) \qquad \dots \quad (2.11)\end{aligned}$$

assuming that z_1 is sum of p/m independent X_i's. We conveniently igonore the fact that p/m may not be an integer. In view of (1.4), (2.11) reduces to

$$E\exp(|\epsilon_1|h/m) \leqslant m\, 2^{p/m} \exp(h^2\sigma^{*2}p/(lm)). \qquad \text{... (2.12)}$$

Replacing h/m by h one gets

$$E\exp(h|\epsilon_1|) \leqslant m\, 2^{p/m} \exp(h^2\sigma^{*2}\, mp/l). \qquad \text{... (2.13)}$$

With (2.13) and proceeding along the proof of Theorem 2.1 of Dasgupta (1992a) one gets

$$P\Big(\sum_{i=1}^{k+1} \epsilon_i > t_n\, n^{1/2}\Big) \leqslant [m\, 2^{p/m} \exp(h^2\sigma^{*2}mp/(ln))]^{k+1}\exp(-hs_n t_n) \qquad \text{... (2.14)}$$

where $\qquad s_n^2 = \text{var}\Big(\sum_{i=1}^{k+1} \epsilon_i\Big) \geqslant kp \sim n.$

Take $h = t/(kp)^{1/2}$, then for $t_n = t \pm f_n(t) = t(1+o(1))$

$$P\Big(\sum_{i=1}^{k+1} \epsilon_i > t_n\, n^{1/2}\Big) \leqslant b\, m^{k+1}\, 2^{n/m} \exp(mt^2\sigma^{*2}/l)\exp(-t^2(1+o(1)))$$

$$\leqslant b\, m^{k+1}\, 2^{n/m} \exp(-\gamma_1 t^2) \qquad \text{... (2.15)}$$

where $\gamma_1 = \gamma_1(m\sigma^{*2}/l) > 0$, provided

$$m\sigma^{*2}/l < 1. \qquad \text{... (2.16)}$$

Now from (1.1) $\sigma \bar{1}^{*2}(+2(m-1)\rho) = 1$. Hence (2.16) is satisfied under (2.1). Since $\Phi(-t_n) \sim (2\pi)^{-1/2}\, t_n^{-1} \exp(-t_n^2/2),\ t_n \to \infty$,

$$\Big|P\Big(\sum_{i=1}^{k+1} \epsilon_i > t_n\, n^{1/2}\Big) - \Phi(-t_n)\Big| \leqslant b\, m^{k+1}\, 2^{n/m} \exp(-\gamma_1 t^2). \qquad \text{... (2.17)}$$

Now

$$\text{r.h.s. of (2.17)} \leqslant b\, n^{-1/2} \exp(-\gamma_2 t^2),\ \gamma_2 < \gamma_1 \qquad \text{... (2.18)}$$

if $t^2 \geqslant (\gamma_1 - \gamma_2)\dfrac{n}{m} \log 2(1+o(1))$, i.e., if

$$t^2 \geqslant \lambda n \text{ for some } \lambda > 0. \qquad \text{... (2.19)}$$

Now for the region $o(n^{1/2}) \leqslant t \leqslant \lambda n^{1/2}$ proceed along the remark 2.11 of Dasgupta (1989) with $g(x) = \exp(|x|)$. Then for the region $t_n^2 \geqslant 2 \log(|t_n| + \log g(rs_n t_n K_n)),\ K_n \to 0$ i.e., $t_n^2 \geqslant rs_n t_n K_n,\ t_n = t(1+o(1))$ i.e., $t \geqslant rs_n K_n$

$= o(n^{1/2})$, since $s_n^2 = \text{var}\left(\sum_1^{k+1} \epsilon_i\right) \sim n$, one obtains from (2.32) of Dasgupta (1989), after deleting the second term therein since truncation is not necessary when m.g.f. exists, the following :

$$\left| P\left(\sum_{i=1}^{k+1} \epsilon_i > n^{1/2} t_n \right) - \Phi(-t_n) \right| = O(t\, e^{rn^{1/2} tK_n})^{-1+O(K_n)}$$

$$\leqslant bn^{-1/2}\, e^{-\gamma t^2},\ o(n^{1/2}) < t < \lambda n^{1/2} \qquad \ldots \quad (2.20)$$

A similar argument is used in (2.23) of Dasgupta (1992a).

Now take $f(n) = n^{-1/2}$ and $p = n^{\alpha}$, for some $\alpha > 0$, this satisfies (2.10) for $t > o(n^{1/2})$. Hence (2.2) is true for $t > o(n^{1/2})$, since (2.5)—(2.7) are then satisfied.

Next we proceed to show (2.2) for $t = o(n^{1/2})$. Since ϵ_1 is sum of p m-dependent variates, proceeding as in (2.13) we may have

$$\phi(n) = E \exp\left(h \left| \sum_{i=1}^{n} X_i \right| \right) \leqslant m\, 2^{n/m} \exp(h^2 \sigma^{*2} mn/l),$$

$$\sigma^{*2} = EX_1^2 < 1 \text{ as } \bar{\rho} > 0. \qquad \ldots \quad (2.21)$$

Therefore from the remark in page 134 of Statulevicius (1966) the lemma on deviation therein holds for $\left(\sum_{i=1}^{n} X_i \right)$ provided

$$|\log \phi(h)|_{|h| = \frac{\Delta}{\sigma}} \leqslant H\Delta^2, \sigma^2 = \text{var}\left(\sum_1^n X_i \right) = n. \qquad \ldots \quad (2.22)$$

Now take $\Delta = n^{1/2} = \sigma$. Then from (2.21)

$$|\log \phi(1)| \leqslant \log m + \frac{n}{m} \log 2 + n\sigma^{*2} m/l \leqslant H\, n = H\Delta^2 \qquad \ldots \quad (2.23)$$

for some $H = H(m, \sigma^{*2}, l) > 0$. Therefore from (2) of Statulevicius (1966) we obtain the following :

$$1 - F_n(t) = (1 + o(1)) \exp((t^3/n)\lambda(t/n))\,(1 - \Phi(t)) \text{ for } 1 \leqslant t \leqslant o(n^{1/2}) \quad \ldots \quad (2.24)$$

where $\lambda(x)$ is the power series of Cramér. Hence

$$|F_n(t) - \Phi(t)| = |\exp((t^3/n)\lambda(t/n)) - 1|\,(1 - \Phi(t))\,(1 + o(1)). \qquad \ldots \quad (2.25)$$

Now for $t = o(n^{1/2})$, $|\exp((t^3/n)\lambda(t/n)) - 1| \leqslant n^{-1} t^3\, \lambda(t/n) e^{\lambda(t/n) t^3/n} \leqslant bn^{-1} t^3 \exp(o(t^2))$. Hence

$$|F_n(t) - \Phi(t)| \leqslant bn^{-1} t^2\, e^{-t^2/2 + o(t^2)} \leqslant bn^{-1/2}\, e^{-\gamma t^2}\ ;\ 0 < \gamma < 1/2 \quad \ldots \quad (2.26)$$

where $t > t_0$, for some $t_0 > 0$.

Now for $0 < t \leqslant t_0$, (2.2) follows from the uniform bound $O(n^{-1/2})$ due to Stein (1972). Hence (2.2), for $0 < t = o(n^{1/2})$. For the negative values of t, $|t| = o(n^{1/2})$ one uses (2′) of Statulevicius (1966) and proceeds similarly.

Hence the theorem.

We can bound the l.h.s. of (2.21) just by assuming existence of m.g.f. of X_1 in a neighbourhood of zero, see (2.11)—(2.13). Therefore, since (2.21) implies (2.24) we have the following

Corollary 2.1. *Let* $\{X_n \, ; n \geqslant 1\}$ *be a stationary m dependent process satisfying* (1.1) *and* $E\, e^{r|X_1|} < \infty$ *for some* $r > 0$, *then*

$$1-F_n(t_n) \sim \Phi(-t_n) \sim F_n(-t_n) \quad \textit{for } t_n = o(n^{1/6}). \qquad \ldots \quad (2.27)$$

A few consequence of Theorem 2.1 are moment type convergences and L_q versions of the Berry Essen theorem. The proof of the followings are similar to those of Theorem 2.5 and Corollary 2.1 of Dasgupta (1992a).

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *are satisfied. Let* $g : (-\infty, \infty) \to [0, \infty)$ *be an even function with* $g(0) = 0$, *and* $E\, g(T) < \infty$, $T = N(0, 1)$ *and*

$$g'(x) = O(exp(x^2\gamma^*)), \; x > 0, 0 < \gamma^* < \gamma. \qquad \ldots \quad (2.28)$$

Then the following holds

$$\left| E\, g\left(n^{-1/2} \sum_{i=1}^{n} X_i\right) - E\, g(T) \right| = O(n^{-1/2}). \qquad \ldots \quad (2.29)$$

Corollary 2.2. *Let the assumptions of Theorem* 2.1 *be satisfied. Then for* $\delta > 1$ *and* $q \geqslant 1$ *one has*

$$\| (1+|t|)^{-\delta/q}\, exp\,(\gamma t^2)\, (F_n(t)-\Phi(t)) \|_q = O(n^{-1/2}). \qquad \ldots \quad (2.30)$$

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