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Some results on codimension-one A^1-fibrations

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Abstract

We prove a few results on the sufficiency of generic and codimension-one fibre conditions for determination of the structure of algebras of transcendence degree one. We first show that over a Noetherian normal domain R, a faithfully flat subalgebra of a finitely generated algebra whose generic and codimension-one fibres are A^1 is necessarily the symmetric algebra of an invertible ideal of R. We next prove a structure theorem for a faithfully flat algebra over a locally factorial Krull domain R whose generic and codimension-one fibres are A^1. For R local, we deduce a minimal sufficient condition for the algebra to be finitely generated and hence A^1.

Keywords: A^1-fibration; Codimension-one; Subalgebra; Finitely generated algebra; Krull domain; Direct limit

1. Introduction

Let R be a commutative ring. For a prime ideal P of R, $k(P)$ denotes the field R_P/PR_P. A polynomial ring in n variables over R is denoted by $R^{[n]}$. A finitely generated flat R-algebra A will be called an A^1-*fibration* if $k(P) \otimes_R A = k(P)^{[1]}$ for every $P \in \operatorname{Spec} R$.

The following results on A^1-fibration were proved in [2, 3.4] and [1, 3.10], respectively.

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Theorem 1.1. *Let R be a Noetherian normal domain with quotient field K and A a faithfully flat finitely generated R-algebra such that $K \otimes_R A = K^{[1]}$ and $k(P) \otimes_R A$ is geometrically integral for every prime ideal P in R of height one. Then $A \cong R[IX]$ for an invertible ideal I of R.*

Theorem 1.2. *Let R be a Noetherian normal domain with quotient field K and A a flat R-subalgebra of $R^{[m]}$ such that $K \otimes_R A = K^{[1]}$ and $k(P) \otimes_R A$ is an integral domain for every prime ideal P in R of height one. Then $A \cong R[IX]$ for an invertible ideal I of R.*

A striking feature of both these theorems is that conditions on merely the generic and codimension-one fibres completely determine an $\mathbf{A}^1$-fibration. For a better insight into such codimension-one $\mathbf{A}^1$-fibration, we first explore if both results emanate from a common general theorem.

Note that in Theorem 1.2, when A is given to be a subalgebra of a polynomial algebra, we do not need the hypothesis that A is finitely generated (used in Theorem 1.1)—it turns out to be a consequence! One wonders whether the two hypotheses "A is finitely generated" and "$A \hookrightarrow R^{[m]}$" can be replaced by the common hypothesis "A is a subalgebra of a finitely generated algebra." The following example of Bhatwadekar [2, 4.1] seems to indicate that Theorem 1.1 cannot be so generalised in its above form.

Example 1.3. Let k be a field and $R = k[x]$ $(= k^{[1]})$. Fix $y \in k[[x]]$ such that y is transcendental over $k(x)$. Let $A = k[[x]] \cap k[x, x^{-1}, y]$, an R-subalgebra of the finitely generated R-algebra $R[x^{-1}, y]$. Then A is a Noetherian factorial domain such that A is faithfully flat over R, the generic fibre $K \otimes_R A$ is $K^{[1]}$, and all codimension-one fibres (i.e., closed fibres) are geometrically integral. But A is not an $\mathbf{A}^1$-fibration over R; A is not even finitely generated.

However, in the above example, the codimension-one fibres are not all of the correct (i.e., one) dimension. Here, $A/(x-\lambda)A = k^{[1]}$ for every non-zero λ in k but $A/xA = k$. In this paper, we first show that this degeneracy is the only obstruction for Theorem 1.1 to be extended to subrings of finitely generated algebras. We prove (Theorem 3.5):

Theorem A. *Let R be a Noetherian normal domain with quotient field K and A a faithfully flat R-algebra such that A is an R-subalgebra of a finitely generated R-algebra B and such that A satisfies the fibre conditions:*

(i) $K \otimes_R A = K^{[1]}$.
(ii) *For every prime ideal P in R of height one, $k(P) \otimes_R A$ is an integral domain with* $\operatorname{tr.deg}_{k(P)} k(P) \otimes_R A > 0$ *and $k(P)$ is algebraically closed in $k(P) \otimes_R A$.*

Then $A \cong R[IX]$ for an invertible ideal I of R.

In particular:

Corollary. *Over a Noetherian normal domain R, any faithfully flat algebra A, whose generic and codimension-one fibres are $\mathbf{A}^1$, and which is an R-subalgebra of a finitely generated R-algebra, is isomorphic to the Rees algebra of an invertible ideal of R.*

When $R \hookrightarrow A \hookrightarrow R^{[m]}$, we shall prove (Proposition 3.7):

Proposition. *Let R be a Noetherian domain, A ($\neq R$) an R-subalgebra of $R^{[m]}$, and P a prime ideal in R such that PA is a prime ideal in A. Then* $\text{tr.deg}_{R/P} A/PA > 0$ *and R/P is algebraically closed in A/PA.*

Using the result, we shall show (Remark 3.8) that Theorems 1.1 and 1.2 are really special cases of Theorem 3.5. The two results (3.5 and 3.7) also put in perspective the extreme mild hypothesis of integrality of codimension-one fibres in Theorem 1.2 and the somewhat stronger hypothesis in Theorem 1.1.

We now take a closer look at the hypothesis "A is an R-subalgebra of a finitely generated R-algebra." Its necessity in Theorem A can be seen from the well-known example of the $\mathbb{Z}$-algebra $A = \mathbb{Z}[\{\frac{X}{p} \mid p \text{ a prime in } \mathbb{Z}\}]$. Here $A_P = {R_P}^{[1]}$ for every prime ideal P in the ring of integers $\mathbb{Z}$ and yet A is not finitely generated. Now a question arises:

Question. In Theorem A, can the hypothesis of finite generation (i.e., A being dominated by a finitely generated R-algebra B) be dropped in the case of a nice *local* domain R (say when R is a regular local ring)?

Over a discrete valuation ring (R, π), any faithfully flat algebra satisfying (i) is a subalgebra of the finitely generated R-algebra $A[1/\pi]$ ($= R[1/\pi]^{[1]}$). But the following example shows that even over a regular local ring R of dimension two, there could exist a faithfully flat algebra A which satisfies $A_P = {R_P}^{[1]}$ for every prime ideal P in R of height $\leqslant 1$ but which is not finitely generated over R.

Example 1.4. Let k be an infinite field and $R = k[[t_1, t_2]]$ where t_1, t_2 are algebraically independent over k. Let $A = R[\{\frac{X}{q} \mid q \text{ a square-free non-unit in } R\}]$. For every height one prime ideal P of the factorial domain R, $P = pR$ for some prime element p; thus $A_P = R_P[\frac{X}{p}] = {R_P}^{[1]}$. However $A/(t_1, t_2)A = k$. Note that A, being a direct limit of the polynomial rings $R[\frac{X}{q}]$, is flat over R and hence faithfully flat over R (since $(t_1, t_2)A \neq A$). A is not finitely generated.

In Section 4, we shall show (Theorem 4.6) that Example 1.4 is really a prototype of the general structure of a *faithfully flat algebra A over a factorial domain R satisfying the generic and codimension-one fibre conditions* (i) *and* (ii). In fact, Theorem 4.6 shows that any such algebra occurs as the *direct limit of polynomial algebras*. A consequence of the structure theorem is that *at each point P of* Spec R, *the fibre ring $k(P) \otimes_R A$ is either $k(P)$ or $k(P)^{[1]}$* (Corollary 4.11).

In Example 1.4, the closed fibre $A/(t_1, t_2)A$ ($= k$) does not have the correct dimension. This appears to be the only obstacle to an affirmative answer to the above Question. We shall deduce from the structure theorem that Theorem A indeed holds over a factorial local domain without the hypothesis that A is dominated by a finitely generated algebra if, along with the conditions on generic and codimension-one fibres, we also assume that the closed fibre has correct dimension. More precisely, we prove (Theorem 4.12):

Theorem B. *Let (R, m) be a factorial local domain with quotient field K. Suppose that A is a faithfully flat R-algebra satisfying*:

(i) $K \otimes_R A = K^{[1]}$.
(ii) *For every prime ideal P in R of height one, $k(P) \otimes_R A$ is an integral domain with* $\text{tr.deg}_{k(P)} k(P) \otimes_R A > 0$ *and $k(P)$ is algebraically closed in $k(P) \otimes_R A$.*

Then the following conditions are equivalent:

(1) *A is finitely generated over R.*
(2) $\operatorname{tr.deg}_{R/m} A/mA > 0$.
(3) $\dim A/mA > 0$.
(4) $A = R^{[1]}$.

Theorems A and B will be proved in Sections 3 and 4, respectively. In Section 2, we quote two local–global results and prove a few general results which will be used in Sections 3 and 4.

2. Preliminary results

The following criterion [5, Theorem 2.20] reduces the question of finite generation of a subalgebra of a polynomial algebra to the local situation.

Theorem 2.1. *Let R be a Noetherian domain and A an overdomain of R such that*

(I) *There exists a non-zero $f \in A$ for which A_f is a finitely generated R-algebra.*
(II) *A_m is a finitely generated R_m-algebra for all maximal ideals m of R.*

Then A is a finitely generated R-algebra.

Condition (I) will be satisfied when A is contained in a finitely generated R-algebra ([4, 2.1] or [5, 2.11]) so that we have

Corollary 2.2. *Let R be a Noetherian domain and A a subalgebra of a finitely generated R-algebra B. If A is locally finitely generated over R, then A is finitely generated over R.*

As a consequence, a well-known result of Eakin and Heinzer [3] may be stated in the following form:

Theorem 2.3. *Let R be a Noetherian domain and A an overdomain of R which is contained in a finitely generated R-algebra B. If $A_m = {R_m}^{[1]}$ for every maximal ideal m of R, then A is isomorphic to the symmetric algebra of an invertible ideal I of R.*

We now present a modified version of the Russell–Sathaye criterion [6, 2.3.1] for an algebra to be a polynomial ring in one variable.

Theorem 2.4. *Let $R \subset A$ be integral domains. Let p be a prime element in R such that p remains prime in A, $pA \cap R = pR$, $A[1/p] = R[1/p]^{[1]}$ and R/pR is algebraically closed in A/pA. Then there exists a sequence of rings*

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots \subseteq A \tag{2.1}$$

such that $A_n = R^{[1]}$ and $pA \cap A_n \subseteq pA_{n+1}$ for each $n \geqslant 0$, and $A = \bigcup_i A_i$.

Moreover, the following conditions are equivalent:

(1) *A is finitely generated over R.*
(2) $\operatorname{tr.deg}_{R/pR} A/pA > 0$.
(3) $A = R^{[1]}$.

Proof. By assumption there exists $x \in A$ such that x is transcendental over R and $A[1/p] = R[1/p][x]$. Set $A_0 = R[x]$. Then $A_0 \subseteq A$ and $A_0[1/p] = A[1/p]$. If $pA \cap A_0 = pA_0$, then $A = A_0$ and we are through (taking $A_n = A_0$ for every $n \geqslant 1$).

Suppose that $pA \cap A_0 \neq pA_0$. Note that

$$A_0/pA_0 = (R/pR)[x] = (R/pR)^{[1]}.$$

Let $\bar{x}$ denote the image of x in A/pA. Since $pA_0 \subsetneqq pA \cap A_0$, the natural map $A_0/pA_0 \to A/pA$ is not injective. Hence it follows that $\bar{x}$ is algebraic over R/pR. Thus $\bar{x} \in R/pR$ as R/pR is algebraically closed in A/pA. Therefore, $x - c_0 \in pA$ for some $c_0 \in R$. For such c_0, the induced map $A_0/(p, x - c_0)A_0 \to A/pA$ is injective since $A_0/(p, x - c_0)A_0 = R/pR$ and $pA \cap R = pR$. Hence $pA \cap A_0 = (p, x - c_0)A_0$. Let $x_1 = (x - c_0)/p$ ($\in A$) and set $A_1 = R[x_1]$. Then $A_0 \subseteq A_1 \subseteq A$, $pA \cap A_0 \subseteq pA_1$ and $A_1[1/p] = A[1/p]$. If $pA \cap A_1 = pA_1$, then $A = A_1$ and we are done (taking $A_n = A_0$ for every $n \geqslant 2$).

If not, then we can apply the same process as above to get $c_1 \in R$ such that $pA \cap A_1 = (p, x_1 - c_1)A_1$. Set $x_2 = (x_1 - c_1)/p$ and $A_2 = R[x_2]$, and repeat the above construction.

Thus we get the sequence of rings in (2.1) with $A_n = R[x_n]$ ($= R^{[1]}$) and $pA \cap A_n \subseteq pA_{n+1}$ for each n. Set $C = \bigcup_i A_i$.

We show that $A = C$. Since $C[1/p] = A[1/p]$, it suffices to show that $pA \cap C = pC$. Let y be an arbitrary element of $pA \cap C$. Then $y \in A_i$ for some i, so that $y \in pA \cap A_i \subseteq pA_{i+1} \subseteq pC$. Thus $pA \cap C = pC$, and hence $A = C$.

We now prove the equivalence of (1), (2) and (3). It suffices to show (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3).

(1) $\Rightarrow$ (3). Write $A = R[f_1, \ldots, f_r]$. Since $A = \bigcup_i A_i$, there exists $n > 0$ such that $f_1, \ldots, f_r \in A_n$. It then follows that $A = A_n = R^{[1]}$, as desired.

(2) $\Rightarrow$ (3). Note that, by construction, $A_i = R[x_i]$, and if $A_i \neq A$, then $pA \cap A_i = (p, x_i - c_i)A_i$, so that $A_i/(pA \cap A_i) = R/pR$. Suppose that the sequence (2.1) is of infinite length, that is, $A \neq A_n$ for each $n > 0$. Then, identifying $A_i/(pA \cap A_i)$ with its natural isomorphic image in A/pA, we have

$$A/pA = \bigcup_i A_i/(pA \cap A_i) = R/pR,$$

which contradicts $\operatorname{tr.deg}_{R/pR} A/pA > 0$. Therefore $A = A_n = R^{[1]}$ for some n. This completes the proof. □

The following result will be used in Sections 3 and 4.

Lemma 2.5. *Let R be an integral domain and let a, d be non-zero elements of R. Set $I = dR[1/a] \cap R$. Then the following assertions hold.*

(1) *If I is an invertible ideal of R, then $I^n = d^n R[1/a] \cap R$ for every positive integer n.*
(2) *Suppose that d is a unit in $S^{-1}R[1/a]$, where $S = \{s \in R \mid s \text{ is not a zero-divisor in } R/aR\}$. If I is flat over R, then I is an invertible ideal of R.*

Proof. (1) Let $J_n = d^n R[1/a] \cap R$. We prove $I^n = J_n$ by induction on n. It suffices to show that $(I^n)_m = (J_n)_m$ for every maximal ideal m of R. Since $(J_n)_m = d^n R_m[1/a] \cap R_m$, replacing R by R_m, we may suppose that R is local. Since I is invertible, it now follows that I is principal. Write $I = tR$. Then $t^{-1}d \in R$ because $d \in I$.

Now let w be an element of J_n. Then $w \in J_n \subseteq I = tR$, and hence $t^{-1}w \in R$. Thus $t^{-1}w \in t^{-1}d^n R[1/a] \cap R \subseteq d^{n-1}R[1/a] \cap R$, so that $t^{-1}w \in t^{n-1}R$ by the induction hypothesis. Therefore $w \in t^n R$. Thus $J_n \subseteq I^n$. Since the converse inclusion is obvious, we have $I^n = J_n$, as desired.

(2) For an R-module M, we denote $S^{-1}M$ by M_S. Note that $R[1/a] \cap R_S = R$. Hence, putting $J = R \cap dR_S$, we have $I \cap J = dR$ and hence $IJ \subseteq dR$. We show that $IJ = dR$. Since I is flat over R, I/IJ is flat over R/J. By construction of J, the map $R/J \to (R/J)_S$ $(= R_S/dR_S)$ is injective. Hence, by flatness of I/IJ over R/J, the map $I/IJ \to (I/IJ)_S$ is injective; in particular, the induced map $dR/IJ \to (dR/IJ)_S$ is injective. Now $I_S = R_S$ (since d is a unit in $S^{-1}R[1/a]$) and $J_S = dR_S$, so that $(IJ)_S = I_S J_S = dR_S$. Thus $(dR/IJ)_S = 0$ and hence $dR/IJ = 0$, i.e., $IJ = dR$. Therefore, I is invertible. □

Remark 2.6. Note that the condition "d is a unit in $S^{-1}R[1/a]$" in (2) of Lemma 2.5 is automatically satisfied when aR has a primary decomposition (for instance, when R is a Noetherian or a Krull domain). In fact, if $aR = Q_1 \cap \cdots \cap Q_n$ is an irredundant primary decomposition, then, with the same notations as in Lemma 2.5, $S = R \setminus (P_1 \cup \cdots \cup P_n)$ where $P_i = \sqrt{Q_i}$, so that $S^{-1}R[1/a] = K$, the quotient field of R.

We now prove two results on Krull domains which will be used in Section 4.

Lemma 2.7. *For a Krull domain R, the following conditions are equivalent*:

(1) *R is locally factorial.*
(2) *$dR[1/a] \cap R$ is an invertible ideal of R for every non-zero $a, d \in R$.*

Proof. $(1) \Rightarrow (2)$. It is enough to show that $I := dR[1/a] \cap R$ is locally principal; if this is the case, then I is flat over R, and hence I is invertible by Lemma 2.5 (cf. Remark 2.6). Thus we may assume that R is local and factorial. Now we may write $d = {p_1}^{r_1} \cdots {p_n}^{r_n} {p_{n+1}}^{r_{n+1}} \cdots {p_m}^{r_m}$, where $p_1, \ldots, p_m$ are distinct prime elements in the factorial domain R such that $p_i \mid a$ for $1 \leqslant i \leqslant n$ and $p_i \nmid a$ for $n+1 \leqslant i \leqslant m$. Set $u = {p_{n+1}}^{r_{n+1}} \cdots {p_m}^{r_m}$. Then a is coprime to u and it follows easily that $I = uR$. Thus I is principal in the local ring R.

$(2) \Rightarrow (1)$. Suppose that $dR[1/a] \cap R$ is invertible whenever a, d are non-zero elements of R. Replacing R by R_m, we may assume that R is local and show that R is factorial.

Let P be a prime ideal in R of height one, and let $p \in P$ be an element satisfying $PR_P = pR_P$. Then the primary decomposition of pR is of the form $pR = P \cap Q_1 \cap \cdots \cap Q_n$. Let $P_i = \sqrt{Q_i}$ for each i, and take $a \in (P_1 \cap \cdots \cap P_n) \setminus P$. Then we have $P = PR[1/a] \cap R = pR[1/a] \cap R$. Since $pR[1/a] \cap R$ is principal by assumption, we know that P is principal. Thus R is factorial. □

Lemma 2.8. *Let $R \subseteq A$ be integral domains such that R is a Krull ring and A is flat over R. Then we have*

$$A = \bigcap_{P \in \Delta} A_P, \tag{2.2}$$

where Δ is the set of all prime ideals in R of height one.

Proof. First of all recall that, for $P_1, \dots, P_n$ of Δ, setting $S = R \setminus (P_1 \cup \dots \cup P_n)$, we have

$$A_{P_1} \cap \dots \cap A_{P_n} = S^{-1}A. \tag{2.3}$$

To verify the equality in (2.3), let $f \in A_{P_1} \cap \dots \cap A_{P_n}$ and $I = (A :_R f)$, which is an ideal of R. Then $I \not\subseteq P_i$ for each i, so that $I \not\subseteq P_1 \cup \dots \cup P_n$. Take $s \in I \setminus (P_1 \cup \dots \cup P_n)$. Then $f = sf/s \in S^{-1}A$, as desired.

Now we prove the equality in (2.2). Let f be an element of $\bigcap A_P$. Then $f = a/t$ for some $a \in A$ and $0 \neq t \in R$. Since R is a Krull domain, there exist only finitely many prime ideals of height one containing t, say $P_1, \dots, P_n$. Then $f \in A_{P_1} \cap \dots \cap A_{P_n}$, so that, by Eq. (2.3), $f = b/s$ for some $b \in A$ and $s \in R \setminus (P_1 \cup \dots \cup P_n)$. Note that s is a non-zero divisor in R/tR and hence in A/tA as A is flat over R. Since $sa = tb$, it then follows that $a \in tA$. Thus $f = a/t \in A$, and the assertion is verified. □

For convenience, we state below an easy lemma.

Lemma 2.9. *Let $A = R[IX]$ be the Rees algebra of an invertible ideal I of R and let Q be a prime ideal of R. Then QA is a prime ideal of A.*

Proof. A, being a Rees algebra, is an $\mathbb{A}^1$-fibration. In particular, A is flat over R and $k(Q) \otimes_R A$ $(= k(Q)^{[1]})$ is a domain. Thus $A/QA (\hookrightarrow k(Q) \otimes_R A)$ is a domain, i.e., $QA \in \operatorname{Spec} A$. □

3. On $\mathbb{A}^1$-fibrations of subalgebras of finitely generated algebras

We use the techniques of [1,2] to prove a version of the Patching Lemma which reduces Theorem A to the case of semi-local PID.

Lemma 3.1. *Let $R \subset A$ be integral domains with A being a faithfully flat R-algebra. Let $a \neq 0$ be an element of R, $S = \{s \in R \mid s$ is not a zero-divisor in $R/aR\}$, and $L = S^{-1}R[1/a]$. Suppose that x and y are transcendental elements of A over R such that $x \in S^{-1}R[y]$ and $L[x] = L[y]$. Set $D = R[1/a][x] \cap S^{-1}R[y]$. If D is flat over R, or if R is locally factorial, then*

$$D = R\left[I\left(\frac{x-c}{d}\right)\right]$$

for an invertible ideal I of R and $c, d \in R$.

Proof. If a is a unit in R, then $D = R[x]$ and all assertions follow trivially; so we assume that a is a non-unit.

Since $x \in S^{-1}R[y]$ and $L[x] = L[y]$, we can write

$$x = \frac{dy + d'}{s},$$

for some $d, d' \in R$ and $s \in S$ such that d is a unit in L. Then

$$d' = sx - dy \in (s, d)A \cap R = (s, d)R$$

by faithful flatness of A over R, so that $d' = sc - dc'$ for some $c, c' \in R$. It then follows that

$$x = \frac{dy + sc - dc'}{s} = \frac{d(y - c')}{s} + c.$$

Hence, setting $z = (x - c)/d = (y - c')/s$, we have $S^{-1}R[y] = S^{-1}R[z]$ and

$$D = S^{-1}R[z] \cap R[1/a][dz] = \bigoplus_{n \geqslant 0} \big(S^{-1}R \cap d^n R[1/a]\big) z^n = \bigoplus_{n \geqslant 0} \big(R \cap d^n R[1/a]\big) z^n$$

using $R = S^{-1}R \cap R[1/a]$. Let $I = R \cap dR[1/a]$.

We show that I is an invertible ideal of R. If D is flat over R, then I, being isomorphic to a direct summand of D, is also flat over R and hence, as d is a unit in L, I is invertible by (2) of Lemma 2.5. If R is locally factorial, then I is invertible by Lemma 2.7.

It thus follows from (1) of Lemma 2.5 that

$$D = R[Iz] = R\left[I\left(\frac{x - c}{d}\right)\right].$$

This completes the proof. □

Corollary 3.2. *Let $R \subset A$ be integral domains with A being a faithfully flat R-algebra. Suppose that there exists a non-zero element $a \in R$ such that*

(I) $A[1/a] = R[1/a]^{[1]}$.
(II) $S^{-1}A = (S^{-1}R)^{[1]}$, *where $S = \{s \in R \mid s$ is not a zero-divisor in $R/aR\}$.*

Then there exists an invertible ideal I in R such that $A \cong R[IX]$.

Proof. Since $R = R[1/a] \cap S^{-1}R$, from flatness of A, it follows that $A = A[1/a] \cap S^{-1}A$. By (I) and (II), there exist $x, y \in A$ such that $A[1/a] = R[1/a][x]$ and $S^{-1}A = S^{-1}R[y]$. Then $A = R[1/a][x] \cap S^{-1}R[y]$. Note that $S^{-1}A[1/a] = L[x] = L[y]$, where $L = S^{-1}R[1/a]$. Thus, by Lemma 3.1, $A \cong R[IX]$ for an invertible ideal I in R. □

Remark 3.3. Note that if A is a subalgebra of a faithfully flat R-algebra (for instance, when $A \hookrightarrow R^{[m]}$), then $JA \cap R = J$ for every ideal J in R, so that A is faithfully flat over R if and only if A is flat over R.

The next result follows from [3]. We give below a proof based on Corollary 3.2.

Corollary 3.4. *Let R be a semi-local domain with maximal ideals $m_1, \ldots, m_n$. Suppose that A is an overdomain of R such that $A_{m_i} = R_{m_i}{}^{[1]}$ for $i = 1, \ldots, n$. Then $A = R^{[1]}$.*

Proof. Clearly A is finitely generated and faithfully flat over R. We prove the result by induction on n. The case $n = 1$ is obvious. Let $T = R \setminus (m_1 \cup \cdots \cup m_{n-1})$. By induction hypothesis, $T^{-1}A = (T^{-1}R)^{[1]}$. Since A is finitely generated, there exists $a \in T$ such that $A[1/a] = R[1/a]^{[1]}$. Every element of $R \setminus m_n$ is comaximal with a. Hence, from condition $A_{m_n} = R_{m_n}{}^{[1]}$, we have $S^{-1}A = (S^{-1}R)^{[1]}$ where S is as in Corollary 3.2. Therefore, $A = R^{[1]}$ by Corollary 3.2. □

We now prove Theorem A.

Theorem 3.5. *Let R be a Noetherian normal domain with quotient field K. Let A be a faithfully flat R-algebra such that A is an R-subalgebra of a finitely generated R-algebra B. Suppose that A satisfies the fibre conditions*:

(i) $K \otimes_R A = K^{[1]}$.
(ii) *For every prime ideal P in R of height one, $k(P) \otimes_R A$ is an integral domain with* $\operatorname{tr.deg}_{k(P)} k(P) \otimes_R A > 0$ *and $k(P)$ is algebraically closed in $k(P) \otimes_R A$.*

Then $A \cong R[IX]$ for an invertible ideal I of R.

Proof. Since A is contained in a finitely generated R-algebra B, by Theorem 2.3, it is enough to show that $A_m = R_m{}^{[1]}$ for every maximal ideal m of R. Note that if T is a multiplicatively closed subset of R, then $T^{-1}A \subseteq T^{-1}B$ and $T^{-1}A$ is a faithfully flat $T^{-1}R$-algebra satisfying the fibre conditions (i) and (ii). Thus, replacing R by R_m, we may assume that R is a local ring with maximal ideal m.

We use induction on $\dim R$. The case $\dim R = 1$ follows from Theorem 2.4. Assume that $\dim R \geqslant 2$. Choose an arbitrary non-zero element t in m. Then $\dim R[1/t] < \dim R$, so that by the induction hypothesis $A[1/t]$ is locally finitely generated over $R[1/t]$ and hence finitely generated by Corollary 2.2.

Since $K \otimes_{R[1/t]} A[1/t] = K^{[1]}$ by (i), there exists $a \in R$ such that $A[1/a] = R[1/a]^{[1]}$. Let $P_1, \ldots, P_n$ be the associated prime ideals of aR, and let $S = R \setminus (P_1 \cup \cdots \cup P_n)$. Then $S^{-1}R$ is a semi-local PID because R is a Noetherian normal domain and R_{P_i} is a discrete valuation ring for each i. Now $A_{P_i} = R_{P_i}{}^{[1]}$ for each i (using Theorem 2.4). Hence $S^{-1}A = (S^{-1}R)^{[1]}$ by Corollary 3.4. Hence, by Corollary 3.2, $A = R^{[1]}$, as desired. □

Remark 3.6. We give below some examples to illustrate the hypotheses in Theorem 3.5.

(1) The hypothesis on flatness is needed even when A is a finitely generated subalgebra of $R^{[1]}$. For instance, consider $R = k[[t_1, t_2]]$ and $A = R[t_1X, t_2X] \cong R[U, V]/(t_2U - t_1V)$.

(2) The hypothesis on faithful flatness is also necessary. Consider $R = k[[t_1, t_2]]$ and $A = R[U, V]/(t_1U + t_2V - 1)$.

(3) It is also easy to see that the condition "$k(P)$ is algebraically closed in $k(P) \otimes_R A$ for each height one prime ideal P" in hypothesis (ii) is necessary. Let $R = \mathbb{R}[[t]]$ be the power series ring over the field $\mathbb{R}$ of real numbers and $A = R[U, V]/(tU + V^2 + 1)$. Then A is a finitely generated flat R-algebra, the generic fibre of A is $\mathbf{A}^1$ and $A/tA = \mathbb{C}^{[1]}$ is an integral domain of positive transcendence degree over R/t $(= \mathbb{R})$. But $A \neq R^{[1]}$. The condition may be dropped when $A \hookrightarrow R^{[m]}$ (see Remark 3.8).

(4) A slight modification of Example 1.3 shows the necessity of condition "$\operatorname{tr.deg}_{k(P)} k(P) \otimes_R A > 0$" in hypothesis (ii) even over a discrete valuation ring. Let k be a field, $R = k[x]_{(x)}$, and $A = k[[x]] \cap k(x)[y]$ where $y \in k[[x]]$ is transcendental over $k(x)$. Then A is a flat R-algebra contained in the finitely generated R-algebra $R[1/x, y]$, the generic fibre of A is $\mathbf{A}^1$ and $A/xA = k$. But A is not finitely generated over R. The condition may be dropped if A is given to be finitely generated or $A \hookrightarrow R^{[m]}$ (see Remark 3.8).

For further discussion on Theorem 3.5, we study certain properties of the domain A/PA when $R \hookrightarrow A \hookrightarrow R^{[m]}$ and P is a prime ideal in R which remains prime in A.

Proposition 3.7. *Let R be an integral domain and A an R-subalgebra of B where $B = R^{[m]}$. Let P be a prime ideal in R such that PA is a prime ideal in A. Then R/P is algebraically closed in A/PA. Moreover* $\operatorname{tr.deg}_{R/P} A/PA > 0$ *if R is Noetherian and $A \neq R$.*

Proof. Let $Q = PB \cap A$. Then $PA \subseteq Q$ and $Q \cap R = PB \cap R = P$. We will show that R/P is algebraically closed in A/PA. For $a \in A$, we denote by $\bar{a}$ the image of a in $A/Q(\hookrightarrow (R/P)^{[m]})$ and by a^* the image of a in A/PA. Suppose that a^* is algebraic over R/P. Let $\varphi(X)$ be the monic minimal polynomial of a^* over the quotient field of R/P and let $\lambda \in R/P$ be a non-zero element satisfying $f(X) := \lambda\varphi(X) \in (R/P)[X]$. Then $f(a^*) = 0$, and hence $f(\bar{a}) = 0$, because A/Q is a surjective image of A/PA. Since R/P is algebraically closed in $(R/P)^{[m]}$, it then follows that $\bar{a} \in R/P$. Thus there exists $c \in R$ such that $\bar{c} = \bar{a}$ and $f(\bar{c}) = 0$, which implies $\varphi(X) = X - \bar{c}$. Since $\varphi(a^*) = 0$, from this we have $a^* = \bar{c} \in R/P$, as desired.

Next we will show that $\operatorname{tr.deg}_{R/P} A/PA > 0$ when R is Noetherian and $A \neq R$. By [5, 1.8], we have

$$\operatorname{ht}(Q/PA) + \operatorname{tr.deg}_{R/P} A/Q \leqslant \operatorname{tr.deg}_{R/P} A/PA.$$

Hence if $\operatorname{tr.deg}_{R/P} A/PA = 0$, then $\operatorname{ht}(Q/PA) = 0$, so that $Q = PA$. Thus $R/P \hookrightarrow A/PA \hookrightarrow (R/P)^{[m]}$, which implies $A/PA = R/P$. From this we have $A = R + PA$, and hence $A = R + P^nA$ for every $n > 0$. Since R is Noetherian, it follows that

$$A = \bigcap_{n>0}\left(R + P^nA\right) \subseteq \bigcap_{n>0}\left(R + P^nB\right) = R,$$

a contradiction. Thus $\operatorname{tr.deg}_{R/P} A/PA > 0$, as claimed. □

Remark 3.8. We now show how the earlier results, quoted as Theorems 1.1 and 1.2 in Introduction, can be interpreted in terms of Theorem 3.5.

Theorem 3.5 ⇒ Theorem 1.1. Note that condition (ii) in Theorem 3.5 is satisfied when the fibre ring $k(P) \otimes_R A$ is geometrically integral and of positive transcendence degree over $k(P)$. It suffices to show that when A is finitely generated then one can drop the condition "$\operatorname{tr.deg}_{k(P)} k(P) \otimes_R A > 0$" from hypothesis (ii) of Theorem 3.5. This follows from Theorem 2.4.

Theorem 3.5 ⇒ Theorem 1.2. Recall that A is faithfully flat over R when A is flat over R and $A \hookrightarrow R^{[m]}$. Now it suffices to show that when $A \hookrightarrow R^{[m]}$ then one can drop both conditions "$\operatorname{tr.deg}_{k(P)} k(P) \otimes_R A > 0$" and "$k(P)$ is algebraically closed in $k(P) \otimes_R A$" from hypothesis (ii) of Theorem 3.5.

Observe that since A is flat over R, if a fibre ring $k(P) \otimes_R A$ is a domain for a prime ideal P in R, then so is A/PA. Now the implication follows from Proposition 3.7.

Remark 3.9. When R is a Noetherian or a Krull domain, the patching technique gives an alternative proof of the result [3] that finitely generated locally polynomial R-algebra A is a symmetric algebra. Clearly there exists $a \in R$ such that $A[1/a] = R[1/a]^{[1]}$. Apply the argument in last paragraph of the proof of Theorem 3.5.

4. On generic and codimension-one A^1-fibration over a UFD

Hypotheses and notations

Throughout the section we will assume that

(1) R is a locally factorial Krull domain with quotient field K.
(2) A is a faithfully flat R-algebra satisfying the following conditions:
   (i) $K \otimes_R A = K^{[1]}$.
   (ii) For every prime ideal P in R of height one, $k(P) \otimes_R A$ is an integral domain with $\text{tr.deg}_{k(P)} k(P) \otimes_R A > 0$, and $k(P)$ is algebraically closed in $k(P) \otimes_R A$.

The results in this section were originally proved over a factorial domain R. The authors thank S.M. Bhatwadekar for his suggestive query on generalisation to the locally factorial case.

For ready reference, we list below the notations that will be frequently used.

- $\Delta = \{P \in \operatorname{Spec} R \mid \operatorname{ht} P = 1\}$.
- $\Gamma_a = \{P \in \Delta \mid a \in P\}$ where $0 \neq a \in R$. Note that $\Gamma_{ab} = \Gamma_a \cup \Gamma_b$ for $a, b \in R$. Note also that $pR \in \Gamma_a \Leftrightarrow p \mid a$ for a prime element p in R.
- x: A fixed element of A such that x is transcendental over R and

$$T^{-1}A = K[x],$$

  where $T = R \setminus \{0\}$. (Such an x exists by condition (i).)
- e_P: For each $P \in \Delta$, e_P will denote the unique non-negative integer e such that

$$A_P = R_P\left[\frac{x-c}{p^e}\right]$$

  for some $c \in R$, where p is a uniformizing parameter of the discrete valuation ring R_P. (Existence of e_P will be shown in Corollary 4.3.)
- $\Delta_0 = \{P \in \Delta \mid e_P > 0\}$.
- $\Delta_0(m) = \{P \in \Delta_0 \mid P \subseteq m\}$ where m is a maximal ideal of R.
- A_{Γ_a}: For each $0 \neq a \in R$, A_{Γ_a} will denote the R-subalgebra of A defined in Lemma 4.1.
- $\Sigma = \{\Gamma_a \mid 0 \neq a \in R\}$.
- $\varinjlim A_{\Gamma_a}$: Direct limit of the direct system $\{A_{\Gamma_a} \mid \Gamma_a \in \Sigma\}$ (defined in Lemma 4.5).

Lemma 4.1. *For $0 \neq a \in R$, let $S = R \setminus \bigcup_{P \in \Gamma_a} P$ and define*

$$A_{\Gamma_a} = S^{-1}A \cap R[1/a][x].$$

Then

$$A_{\Gamma_a} = R\left[I\left(\frac{x-c}{d}\right)\right] \tag{4.1}$$

for an invertible ideal I of R and $c, d \in R$. In particular, A_{Γ_a} is faithfully flat over R. Furthermore, we have:

(1) $A_{\Gamma_a} \subseteq A$.
(2) $(A_{\Gamma_a})_P = A_P$ *for* $P \in \Gamma_a$.
(3) $(A_{\Gamma_a})_P = R_P[x]$ *for* $P \notin \Gamma_a$.

Proof. Since all assertions follow trivially when a is a unit in R, we assume that a is a non-unit.

Note that $S^{-1}R$ is a semi-local Krull domain of dimension one, and hence $S^{-1}R$ is a PID. Since $S^{-1}A \subset K[x] = S^{-1}R[1/a][x]$, it follows from Theorem 3.5 that $S^{-1}A = (S^{-1}R)^{[1]}$. Hence $S^{-1}A = S^{-1}R[y]$ for some $y \in A$. Then $K[x] = K[y]$ and $x \in A \subseteq S^{-1}A = (S^{-1}R)[y]$. Thus, by Lemma 3.1, there exist an invertible ideal I in R and elements $c, d \in R$ satisfying (4.1).

The assertions (2) and (3) follow from the definition of A_{Γ_a}. For the assertion (1), note that $R = S^{-1}R \cap R[1/a]$ since R is a Krull domain. Since A is flat over R, it then follows that $A = S^{-1}A \cap A[1/a]$, so that $A_{\Gamma_a} \subseteq A$, because $R[1/a][x] \subseteq A[1/a]$. □

Remark 4.2. For the ring A_{Γ_a} defined above, by Lemma 2.8, we have

$$A_{\Gamma_a} = \bigcap_{P \in \Delta} (A_{\Gamma_a})_P = \left(\bigcap_{P \in \Gamma_a} A_P\right) \cap \left(\bigcap_{P \notin \Gamma_a} R_P[x]\right), \tag{4.2}$$

which shows that such a faithfully flat R-algebra A_{Γ_a} that satisfies the three conditions (1)–(3) in Lemma 4.1 is unique.

Corollary 4.3. *For* $P \in \Delta$, *letting* $p \in P$ *be an element such that* $PR_P = pR_P$, *we have*

$$A_P = R_P\left[\frac{x-c}{p^e}\right]$$

for some $c \in R$ *and* $e \geqslant 0$. *Furthermore, the integer* e *is uniquely determined for* P.

Proof. Since $P \in \Gamma_p$, it follows from Lemma 4.1 that

$$A_P = (A_{\Gamma_p})_P = R_P\left[I\left(\frac{x-c}{d}\right)\right] = R_P\big[\alpha(x-c)\big],$$

where I is an invertible ideal of R; $c, d \in R$; and α is an element of K satisfying $(d^{-1}I)_P = \alpha R_P$. Note that $x \in A_P$. Hence $v_P(\alpha) \leqslant 0$, where v_P is the valuation of K whose valuation ring is R_P, so that we may take $\alpha = p^{-e}$ with $e \geqslant 0$. Since R_P is a discrete valuation ring with uniformizing parameter p, the uniqueness of e is obvious. □

For $P \in \Delta$, we denote by e_P the (unique) integer e given in Corollary 4.3 above.

Corollary 4.4. *Suppose that* R *is factorial. Let* a *be an element in* R *with prime factorisation* ${p_1}^{m_1} \cdots {p_n}^{m_n}$, *let* $P_i = p_iR$, *and set* $e_i = e_{P_i}$. *Then*

$$A_{\Gamma_a} = R\left[\frac{x-c}{p_1^{e_1} \cdots p_n^{e_n}}\right]$$

for some $c \in R$.

Proof. Note that $\Gamma_a = \{P_1, \dots, P_n\}$. The proof of 4.1 shows that there exist $c, d \in R$ such that $A_{\Gamma_a} = R[I(\frac{x-c}{d})]$ where $I = R \cap dR[1/a]$ is an invertible ideal of R. Since R is factorial, I is actually a principal ideal, say, $I = uR$. The proof of Lemma 2.7 shows that $u/d = p_1^{-r_1} \cdots p_n^{-r_n}$ for some non-negative integers $r_1, \dots, r_n$. Thus

$$R\left[I\left(\frac{x-c}{d}\right)\right] = R\left[\frac{x-c}{p_1^{r_1} \cdots p_n^{r_n}}\right].$$

Then, again by Lemma 4.1, we have

$$A_{P_i} = (A_{\Gamma_a})_{P_i} = R_{P_i}\left[\frac{x-c}{p_i^{r_i}}\right],$$

so that $r_i = e_i$ for each i by Corollary 4.3. This completes the proof. □

Lemma 4.5. *Let a, b be non-zero elements of R. If $\Gamma_a \subseteq \Gamma_b$, then $A_{\Gamma_a} \subseteq A_{\Gamma_b}$.*

Proof. Since $R_P[x] \subseteq A_P$ for $P \in \Delta$, the assertion easily follows from Eq. (4.2). □

Let $\Sigma = \{\Gamma_a \mid 0 \neq a \in R\}$. Then Lemma 4.5 shows that the rings A_{Γ_a}, together with inclusion maps, form a direct system $\{A_{\Gamma_a} \mid \Gamma_a \in \Sigma\}$ indexed by Σ. We now prove the structure theorem:

Theorem 4.6. $A = \varinjlim A_{\Gamma_a}$.

Proof. Set $C = \varinjlim A_{\Gamma_a}$. Then $C \subseteq A$, because each A_{Γ_a} is a subring of A. First we show that C is faithfully flat over R. For this purpose, let $0 \to M \to N$ be an exact sequence of R-modules. Then the sequence

$$0 \to M \otimes_R A_{\Gamma_a} \to N \otimes_R A_{\Gamma_a}$$

is exact for every Γ_a because A_{Γ_a} is flat over R by Lemma 4.1. Since direct limit is an exact functor, it follows that

$$0 \to \varinjlim(M \otimes_R A_{\Gamma_a}) \to \varinjlim(N \otimes_R A_{\Gamma_a})$$

is exact. Recall that direct limit commutes with tensor product, namely,

$$\varinjlim(M \otimes_R A_{\Gamma_a}) = M \otimes_R (\varinjlim A_{\Gamma_a}) = M \otimes_R C.$$

Thus $0 \to M \otimes_R C \to N \otimes_R C$ is an exact sequence, and hence C is flat over A. Moreover, for any maximal ideal m of R, we have $mC \subseteq mA \neq A$, so that $mC \neq C$. This shows that C is faithfully flat over R.

Next we show that $C_P = A_P$ for every $P \in \Delta$. In fact, note that $(A_{\Gamma_a})_P = A_P$ or $(A_{\Gamma_a})_P = R_P[x]$ for every $\Gamma_a \in \Sigma$ and $P \in \Delta$, and $R_P[x] \subseteq A_P$ for any $P \in \Delta$. It then follows from commutativity of direct limit with tensor product that

$$C_P = R_P \otimes_R C = R_P \otimes_R (\varinjlim A_{\Gamma_a}) = \varinjlim(R_P \otimes_R A_{\Gamma_a}) = \varinjlim(A_{\Gamma_a})_P = A_P,$$

as desired. Now, from Lemma 2.8, we have

$$C = \bigcap_{P \in \Delta} C_P = \bigcap_{P \in \Delta} A_P = A,$$

which completes the proof. □

As a consequence of the theorem, extensions of all prime ideals in R remain primes in A.

Corollary 4.7. *If Q is a prime ideal of R, then QA is a prime ideal of A.*

Proof. Let $f, g \in A$ be elements such that $fg \in QA$, and write $fg = a_1h_1 + \cdots + a_nh_n$ with elements $a_1, \ldots, a_n \in Q$ and $h_1, \ldots, h_n \in A$. Then, by Theorem 4.6, there exists $\Gamma_a \in \Sigma$ such that $f, g, h_1, \ldots, h_n \in A_{\Gamma_a}$, so that $fg \in QA_{\Gamma_a}$. Note that A_{Γ_a} is the Rees algebra of an invertible ideal of R, so that QA_{Γ_a} is a prime ideal of A_{Γ_a} (Lemma 2.9). Thus we have $f \in QA_{\Gamma_a} \subseteq QA$ or $g \in QA_{\Gamma_a} \subseteq QA$. Hence QA is a prime ideal of A. □

From Theorem 4.6, we shall now deduce that finite generation of A is equivalent to the finiteness of the set

$$\Delta_0 = \{P \in \Delta \mid e_P > 0\}.$$

Note that, for $P \in \Delta$, we have $P \in \Delta_0$ if and only if $A_P \neq R_P[x]$.

Lemma 4.8. *Let Γ_a, Γ_b be elements of Σ such that $\Gamma_a \subseteq \Gamma_b$. Then $A_{\Gamma_a} \subsetneqq A_{\Gamma_b}$ if and only if there exists $P \in \Gamma_b \setminus \Gamma_a$ such that $P \in \Delta_0$.*

Proof. First suppose that there exists $P \in \Gamma_b \setminus \Gamma_a$ such that $P \in \Delta_0$. Then we have $(A_{\Gamma_a})_P = R_P[x]$ and $(A_{\Gamma_b})_P \neq R_P[x]$, so that $A_{\Gamma_a} \neq A_{\Gamma_b}$. Since $A_{\Gamma_a} \subseteq A_{\Gamma_b}$ in general, we have $A_{\Gamma_a} \subsetneqq A_{\Gamma_b}$.

Next suppose that there does not exist $P \in \Gamma_b \setminus \Gamma_a$ such that $P \in \Delta_0$. Then, for $P \in \Gamma_b \setminus \Gamma_a$, we have $(A_{\Gamma_b})_P = R_P[x]$, so that $A_{\Gamma_a} = A_{\Gamma_b}$ by (4.2). □

Corollary 4.9. *The following conditions are equivalent:*

(1) *A is finitely generated over R.*
(2) *Δ_0 is a finite set.*
(3) *$A \cong R[IX]$ for an invertible ideal I of R.*

Proof. (1) ⇒ (2). Recall that, by Theorem 4.6, we have

$$A = \varinjlim A_{\Gamma_a} = \bigcup_{\Gamma_a} A_{\Gamma_a}. \tag{4.3}$$

Let $A = R[f_1, \ldots, f_n]$. By (4.3), for each i there exists $0 \neq a_i \in R$ such that $f_i \in A_{\Gamma_{a_i}}$. Then, setting $a = a_1 \cdots a_n$, we have $f_i \in A_{\Gamma_a}$ for each i, which implies $A = A_{\Gamma_a}$. Now suppose that Δ_0 is an infinite set. Then there exists $P \in \Delta_0 \setminus \Gamma_a$, because Γ_a is a finite set. Let t be a non-zero element of P and let $b = at$. Then $\Gamma_a \subseteq \Gamma_b$ and $P \in \Gamma_b \setminus \Gamma_a$. Thus $A_{\Gamma_a} \neq A_{\Gamma_b}$ by Lemma 4.8. On the other hand, by Lemma 4.5, we have

$$A = A_{\Gamma_a} \subseteq A_{\Gamma_b} \subseteq A,$$

so that $A_{\Gamma_a} = A_{\Gamma_b}$, a contradiction.

(2) $\Rightarrow$ (3). Let $\Delta_0 = \{P_1, \ldots, P_m\}$ and let a be a non-zero element of $P_1 \cap \cdots \cap P_m$. Then $\Delta_0 \subset \Gamma_a$, and hence, by Lemma 4.8, we have $A_{\Gamma_a} = A_{\Gamma_b}$ for every $0 \neq b \in R$ such that $\Gamma_a \subseteq \Gamma_b$. It thus follows from (4.3) and Lemma 4.1 that $A = A_{\Gamma_a} \cong R[IX]$ for some invertible ideal I of R. $\square$

We now show how, for any maximal ideal m of R, the structure of the closed fibre A/mA is related to the finiteness or otherwise of the set

$$\Delta_0(m) = \{P \in \Delta_0 \mid P \subseteq m\}.$$

Corollary 4.10. *Let m be a maximal ideal of R and $k = R/m$.*

(1) *If $\Delta_0(m)$ is a finite set, then $A/mA = k^{[1]}$.*
(2) *If $\Delta_0(m)$ is an infinite set, then $A/mA = k$.*

Proof. Note that $A_m/mA_m = A/mA$ because m is maximal. Hence, replacing R and A by R_m and A_m, respectively, we may assume that R is a local ring with maximal ideal m and $\Delta_0(m) = \Delta_0$. In particular, R is factorial.

(1) This is an immediate consequence of Corollary 4.9.

(2) For $\Gamma_a \in \Sigma$, we have $A_{\Gamma_a} = R[x_a]$, where x_a is of the form

$$x_a = \frac{x - c}{p_1^{e_1} \cdots p_n^{e_n}}$$

as given in Corollary 4.4. Since Δ_0 is an infinite set, there exists $P \in \Delta_0$ such that $P \notin \Gamma_a$. Let $P = pR$ and $b = ap$. Then $\Gamma_b = \Gamma_a \cup \{P\}$ so that, again by Lemma 4.4,

$$x_b = \frac{x - c'}{p_1^{e_1} \cdots p_n^{e_n} p^e}$$

for some $c' \in R$ and $e = e_P$. Let $w = p_1^{e_1} \cdots p_n^{e_n}$. Then

$$p^e x_b = x_a - \frac{c' - c}{w},$$

which implies that $(c' - c)/w \in R$, because $x_a \in A_{\Gamma_a} \subset A_{\Gamma_b} = R[x_b]$. Let $\lambda = (c' - c)/w$. Since $p \in m$ and $e \geqslant 1$, it follows that

$$x_a = p^e x_b + \lambda \equiv \lambda \pmod{mA_{\Gamma_b}}. \tag{4.4}$$

As $A_{\Gamma_a}/mA_{\Gamma_a} = R/m \otimes_R A_{\Gamma_a}$, the direct system $\{A_{\Gamma_a} \mid \Gamma_a \in \Sigma\}$ with direct limit A induces a direct system $\{A_{\Gamma_a}/mA_{\Gamma_a} \mid \Gamma_a \in \Sigma\}$ with direct limit $R/m \otimes_R A = A/mA$. Equation (4.4) shows that under the homomorphism

$$f_{ab}: A_{\Gamma_a}/mA_{\Gamma_a} \to A_{\Gamma_b}/mA_{\Gamma_b}$$

we have $f_{ab}(A_{\Gamma_a}/mA_{\Gamma_a}) = k$. Therefore

$$A/mA = \varinjlim (A_{\Gamma_a}/mA_{\Gamma_a}) = k,$$

as claimed. This completes the proof. □

Corollary 4.11. *At each prime ideal $Q \in \operatorname{Spec} R$, either $k(Q) \otimes_R A = k(Q)^{[1]}$ or $k(Q) \otimes_R A = k(Q)$.*

Proof. Note that $k(Q) \otimes_R A = A_Q/QA_Q$. Replace the extension $R \subset A$ by $R_Q \subset A_Q$ and apply Corollary 4.10. □

The preceding results lead to Theorem B.

Theorem 4.12. *Suppose that R is a local ring with maximal ideal m and residue field k $(= R/m)$. Then the following conditions are equivalent.*

(1) *A is finitely generated over R.*
(2) $\operatorname{tr.deg}_k A/mA > 0$.
(3) $\dim A/mA > 0$.
(4) $A = R^{[1]}$.

Proof. The assertion is an immediate consequence of Corollaries 4.9 and 4.10. □

Corollary 4.13. *Under the hypotheses of this section, if all closed fibres of the extension $R \subset A$ are of positive dimension (or positive transcendence degree), then A is a locally polynomial algebra.*

We record below the initial example constructed by the authors, during the investigations of non-finitely generated codimension-one $\mathbf{A}^1$-fibrations, that eventually led to Theorem 4.6. The example could give a concrete illustration of how infinitely generated algebras described in Theorem 4.6 can arise from a given infinite set of primes.

Example 4.14. Let k be an infinite field and $R = k[[t_1, t_2]]$ where t_1, t_2 are algebraically independent over k. Let

$$\Pi = \{at_1 + bt_2 \mid a, b \in k,\ (a, b) \neq (0, 0)\}$$

and

$$\Omega = \{p_1 p_2 \cdots p_n \mid n \geqslant 1,\ p_i \in \Pi,\ p_i R \neq p_j R \text{ for } i \neq j\}.$$

Thus Π is a set of primes in R and Ω the set of elements of R which can be expressed as a product of *distinct* (non-associate) primes from Π.

Now for $i = 1, 2$, let $C_i = R[1/t_i][\{\frac{X}{p} \mid p \in \Pi\}]$ and let $C = C_1 \cap C_2$. Note that $R[\{\frac{X}{p} \mid p \in \Pi\}] \subsetneqq C$; for instance, $\frac{X}{t_1 t_2} \in C \setminus R[\{\frac{X}{p} \mid p \in \Pi\}]$.

We show that $C = R[\{\frac{X}{q} \mid q \in \Omega\}]$. It is enough to prove that $C_i = R[1/t_i][\{\frac{X}{q} \mid q \in \Omega\}]$ for $i = 1, 2$. It suffices to check that $\frac{X}{q} \in C_i$ for each $q \in \Omega$ and each i. Let $q = p_1 p_2 \cdots p_n$ where $p_1, \ldots, p_n \in \Pi$. We show that $\frac{X}{q} \in C_i$ by induction on n. Now $p_n = at_1 + bt_2$ and $p_{n-1} = ct_1 + dt_2$ for some $a, b, c, d \in k$ such that $ad - bc \neq 0$. Set $r = p_1 p_2 \cdots p_{n-2}$. Let

$$y = \frac{X}{rp_{n-1}} - \frac{X}{rp_n}, \qquad z = \frac{X}{rp_{n-1}} + \frac{X}{rp_n}.$$

By induction hypothesis, both $\frac{X}{rp_{n-1}}, \frac{X}{rp_n} \in C_i$, so that $y, z \in C_i$. Hence $(b+d)y - (b-d)z \in C_i$. Now

$$(b+d)y - (b-d)z = 2(ad - bc)t_1 \frac{X}{q}.$$

Since t_1 is a unit in C_1, it follows that $\frac{X}{q} \in C_1$. Again, $(a+c)y - (a-c)z \in C_i$. Now

$$(a+c)y - (a-c)z = 2(bc - ad)t_2 \frac{X}{q}.$$

Therefore, as t_2 is a unit in C_2, it follows that $\frac{X}{q} \in C_2$. The description $C = R[\{\frac{X}{q} \mid q \in \Omega\}]$ shows that C is a faithfully flat R-algebra satisfying $C_P = R_P{}^{[1]}$ for every height one prime ideal P of R, but C is not finitely generated over R. Here C is, in fact, a direct limit of finite subsets of the set of principal prime ideals of the form pR where $p \in \Pi$. For $P \in \Delta$, $e_P = 1$ if $P = pR$ for some prime p in Π; otherwise $e_P = 0$. Note that C is a subalgebra of the R-algebra A in Example 1.4.

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