Communications on Stochastic Analysis
Vol. 1, No. 1 (2007) 87-101

DILATION OF A CLASS OF QUANTUM DYNAMICAL SEMIGROUPS

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Abstract. Hudson-Parthasarathy (H-P) type quantum stochastic dilation of a class of C_0 semigroups of completely positive maps (quantum dynamical or Markov semigroups) on a von Neumann or C^* algebra, with unbounded generators, is constructed under some assumptions on the semigroup and its generator. The assumption of symmetry with respect to a semifinite trace allows the use of Hilbert space techniques,while that of covariance with respect to an action of a Lie group on the algebra gives a better control on the domain of the generator. A dilation of the dynamical semigroup is obtained, under some further assumptions on the domain of the generator, with the help of a conjugation by a unitary quantum stochastic process satisfying Hudson-Parthasarathy equation in Fock space.

1. Introduction

In an earlier series of papers ([13], [14]), we had constructed a theory of stochastic dilation "naturally" associated with a given quantum dynamical semigroup (q.d.s.) on a von Neumann or C^* algebra with bounded generator. There the computations involved C^* or von Neumann Hilbert modules, using the results of [3], map-valued quantum stochastic processes on modules and stochastic integration with respect to them ([15], [19]). It is then natural to consider the case of a q.d.s. with unbounded generator and ask the same questions about the associated stochastic dilations. As one would expect, the problem is too intractable in this generality and we impose some further structures on it, viz. we assume that the semigroup is symmetric with respect to a semifinite trace and covariant under the action of a Lie group on the algebra. This additional hypothesis enables us to control the domains of the various operator coefficients appearing in the quantum stochastic differential equations so that the Mohari-Sinha conditions ([17],[16]) can be applied. As precursors of this work, we may mention those in [10] and [1]. While the first one deals with a general EH flow with unbounded structure maps under some additional hypotheses, the second one treats the problem in a different spirit.

A remark about notation: we shall denote by $\mathrm{Lin}(\mathcal{V}, \mathcal{W})$ the space of linear maps from a vecor space $\mathcal{V}$ to another vector space $\mathcal{W}$, and by $\mathrm{Dom}(L)$ the domain of a possibly unbounded operator on a Banach (or more generally, locally convex) space. Tensor product of Hilbert spaces or of operators will be usually denoted

2000 *Mathematics Subject Classification.* Primary 81S25; Secondary 46L53.
Key words and phrases. completely positive semigroups, quantum stochastic dilation.

by $\otimes$, and sometimes $\otimes_{\rm alg}$ is used to denote algebraic tensor product (i.e. without any kind of topological completion). Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces and A be a (possibly unbounded) linear operator from $\mathcal{H}_1$ to $\mathcal{H}_1 \otimes \mathcal{H}_2$ with domain $\mathcal{D}$. For each $f \in \mathcal{H}_2$, we define a linear operator $\langle f, A \rangle$ with domain $\mathcal{D}$ and taking value in $\mathcal{H}_1$ such that,

$$\langle \langle f, A \rangle u, v \rangle = \langle Au, v \otimes f \rangle \tag{1.1}$$

for $u \in \mathcal{D},\ v \in \mathcal{H}_1$. We shall denote by $\langle A, f \rangle$ the adjoint of $\langle f, A \rangle$, whenever it exists. Similarly, for any $T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and $f \in \mathcal{H}_2$, one can define $T_f \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1 \otimes \mathcal{H}_2)$ by setting $T_f u = T(u \otimes f)$.

2. Preliminaries

Let $\mathcal{A}$ be a separable C^*-algebra and τ be a densely defined, semifinite, lower semicontinuous and faithful trace on $\mathcal{A}$. Let $\mathcal{A}_\tau \equiv \{x : \tau(x^*x) < \infty\}$. Let $h = L^2(\tau)$, and $\mathcal{A}$ is naturally imbedded in $\mathcal{B}(h)$. We denote by $\bar{\mathcal{A}}$ the von Neumann algebra obtained by taking the closure of $\mathcal{A}$ with respect to the ultraweak topology inherited from $\mathcal{B}(h)$. Clearly $\mathcal{A}_\tau$ is ultraweakly dense in $\bar{\mathcal{A}}$. Assume furthermore that G is a second countable Lie group with $(\chi_i, i = 1, ...N)$ a basis of its Lie algebra, $g \mapsto \alpha_g \in Aut(\mathcal{A})$ a strongly continuous representation. Suppose that $\alpha_g(\mathcal{A}_\tau) \subseteq \mathcal{A}_\tau$ and $\tau(\alpha_g(x^*y)) = \tau(x^*y)$ for $x \in \mathcal{A}_\tau, y \in \mathcal{A}$, $g \in G$, which, by a standard polarization argument, is equivalent to the assumption that $\tau(\alpha_g(x^*x)) = \tau(x^*x)$ for $x \in \mathcal{A}_\tau$. This allows one to extend α_g to a unitary linear operator (to be denoted by u_g) on h and clearly $\alpha_g(x) = u_g x u_g^*$ for $x \in \mathcal{A}$. It is indeed easy to verify this relation on vectors in $\mathcal{A}_\tau$ and then it extends to the whole of h by the fact that h is the completion of $\mathcal{A}_\tau$. For $f \in C_c^\infty(G)$ (i.e. f is smooth complex-valued function with compact support on G) and an element $x \in \mathcal{A}$, let us denote by $\alpha(f)(x)$ the norm-convergent integral $\int_G f(g)\alpha_g(x)dg$, where dg denotes the left Haar measure on G.

Lemma 2.1. *$g \mapsto u_g$ is strongly continuous with respect to the Hilbert-space topology of h.*

Proof. Let $\mathcal{A}_1 \equiv \{x \in \mathcal{A} \mid \tau(|x|) < \infty\}$. It is known that $\mathcal{A}_1$ is dense in h in the topology of h. Furthermore, for $x \in \mathcal{A}_\tau$ and $y \in \mathcal{A}_1$, $|\tau((u_g(x) - x)^*y)| \leq \|(u_g(x) - x)^*\| \tau(|y|)$, which proves that $g \mapsto \tau((\alpha_g(x) - x)^*y)$ is continuous, by the strong continuity of α with respect to the norm topology of $\mathcal{A}$. But by the density of $\mathcal{A}_1$ and $\mathcal{A}_\tau$ in h and the fact that u_g is unitary, we conclude that for fixed $\xi \in h$, $g \mapsto u_g\xi$ is continuous with respect to the weak topology of h, and hence is strongly continuous. □

The above lemma allows us to define $\alpha(f)(\xi) = \int f(g)u_g(\xi)dg \in h$ for $f \in C_c^\infty(G), \xi \in h$. Furthermore, from the expression $\alpha_g(x) = u_g x u_g^*$, it is possible to extend α_g to the whole of $\mathcal{B}(h)$ as a normal automorphism group implemented by the unitary group u_g on h and we shall denote this extended automorphism group too by the same notation. Let $\mathcal{A}_\infty \equiv \{x \in \mathcal{A} : g \mapsto \alpha_g(x)$ is infinitely differentiable with respect to the norm topology $\}$, i.e. $\mathcal{A}_\infty$ is the intersection of the domains of $\partial_{i_1}\partial_{i_2}...\partial_{i_k}; k \geq 1$, for all possible $i_1, i_2, ... \in \{1, 2, ...N\}$, where ∂_i denotes the closed $*$-derivation on $\mathcal{A}$ given by the generator of the one-parameter automorphism group $\alpha_{exp(t\chi_i)}$, where *exp* denotes the usual exponential map for the Lie group G. The following result is essentially a consequence of the results obtained in [11], [18].

Proposition 2.2. *(i) $\mathcal{A}_\infty$ is dense $*$-subalgebra of $\mathcal{A}$.*
(ii) Similarly, we denote by d_k the self-adjoint generator of the unitary group $u_{exp(t\chi_k)}$ on h such that $u_{exp(t\chi_k)} = e^{itd_k}$, and consider the subspace $h_\infty \equiv \bigcap_{i_1,i_2,\ldots} \mathrm{Dom}(d_{i_1}d_{i_2}...d_{i_k}; k = 1, 2, ...)$. Then h_∞ is dense in h.
(iii) If we equip $\mathcal{A}_\infty$ with a family of norms $\|.\|_{\infty,n};\ n = 0, 1, 2, ...$ given by:

$$\|x\|_{\infty,n} = \sum_{i_1,i_2,\ldots i_k;\ k\leq n} \|\partial_{i_1}...\partial_{i_k}(x)\|;$$

for $n \geq 1$, and $\|x\|_{\infty,0} = \|x\|$, and similarly define a family of Hilbertian norms $\|.\|_{2,n};\ n = 0, 1, 2, ...$ on h_∞ by:

$$\|\xi\|_{2,n}^2 \equiv \sum_{i_1,i_2,\ldots i_k;k\leq n} \|d_{i_1}d_{i_2}...d_{i_k}(\xi)\|^2$$

on h_∞, then $\mathcal{A}_\infty$ and h_∞ are complete with respect to the locally convex topologies induced by the respective (countable) family of norms as defined above. In other words, $\mathcal{A}_\infty$ and h_∞ are Frechet spaces in the topologies (to be called "Frechet topologies" from now on) described above.
(iv) $\alpha_g(\mathcal{A}_\infty) \subseteq \mathcal{A}_\infty$, $u_g(h_\infty) \subseteq h_\infty$ for all $g \in G$. Furthermore, $g \mapsto \alpha_g(x), g \mapsto u_g(\xi)$ are smooth (C^∞) in the respective Frechet topologies for $x \in \mathcal{A}_\infty, \xi \in h_\infty$.
(v) Let $\mathcal{A}_{\infty,\tau} = \mathcal{A}_\infty \bigcap h_\infty$. It is a $*$-closed two-sided ideal in $\mathcal{A}_\infty$ and is dense in $\mathcal{A}$, $\mathcal{A}_\infty$, h and h_∞ with respect to the relevant topologies.

Proof. The proof of (i) and (ii) will follow immediately from the references cited before the statement of this proposition. The proof of (iii) is quite standard, which uses the fact that ∂_i, d_i's are closed maps in $\mathcal{A}$ and h respectively.

Next we indicate briefly the proof of (iv) for $\mathcal{A}_\infty$ only, since it is similar for h_∞. First of all, by the definition of $\mathcal{A}_\infty$ and the fact that $G \times G \ni (g_1, g_2) \mapsto g_1g_2 \in G$ is C^∞ map, we observe that for $x \in \mathcal{A}_\infty$ the map $(g_1, g) \mapsto \alpha_{g_1}(\alpha_g(x)) = \alpha_{g_1g}(x)$ is C^∞ on $G \times G$, hence in particular for fixed g, $G \ni g_1 \mapsto \alpha_{g_1}(\alpha_g(x))$ is C^∞, i.e. $\alpha_g(x) \in \mathcal{A}_\infty$. Similarly, for fixed $x \in \mathcal{A}_\infty$ and any positive integer k, the map $F : R^k \times G \to \mathcal{A}$ given by $F(t_1, ...t_k, g) = \alpha_{exp(t_1\chi_{i_1})...exp(t_{i_k}\chi_k)g}(x)$ is C^∞. By differentiating F in its first k components at 0, we get that $\partial_{i_1}...\partial_{i_k}(\alpha_g(x))$ is C^∞ in g.

To prove (v), we need to note first that the elements of the form $\alpha(f)(\xi)$, with $f \in C_c^\infty(G)$ and $\xi \in \mathcal{A}_\tau$ are clearly in $\mathcal{A}_{\infty,\tau}$. Let us first consider the density in h and h_∞. Since the topology of h_∞ is stronger than that of h and since h_∞ is dense in h in the topology of h, it suffices to prove that the set of elements of the above form is dense in h_∞ in the Frechet topology. For this, we take $\xi \in h_\infty$, and choose a net x_ν of elements from $\mathcal{A}_\tau$ which converges in the topology of the Hilbert space h to ξ, and then it is clear that $\alpha(f)(x_\nu) \to \alpha(f)(\xi)\ \forall f \in C_c^\infty(G)$ with respect to the Frechet topology of h_∞, since $d_{i_1}...d_{i_k}\alpha(f)(x_\nu - \xi) = (-1)^k\alpha(\chi_{i_1}...\chi_{i_k}f)(x_\nu - \xi)$. Thus, it is enough to show that $\{\alpha(f)(\xi), f \in C_c^\infty(G), \xi \in h_\infty\}$ is dense in h_∞ in the Frechet topology. For this, we choose a net $f_p \in C_c^\infty(G)$ such that $\int_G f_p dg = 1 \forall p$

and the support of f_p converges to the singleton set containing the identity element of the group G, and then it is simple to see that $\alpha(f_p)(\xi) \to \xi$ in the Frechet topology. Finally, the norm-density of $\mathcal{A}_{\infty,\tau}$ in $\mathcal{A}$ and the Frechet density in $\mathcal{A}_\infty$ will follow by similar arguments. $\square$

Remark 2.3. It may be noted that for $x \in \mathcal{A}_{\infty,\tau}$, $\delta_{i_1}...\delta_{i_k}(x) = d_{i_1}...d_{i_k}(x) \in \mathcal{A} \bigcap h$. This follows from the fact that if y_p is a net in $\mathcal{A} \bigcap h$ which converges both in the norm topology of $\mathcal{A}$ as well as in the Hilbert space topology of h, then the norm-limit belongs to h and the two limits must coincide as vectors of h.

Now we shall introduce some more useful notation and terminology and prove some preparatory results. If $\mathcal{H}$ is any Hilbert space with a strongly continuous unitary representation of G given by U_g, we denote by $\mathcal{H}_\infty$ the intersection of the domains of the self-adjoint generators of different one-parameter subgroups, just as we did in case of h. We denote the corresponding family of "Sobolev-like" norms again by the same notation as in case of h and consider $\mathcal{H}_\infty$ as a Frechet space as earlier. We call such a pair $(\mathcal{H}, U_g)$ a Sobolev-Hilbert space and for two such pairs $(\mathcal{H}, U_g)$ and $(\mathcal{K}, V_g)$, we denote by $\mathcal{B}(\mathcal{H}_\infty, \mathcal{K}_\infty)$ the space of all linear maps S from $\mathcal{H}$ to $\mathcal{K}$ such that $\mathcal{H}_\infty$ is in the domain of S, $S(\mathcal{H}_\infty) \subseteq \mathcal{K}_\infty$, and S is continuous with respect to the Frechet topologies of the respective spaces. We call a linear map L from $\mathcal{H}$ to $\mathcal{K}$ to be **covariant** if $\mathcal{H}_\infty \subseteq \mathrm{Dom}(L)$ and $LU_g(\xi) = V_g L(\xi) \forall g \in G, \xi \in \mathcal{H}_\infty$.

Lemma 2.4. *If L from $\mathcal{H}$ to $\mathcal{K}$ is bounded (in the usual Hilbert space sense) and covariant in the above sense, then $L \in \mathcal{B}(\mathcal{H}_\infty, \mathcal{K}_\infty)$.*

Proof. Let $d_i^{\mathcal{H}}$ and $d_i^{\mathcal{K}}$ be respectively the self-adjoint generator of the one parameter subgroup corresponding to χ_i in $\mathcal{H}$ and $\mathcal{K}$. From the relation $LU_g = V_g L$ it follows that (since L is bounded) L maps the domain of $d_i^{\mathcal{H}}$ into the domain of $d_i^{\mathcal{K}}$ and $Ld_i^{\mathcal{H}} = d_i^{\mathcal{K}} L$. By repeated application of this argument it follows that $Ld_{i_1}^{\mathcal{H}}...d_{i_k}^{\mathcal{H}}(\xi) = d_{i_1}^{\mathcal{K}}...d_{i_k}^{\mathcal{K}} L(\xi) \forall \xi \in \mathcal{H}_\infty$, and thus $\|L\xi\|_{2,n} \leq \|L\| \|\xi\|_{2,n}$. $\square$

We shall call an element of $\mathcal{B}(\mathcal{H}_\infty, \mathcal{K}_\infty)$ a "**smooth**" map, and if such a smooth map L satisfies an estimate $\|L\xi\|_{2,n} \leq C\|\xi\|_{2,n+p}$ for all n and for some integer p and a constant C, then we say that L is a smooth map of order p with the bound $\leq C$. From the proof of the above lemma we observe that any bounded covariant map is smooth of order 0 with the bound $\leq \|L\|$. By a similar reasoning we can prove the following:

Lemma 2.5. *Suppose that L is a closed (in the Hilbert space sense), covariant map from $\mathcal{H}$ to $\mathcal{K}$ and $\mathcal{H}_\infty$ is in the domain of L. Under these assumptions, L is smooth of the order p for some p.*

Proof. For simplicity of notation, we shall use the same symbol d_i for both $d_i^{\mathcal{H}}$ and $d_i^{\mathcal{K}}$, and also we use the same symbols for the corresponding one parameter groups of unitaries acting on $\mathcal{H}$ and $\mathcal{K}$. Let L be a map as above. Since L is closed in the Hilbert space sense, and the Frechet topology in $\mathcal{H}_\infty$ is stronger than its Hilbert space topology, it follows that L is closed as a map from the Frechet space $\mathcal{H}_\infty$ to the Hilbert space $\mathcal{K}$, and being defined on the entire $\mathcal{H}_\infty$, it is continuous with respect to the above topologies. By the definition of Frechet space continuity, there exists some C and p such that $\|L(\xi)\|_{2,0} \leq C\|\xi\|_{2,p}$. Now, for any fixed k, let $u_t \equiv u_{exp(t\chi_k)}$. Since u_t maps h_∞ into itself and L is covariant, we have that $L(\frac{u_t(\xi)-\xi}{t}) = \frac{u_t(L\xi)-L\xi}{t}$. Now, since $\frac{u_t(\xi)-\xi}{it} \to d_k(\xi)$ as $t \to 0+$ in the Frechet topology, we have that $L(\frac{u_t(\xi)-\xi}{it}) = \frac{u_t(L\xi)-L\xi}{it}$ converges to $Ld_k\xi$ in the Hilbert space topology of $\mathcal{K}$, and so by the closedness of d_k $L\xi$ must belong to the domain of d_k, with $Ld_k\xi = d_kL\xi$. Repeated use of this argument proves that $L(\mathcal{H}_\infty) \subseteq \mathcal{K}_\infty$ and $L(d_{i_1}...d_{i_k}\xi) = d_{i_1}...d_{i_k}(L\xi)\forall\xi \in \mathcal{H}_\infty$. Now, a direct computation enables one to show that L is of order p with the bound $\leq C$. □

Theorem 2.6. *Let $(\mathcal{H}, U_g), (\mathcal{K}, V_g)$ be two Sobolev-Hilbert spaces as in earlier discussion, and L be a closed (not as Frechet space map but as Hilbert space map) linear map from $\mathcal{H}$ to $\mathcal{K}$. Furthermore, assume that $\mathcal{H}_\infty$ is in the domain of $|L|^2$ and is a core for $|L|^2$, and $LU_g = V_gL$ on $\mathcal{H}_\infty$. Then we have the following conclusions:*
(i) L is a smooth covariant map with some order p and bound $\leq C$ for some C;
(ii) L^* (the densely defined adjoint in the Hilbert space sense) will have $\mathcal{K}_\infty$ in its domain;
(iii) L^* is also a smooth covariant map from $\mathcal{K}_\infty$ to $\mathcal{H}_\infty$; with order p and bound $\leq C$ as in (i).

Proof. Let the polar decomposition of L be given by $L = W|L|$. We claim that both W and $|L|$ are covariant maps. First we note that $\mathcal{H}_\infty$ is also a core for L (being a core for $|L|^2$) and since U_g is a unitary operator that maps $\mathcal{H}_\infty$ into itself, clearly $\mathcal{H}_\infty$ is a core for LU_g and also for V_gL. Thus the relation $LU_g = V_gL$ on $\mathcal{H}_\infty$ implies that the operators LU_g and V_gL have the same domain and they are equal. Now, note that L being closed and V_g being bounded, we have that $(V_gL)^* = L^*V_g^* = L^*V_{g^{-1}}$. Furthermore, since U_g^{-1} maps the core $\mathcal{H}_\infty$ for L into itself, one can easily verify that $(LU_g)^* = U_g^*L^*$ Thus, we get that $U_gL^* = L^*V_g\forall g$. It then follows that $U_g|L|^2 = |L|^2U_g$ and hence by spectral theorem U_g and $|L|$ will commute. By Lemma 2.5, we get that $|L|(\mathcal{H}_\infty) \subseteq \mathcal{H}_\infty$, and $|L|$ is a smooth covariant map of some order.

Now, if P denotes the projection onto the closure of the range of $|L|$, then P clearly commutes with U_g for all g, hence in particular $U_gRan(P)^\perp \subseteq Ran(P)^\perp$. Thus $WU_gP^\perp = WP^\perp U_g = 0 = V_gWP^\perp$. On the other hand, $V_gWP = WU_gP$, because $V_gW|L| = V_gL = LU_g = W|L|U_g = WU_g|L|$. Hence we have that W is a bounded covariant map, and thus by 2.4, it follows that W^* is covariant too, and in particular $W^*(\mathcal{K}_\infty) \subseteq \mathcal{H}_\infty \subseteq \mathrm{Dom}(|L|)$, so that $\mathcal{K}_\infty \subseteq \mathrm{Dom}(L^*) = \mathrm{Dom}(|L|W^*)$. Furthermore, from the fact that W and W^* are smooth maps of order 0 with bound ≤ 1 (as $\|W\| = \|W^*\| = 1$) and $|L|$ is a smooth covariant map of some order p with bound $\leq C$ for some C, clearly both $L = W|L|$ and $L^* = |L|W^*$ are smooth covariant maps of order p and bound $\leq C$, which completes the proof. □

Lemma 2.7. *Let $(\mathcal{H}_i, U_g^i), i = 1,2$ and $(\mathcal{K}_i, V_g^i), i = 1,2$ be Sobolev Hilbert spaces and k be any Hilbert space. Then we can construct Sobolev Hilbert spaces $(\mathcal{H}_i \oplus \mathcal{K}_i, U_g^i \oplus V_g^i)$ and $(\mathcal{H}_i \otimes k, U_g^i \otimes I)$ (with the symbols carrying their usual meanings)*

and if $L \in \mathcal{B}(\mathcal{H}_{1_\infty}, \mathcal{H}_{2_\infty}), M \in \mathcal{B}(\mathcal{K}_{1_\infty}, \mathcal{K}_{2_\infty})$, then we have the following:
(i) $L \oplus M \in \mathcal{B}((\mathcal{H}_1 \oplus \mathcal{K}_1)_\infty, (\mathcal{H}_2 \oplus \mathcal{K}_2)_\infty)$, and
(ii) $(\mathcal{H}_1 \otimes k)_\infty$ is the completion of $\mathcal{H}_{1_\infty} \otimes_{alg} k$ under the respective Frechet topology and the map $L \otimes_{alg} I$ on $\mathcal{H}_{1_\infty} \otimes_{alg} k$ extends as a smooth map on the respective Frechet space (we shall denote this smooth map by $L \otimes I$ or sometimes $\tilde{L}$). Furthermore, if L is of order p with some constant C, so will be $\tilde{L}$.

Proof. (i) is straightforward. To prove (ii), we fix any orthonormal basis $\{e_l\}$ of k and let $\xi = \sum \xi_l \otimes e_l$ be a vector in the domain of the self adjoint generator of the one parameter unitary group $u_t \otimes I$, where u_t is as in the proof of Lemma 2.5 and the summation is over a countable set since $\xi_l = 0$ for all but countably many values of l. So, without loss of generality we may assume that the set of l's with ξ_l nonzero is indexed by $1, 2, \ldots$. Since $\sum (\frac{u_t(\xi_l) - \xi_l}{t}) \otimes e_l$ is Cauchy (in the Hilbert space topology) suppose that $\sum (\frac{u_t(\xi_l) - \xi_l}{t}) \otimes e_l \to \sum \eta_l \otimes e_l$. Clearly, for each l, $\eta_l = \lim_{t \to 0} (\frac{u_t(\xi_l) - \xi_l}{t})$, which implies that $\xi_l \in \mathrm{Dom}(d_k)$ and $d_k \xi_l = \eta_l$. Thus, if $\tilde{d_k}$ denotes the self adjoint generator of the one parameter unitary group $u_t \otimes I$, then we have proved that the domain of it consists of precisely the vectors $\sum \xi_l \otimes e_l$ such that each $\xi_l \in \mathrm{Dom}(d_k)$ and $\sum \|d_k(\xi_l)\|^2 < \infty$. Repeated use of this argument enables us to prove that $(\mathcal{H}_1 \otimes k)_\infty$ consists of the vectors $\xi = \sum \xi_l \otimes e_l$ with the property that $\xi_l \in \mathcal{H}_{1_\infty} \forall l$ and for any n, $\|\xi\|_{2,n}^2 \equiv \sum_l \|\xi_l\|_{2,n}^2 < \infty$. From this, it is clear that $\sum_{l=1}^m \xi_l \otimes e_l$ converges (as $m \to \infty$) to ξ in each of the $\|.\|_{2,n}$ norms, i.e. in the Frechet topology. The rest of the proof follows by observing that for any $\xi = \sum_{\text{finite}} \xi_l \otimes e_l \in \mathcal{H}_{1,\infty} \otimes_{\text{alg}} k$, $\|\tilde{L}(\xi)\|_{2,n}^2 = \sum \|L\xi_l\|_{2,n}^2$. $\square$

3. Review of q.s.d.e. with unbounded coefficients

We assume that the reader is familar with the Hudson-Parthasarathy (H-P) formalism of quantum stochastic calculus (see, for example, [19]) including quantum stochastic differential equation (q.s.d.e.) with unbounded coefficients (see [12], [9]). Let $h = L^2(\tau)$ be as before, and let k_0 be a separable Hilbert space. Recall that a Hudson-Parthasarathy (H-P) dilation of a q.d.s. T_t on $\mathcal{A} \subseteq \mathcal{B}(h)$ is given by a family of unitary operator $(U_t)_{t \geq 0}$ on $h \otimes \Gamma(L^2(\mathbb{R}_+, k_0))$ satisfying a q.s.d.e. of the form $dU_t = U_t L^\alpha_\beta d\Lambda^\beta_\alpha(t)$, with some appropriate (possibly unbounded) operators L^α_β defined on a large enough common domain, and such that $U_0 = I$, and

$$< ve(0), U_t(x \otimes I)U_t^* ue(0) >=< v, T_t(x)u > \ \ \forall t \geq 0, \ x \in \mathcal{A}.$$

For the sake of clarity of exposition, we shall use a coordinate-free formalism of quantum stochastic calculus developed in [13] (for bounded coefficients) and [12] (for unbounded coefficients). We shall recall here a few useful facts about the existence and unitarity of solution of q.s.d.e. with unbounded operator coefficients. For the proofs of these results and for a detailed discussion on q.s.d.e. with unbounded coefficients, we refer to chapter 6 of [12] and also to [17],[16].

Let $\mathcal{D}_0, \mathcal{V}_0$ be dense subspaces of h and k_0 respectively. We denote by $\mathcal{Z}_c$ the set $\{Z \in \mathcal{B}(h \otimes \hat{k_0}) \ : \ Z + Z^* + Z\hat{Q}Z^* \leq 0\}$, where $\hat{k_0} = \mathbb{C} \oplus k_0 \equiv h \oplus (h \otimes k_0)$ and $\hat{Q} = 0|_h \oplus I_{h \otimes k_0}$, as in [12] and [13]. For a quadruple (R, S, T, A) where $A \in \mathrm{Lin}(\mathcal{D}_0, h)$, $R, S \in \mathrm{Lin}(\mathcal{D}_0, h \otimes k_0)$, $T \in \mathrm{Lin}(\mathcal{D}_0 \otimes \mathcal{V}_0, h \otimes k_0)$ satisfying

$\mathcal{D}_0 \subseteq \bigcap_{\xi \in \mathcal{V}_0} \mathrm{Dom}(\langle R, \xi \rangle)$, we introduce a linear map Z, to be called 'coefficient matrix', from $\mathcal{D}_0 \otimes (\mathbb{C} \oplus \mathcal{V}_0)$ to $h \otimes \hat{k_0}$ by

$$Z = \begin{pmatrix} A & R^* \\ S & T \end{pmatrix}.$$

Note here that by assumption $(u \otimes \xi) \in \mathrm{Dom}(R^*)$ for all $u \in \mathcal{D}_0$, $\xi \in \mathcal{V}_0$. We recall from [13] and [12] that for an adapted operator valued process V_t on $h \otimes \Gamma$ $\equiv h \otimes \Gamma(L^2(\mathbb{R}_+, k_0))$, one can define the quantum stochastic integral

$$X_t := \int_0^t V_s(a_R(ds) + a_S^\dagger(ds) + \Lambda_T(ds) + Ads)$$

with respect to a quadruple (R, S, T, A), which satisfies

$$\begin{aligned} &< ve(g), X_t ue(f) >= \\ &\quad \int_0^t < ve(g), V_s\{< R, f(s) > + < g(s), S > + < g(s), T_{f(s)} > + A\} ue(f) > ds. \end{aligned}$$

We denote by $\mathcal{Z}$ the set of the above quadruples (R, S, T, A) with the associated coefficient matrix Z such that we can find a sequence $Z^{(n)} \in \mathcal{Z}_c$, $n = 1, 2, ...,$ satisfying the following for all $\xi, \eta \in \mathcal{V}_0$ and $u \in \mathcal{D}_0$:

$$\lim_{n \to \infty} \langle \hat{\xi}, Z^{(n)}_{\hat{\eta}} \rangle u = \langle \hat{\xi}, Z_{\hat{\eta}} \rangle u, \tag{3.1}$$

$$\sup_{n \geq 1} \| Z^{(n)}_{\hat{\eta}} u \| < \infty. \tag{3.2}$$

For $X \in \mathcal{B}(h \otimes \Gamma)$, $\gamma, \zeta \in \mathbb{C} \oplus \mathcal{V}_0$, we define the bilinear forms $\mathcal{L}^\gamma_\zeta(X)$ on the vector space $\mathcal{D}_0 \otimes \Gamma$ (algebraic tensor product) defined below :

$$\begin{aligned} &\langle v\psi, \mathcal{L}^\gamma_\zeta(X) u\psi' \rangle \\ &\quad = \quad \langle v\psi, X \langle \gamma, Z_\zeta \rangle u\psi' \rangle + \langle \langle \gamma, Z_\zeta \rangle v\psi, X u\psi' \rangle + \langle \hat{Q} Z_\gamma v\psi, X \hat{Q} Z_\zeta u\psi' \rangle, \end{aligned} \tag{3.3}$$

where $u, v \in \mathcal{D}_0$ and $\psi, \psi' \in \Gamma$. Note that we have used the same notation for X and its ampliation $(X \otimes I_{\hat{k_0}})$. Clearly, we have the bound

$$|\langle v\psi, \mathcal{L}^\gamma_\zeta(X) u\psi' \rangle| \leq C(u, v, \gamma, \zeta) \|X\| \|\psi\| \|\psi'\|,$$

where $C(u, v, \gamma, \zeta) := \|v\| \|\gamma\| \|Z(u\zeta)\| + \|u\| \|\gamma\| \|Z(v\zeta)\| + \|Z(v\gamma)\| \|Z(u\zeta)\|$.

We denote $\mathcal{L}^{\hat{0}}_{\hat{0}}(X)$ simply by $\mathcal{L}(X)$, where $\hat{0} = (1 \oplus 0) \in \mathbb{C} \oplus \mathcal{V}_0$. Note that

$$\langle v\psi, \mathcal{L}(X) u\psi' \rangle = \langle v\psi, XAu\psi' \rangle + \langle Av\psi, Xu\psi' \rangle + \langle Rv\psi, XRu\psi' \rangle,$$

so that formally one has $\mathcal{L}(X) = XA + A^*X + R^*XR$. For $x \in \mathcal{B}(h)$, let $\mathcal{L}_0(x)$ denote the bilinear form on $\mathcal{D}_0$ given by

$$\langle v, \mathcal{L}_0(x) u \rangle := \langle v, xAu \rangle + \langle Av, xu \rangle + \langle Rv, xRu \rangle;$$

and it is easy to see that

$$\langle v\psi, \mathcal{L}(X) u\psi' \rangle = \langle v, \mathcal{L}_0(\langle \psi, X'_\psi \rangle) u \rangle$$

for $X \in \mathcal{B}(h \otimes \Gamma)$, $\psi, \psi' \in \Gamma$. We also identify $\mathcal{V}_0$ naturally with $0 \oplus \mathcal{V}_0$, so for $\xi, \eta' \in \mathcal{V}_0$, $\mathcal{L}^{\xi}_{\eta'}(X)$ will mean $\mathcal{L}^{(0\oplus\xi)}_{(0\oplus\eta')}(X)$. For $\lambda > 0$, let us denote by β_λ the set

$$\{x \in \mathcal{B}(h) : \langle v, \mathcal{L}_0(x)u\rangle = \lambda\langle v, xu\rangle \text{ for all } u, v \in \mathcal{D}_0\}.$$

Theorem 3.1. *Let $(R, S, T, A) \in \mathcal{Z}$ with the coefficient matrix Z, and let $Z^{(n)}, n = 1, 2, ...$ be a sequence of elements of $\mathcal{Z}_c$ satisfying 3.1 and 3.2. Assume that (i)*

$$\mathcal{L}^{\gamma}_{\zeta}(I) = 0 \text{ for all } \gamma, \zeta \in \mathbb{C} \oplus \mathcal{V}_0, \tag{3.4}$$

$$\beta_\lambda = \{0\} \text{ for some } \lambda; \tag{3.5}$$

(ii) there exist dense subspaces $\tilde{\mathcal{D}_0} \subseteq h$, $\tilde{\mathcal{V}_0} \subseteq k_0$ such that $\tilde{\mathcal{D}_0} \otimes \tilde{\mathcal{V}_0}$ is contained in the domain of Z^*, and the following conditions hold:

$$\lim_{n\to\infty} \langle \hat{\xi}, {Z^{(n)}}^*_{\hat{\eta}}\rangle u = \langle \hat{\xi}, Z^*_{\hat{\eta}}\rangle u, \quad \text{for all } \xi, \eta \in \tilde{\mathcal{V}_0},\ u \in \tilde{\mathcal{D}_0}; \tag{3.6}$$

$$\sup_{n\ge 1} \|{Z^{(n)}}^*_{\hat{\eta}} u\| < \infty \text{ for all } \eta \in \tilde{\mathcal{V}_0}, u \in \tilde{\mathcal{D}_0}; \tag{3.7}$$

$$\tilde{\mathcal{L}}^{\gamma}_{\zeta}(I) = 0 \text{ for all } \gamma, \zeta \in \mathbb{C} \oplus \tilde{\mathcal{V}_0}; \tag{3.8}$$

$$\tilde{\beta}_\lambda = \{0\} \text{ for some } \lambda > 0; \tag{3.9}$$

where the definitions of $\tilde{\mathcal{L}}^{\xi}_{\eta}$ and $\tilde{\beta}_\lambda$ are similar to the definitions of $\mathcal{L}^{\xi}_{\eta}$ and β_λ, with the replacement of Z, $\mathcal{D}_0$ and $\mathcal{V}_0$ by Z^*, $\tilde{\mathcal{D}_0}$ and $\tilde{\mathcal{V}_0}$ respectively. Then the following q.s.d.e. admits a unitary operator-valued solution.

$$dV_t = V_t(a_R(dt) + a^{\dagger}_S(dt) + \Lambda_T(dt) + Adt),\ V_0 = I. \tag{3.10}$$

4. Assumptions on the semigroup and its generator

Let T_t be a q.d.s. on $\mathcal{A}$ which is τ-symmetric, that is, $\tau(T_t(x)y) = \tau(xT_t(y))$ for all positive $x, y \in \mathcal{A}$, and for all $t \ge 0$. We refer the reader to [5] for a detailed account of such semigroups from the point of view of Dirichlet forms. We shall need some of the results obtained in that reference. As it is mentioned in that reference, T_t can be canonically extended to a normal τ-symmetric q.d.s. on $\bar{\mathcal{A}}$ as well as to C_0-semigroup of positive contractions on the Hilbert space h. We shall denote all these semigroups by the same symbol T_t as long as no confusion can arise. Furthermore, we assume that T_t on $\bar{\mathcal{A}}$ is conservative, i.e. $T_t(1) = 1 \forall t \ge 0$.

Let us denote by $\mathcal{L}$ the C^* generator of T_t on $\mathcal{A}$, and by $\mathcal{L}_2$ the generator of T_t on h. Clearly, $\mathcal{L}_2$ is a negative self-adjoint map on h. We also recall ([5]) that there is a canonical Dirichlet form η on h given by, $\mathrm{Dom}(\eta) = \mathrm{Dom}((-\mathcal{L}_2)^{\frac{1}{2}})$, $\eta(a) = \|(-\mathcal{L}_2)^{\frac{1}{2}}(a)\|^2_{2,0}, a \in \mathrm{Dom}(\eta)$. We recall from [5] that $\mathcal{B} := \mathcal{A} \bigcap \mathrm{Dom}(\eta)$ is a $*$-algebra, called the Dirichlet algebra, which is norm-dense in $\mathcal{A}$.

We now make the following assumptions.

Assumptions:

(**A1**) T_t is covariant, i.e. T_t commutes with α_g for all $t \ge 0, g \in G$.
(**A2**)$\mathcal{L}$ has $\mathcal{A}_\infty$ in its domain.
(**A3**) $\mathcal{L}_2$ has h_∞ in its domain.

Remark 4.1. If the G-action on $\mathcal{A}$ is ergodic, that is, the fixed point subalgebra is trivial, it can be proven (see [12]) that the assumption (**A3**) follows automatically from the other two assumptions.

Lemma 4.2. *(i) $\mathcal{A}_{\infty,\tau}$ is a core for both $\mathcal{L}$ and $\mathcal{L}_2$,*
(ii) $\mathcal{L}(\mathcal{A}_{\infty,\tau}) \subseteq \mathcal{A}_{\infty,\tau}$,
(iii) $\mathcal{L}_2(\mathcal{A}_{\infty,\tau}) \subseteq \mathcal{A}_{\infty,\tau}$.

Proof. By the Proposition 2.2, $\mathcal{A}_{\infty,\tau}$ is dense in $\mathcal{A}$ and h in their respective topologies. The hypothesis of covariance of T_t implies that $\mathcal{A}_{\infty,\tau}$ is invariant under T_t. Furthermore, by (A2)-(A3) $\mathcal{A}_{\infty,\tau}$ is in the domains of $\mathcal{L}$ and $\mathcal{L}_2$. Thus by Theorem 1.9 of [8], one has (i). It follows as in the proof of the Proposition 2.2 that $\mathcal{L}(\mathcal{A}_\infty) \subseteq \mathcal{A}_\infty$. Similarly, h_∞ is invariant under T_t and is a core for $\mathcal{L}_2$, and $\mathcal{L}_2(h_\infty) \subseteq h_\infty$. Since $\mathcal{L}$ and $\mathcal{L}_2$ coincide on $\mathcal{A}_{\infty,\tau} = \mathcal{A} \bigcap h_\infty$, the conclusions follow. □

Modifying slightly the arguments of [5] [20], we describe the structure of $\mathcal{L}$.

Theorem 4.3. *(i) There is a canonical Hilbert space $\mathcal{K}$ equipped with an $\mathcal{A}$-$\mathcal{A}$ bimodule structure, in which the right action is denoted by $(a, \xi) \mapsto \xi a, \xi \in \mathcal{K}, a \in \mathcal{A}$ and the left action by $(a, \xi) \mapsto \pi(a)\xi, \xi \in \mathcal{K}, a \in \mathcal{A}$.*
(ii) There is a densely defined closable linear map δ_0 from $\mathcal{A}$ into $\mathcal{K}$ such that $\mathcal{A}_{\infty,\tau} \subseteq \mathcal{B} = \mathrm{Dom}(\delta_0)$ (where $\mathcal{B}$ is the Dirichlet algebra mentioned earlier), and δ_0 is a bimodule derivation, i.e. $\delta_0(ab) = \delta_0(a)b + \pi(a)\delta_0(b) \forall a, b \in \mathcal{B}$.
(iii) For $a, b \in \mathcal{A}_{\infty,\tau}$, $\|\delta_0(a)b\|_{\mathcal{K}} \le C_a\|b\|_{2,0}$, where $\|.\|_{\mathcal{K}}$ denotes the Hilbert space norm of $\mathcal{K}$, and C_a is a constant depending only on a. Thus, for any fixed $a \in \mathcal{A}_{\infty,\tau}$, the map $\mathcal{A}_{\infty,\tau} \ni b \mapsto \sqrt{2}\delta_0(a)b \in \mathcal{K}$ extends to a unique bounded linear map between the Hilbert spaces h and $\mathcal{K}$, and this bounded map will be denoted by $\delta(a)$.
(iv) For $a, b, c \in \mathcal{A}_{\infty,\tau}$, we have

$$\partial\mathcal{L}(a, b, c) \equiv \delta(a)^*\pi(b)\delta(c) = \mathcal{L}(a^*bc) - \mathcal{L}(a^*b)c - a^*\mathcal{L}(bc) + a^*\mathcal{L}(b)c.$$

(v) $\mathcal{K}$ is the closed linear span of $\{\delta(a)b : a, b \in \mathcal{A}_{\infty,\tau}\}$.
(vi) π extends to a normal $*$-homomorphism on $\bar{\mathcal{A}}$.

Proof. We refer for the proof of (i) and (ii) to [20] and [5]. Now, we note that $\mathcal{A}_{\infty,\tau}$ is contained in the "Dirichlet algebra" (c.f. [5]) and in fact is a form-core for the Dirichlet form η mentioned earlier. Using the calculations made in the proof of Lemma 3.3 of [8, page 8], we see that for $a, b \in \mathcal{A}_{\infty,\tau}$,

$$\|\delta_0(a)b\|^2_{\mathcal{K}} = \frac{1}{2}\tau(-b^*\mathcal{L}(a)^*ab - b^*a^*\mathcal{L}(a)b + b^*\mathcal{L}(a^*a)b).$$

Here, we have also used the fact that $a, a^*, a^*a \in \mathrm{Dom}(\mathcal{L})$. From the above expression (iii) immediately follows. We verify (iv) by direct and straightforward calculations, which we omit. To prove (v), we first recall from [5] that $\mathcal{K}$ can be taken to be the closed linear span of the vectors of the form $\delta_0(a)b, a, b \in \mathcal{B}$. Now, by Lemma 3.3 of [5], $\|\delta_0(a)b\|^2_{\mathcal{K}} \le \|b\|^2_{\infty,0}\eta(a, a)$. Since $\mathcal{A}_{\infty,\tau}$ is on one hand norm-dense in $\mathcal{A}$ and also form core for η on the other hand, (v) follows.

To prove (vi), it is enough to show that whenever we have a Cauchy net $a_\mu \in \mathcal{A}_{\infty,\tau}$ in the weak topology, then $\langle \xi, \pi(a_\mu)\xi \rangle$ is also Cauchy for any fixed ξ belonging to the dense subspace of $\mathcal{K}$ spanned by the vectors of the form $\delta(b)c$, with $b, c \in \mathcal{A}_{\infty,\tau}$. But it is clear that for this, it suffices to show that $a \mapsto \langle \delta(b)b', \pi(a)\delta(b)b' \rangle$ is weakly continuous. Now, by the symmetry of $\mathcal{L}$ and the trace property of τ, we have that for $a \in \mathcal{A}_{\infty,\tau}$,

$$\begin{aligned}&\langle \delta(b)b', \pi(a)\delta(b)b' \rangle \\ &= \langle b, ab\mathcal{L}(b'b'^*) \rangle - \langle b, a\mathcal{L}(bb'b'^*) \rangle - \langle \mathcal{L}(bb'b'^*), ab \rangle + \langle \mathcal{L}(bb'(bb')^*), a \rangle.\end{aligned}$$

The first three terms in the right hand side are clearly weakly continuous in a, so we have to concentrate only on the last term, which is of the form $\tau(\mathcal{L}(xx^*)a)$ for $x \in \mathcal{A}_{\infty,\tau}$. Now, we have,

$$\tau(\mathcal{L}(xx^*)a) = \tau(\mathcal{L}(x)x^*a) + \tau(x\mathcal{L}(x^*)a) + \tau(\delta(x^*)^*\delta(x^*)a),$$

and since $\mathcal{L}(\mathcal{A}_{\infty,\tau}) \subseteq \mathcal{A}_{\infty,\tau}$, the first two terms in the right hand side of the above expression are weakly continuous in a, so we are left with the term $\tau(\delta(x^*)^*\delta(x^*)a)$. Let us choose an approximate identity e_n of the C^* algebra $\mathcal{A}$ such that each e_n belongs to $\mathcal{A}_\tau$ (this is clearly possible, since $\mathcal{A}_\tau$ is a norm-dense $*$-ideal, and for $z \in \mathcal{A}_\tau$, one has that $|z| \in \mathcal{A}_\tau$). By normality of τ, $\tau(\delta(x^*)^*\delta(x^*)) = sup_n \tau(e_n\delta(x^*)^*\delta(x^*)e_n) = 2sup_n\|\delta_0(x^*)e_n\|^2_{\mathcal{K}} \leq 2sup_n\|e_n\|^2_{\infty,0}\eta(x^*, x^*) < \infty$, since $\|e_n\|_{\infty,0} \leq 1$ and $x^* \in \mathcal{A}_{\infty,\tau} \subseteq \text{Dom}(\eta)$. Thus, $\delta(x^*)^*\delta(x^*) = y^2$ for some $y \in \mathcal{A}_\tau$, hence $\tau(\delta(x^*)^*\delta(x^*)a) = \tau(yay)$, which proves the required weak continuity. □

Now we obtain the Christensen-Evans type form of the generator $\mathcal{L}$.

Theorem 4.4. *Let $R : h \to \mathcal{K}$ be defined as follows:*

$$\text{Dom}(R) = \mathcal{A}_{\infty,\tau}, \; Rx \equiv \sqrt{2}\delta_0(x).$$

Then R has a densely defined adjoint R^*, whose domain contains the linear span of the vectors $\delta(x)y, \; x, y \in \mathcal{A}_{\infty,\tau}$ and

$$R^*(\delta(x)y) = x\mathcal{L}(y) - \mathcal{L}(x)y - \mathcal{L}(xy).$$

We denote the closure of R by the same notation R. For $x, y \in \mathcal{A}_{\infty,\tau}$,

$$(R^*\pi(x)R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R)(y) = \mathcal{L}(x)y.$$

Furthermore,

$$\delta(x)y = (Rx - \pi(x)R)(y), x, y \in \mathcal{A}_{\infty,\tau},$$

$$\mathcal{L}_2 = -\frac{1}{2}R^*R.$$

Proof. For $x, y, z \in \mathcal{A}_{\infty,\tau}$, we observe by using the symmetry of $\mathcal{L}$ that

$$\begin{aligned}&\langle \delta(x)y, Rz \rangle \\ &= 2\langle \delta_0(x)y, \delta_0(z) \\ &= \tau(y^*\mathcal{L}(x^*z) - y^*\mathcal{L}(x^*)z - y^*x^*\mathcal{L}(z)) \\ &= \tau(\mathcal{L}(y^*)x^*z - (\mathcal{L}(x)y)^*z - \mathcal{L}(xy)^*z) \\ &= \langle \{x\mathcal{L}(y) - \mathcal{L}(x)y - \mathcal{L}(xy)\}, z \rangle.\end{aligned}$$

This suffices for the proof of the statements regarding R^*. It can be verified by a straightforward computation that $(R^*\pi(x)R - \frac{1}{2}R^*Rx - \frac{1}{2}xR^*R)(y) = \mathcal{L}(x)y$ holds for $x, y \in \mathcal{A}_{\infty,\tau}$. The remaining statements are also verified in a straightforward manner. $\square$

5. H-P Dilation

We shall now prove the existence of a unitary HP dilation for T_t.

Theorem 5.1. *There exist a Hilbert space k_1 and a partial isometry $\Sigma : \mathcal{K} \to h \otimes k_0$ (where $k_0 = L^2(G) \otimes k_1$) such that $\pi(x) = \Sigma^*(x \otimes I_{k_0})\Sigma$ and $\tilde{R} \equiv \Sigma R$ is covariant in the sense that $(u_g \otimes v_g)\tilde{R} = \tilde{R}u_g$ on $\mathcal{A}_{\infty,\tau}$ where $v_g = L_g \otimes I_{k_1}$, L_g denoting the left regular representation of G in $L^2(G)$.*

Proof. The proof is essentially by the ideas as those in [6], so we omit the details. First we construct a strongly continuous unitary representation V_g of G in $\mathcal{K}$ (strong continuity will follow by covariance of $\mathcal{L}$ on a dense set of vectors, and hence by unitarity for every vector) such that π is covariant under this G-action in $\mathcal{K}$. This V_g satisfies $V_g\delta(x) = \delta(\alpha_g(x))$ by the construction, which clearly implies that $V_gR = Ru_g$ on $\mathcal{A}_{\infty,\tau}$. Thus, π is a normal covariant $*$-representation of $\bar{A}$ in $\mathcal{K}$, hence extends to a normal $*$-representation, say $\bar{\pi}$ of the crossed product von Neumann algebra $\mathcal{A} \rtimes G$, which is the weak closure of the algebra generated by $(x \otimes I_{L^2(G)}), x \in \bar{A}$ and $u_g \otimes L_g, g \in G$ in $\mathcal{B}(h \otimes L^2(G))$. Thus there is $\Sigma : \mathcal{K} \to h \otimes L^2(G) \otimes k_1$ (for some k_1) such that $\Sigma^*(X \otimes I_{k_1})\Sigma = \bar{\pi}(X)$, for $X \in \bar{A} \rtimes G$. So in particular $\Sigma^*(x \otimes I_{k_0})\Sigma = \pi(x)$, and $\Sigma^*(u_g \otimes v_g)\Sigma = V_g$. The rest of the proof follows easily from the arguments similar to those in [6]. $\square$

It is clear that for $x \in \mathcal{A}_{\infty,\tau}$, $\mathcal{L}(x) = \tilde{R}^*(x \otimes 1_{k_0})\tilde{R} - \frac{1}{2}\tilde{R}^*\tilde{R}x - \frac{1}{2}x\tilde{R}^*\tilde{R}$. This enables us to write down the candidate for the unitary dilation for the q.d.s. T_t.

Before stating and proving the main theorem concerning H-P dilation, we make a crucial observation. Let us consider the form-generator given by $\mathcal{B}(h) \ni x \mapsto \langle \tilde{R}u, (x \otimes 1)\tilde{R}v\rangle - \frac{1}{2}\langle xu, \tilde{R}^*\tilde{R}v\rangle - \frac{1}{2}\langle \tilde{R}^*\tilde{R}u, xv\rangle$, $u, v \in \mathrm{Dom}(\tilde{R}^*\tilde{R})$. By the construction of Davies ([7]), there exists a unique minimal q.d.s. on $\mathcal{B}(h)$, say $\tilde{T}_t$, such that the predual semigroup of $\tilde{T}_t$, say $\tilde{T}_{t,*}$, has the generator (say $\tilde{\mathcal{L}}_*$) whose domain contains all elements of the form $y = (1 + \tilde{R}^*\tilde{R})^{-1}\rho(1 + \tilde{R}^*\tilde{R})^{-1}$ for $\rho \in \mathcal{B}_1(h)$, and $\tilde{\mathcal{L}}_*(y) = \pi_*(\tilde{R}_1\rho\tilde{R}_1^*) - \frac{1}{2}\tilde{R}_1^*\tilde{R}_1\rho - \frac{1}{2}\rho\tilde{R}_1^*\tilde{R}_1$, where $\tilde{R}_1 = \tilde{R}(1 + \tilde{R}^*\tilde{R})^{-1}$ and π_* denotes the predual of the normal $*$-representation $x \mapsto (x \otimes 1)$ of $\mathcal{B}(h)$ into $\mathcal{B}(h \otimes k_0)$ (i.e. for $T \in \mathcal{B}_1(h \otimes k_0)$, $\pi_*(T) = \sum_i T_{ii}$, $T_{ii} \in \mathcal{B}_1(h)$ being the diagonal elements of T expressed in a block-operator form with respect to an orthonormal basis of k_0, and the sum is in the trace-norm).

Lemma 5.2. *$\tilde{T}_t$ is conservative.*

Proof. Let $\tilde{\mathcal{L}}$ denote the generator of $\tilde{T}_t$. We claim that $\mathcal{A}_{\infty,\tau} \subseteq \mathrm{Dom}(\tilde{\mathcal{L}})$ and $\tilde{\mathcal{L}} = \mathcal{L}$ on $\mathcal{A}_{\infty,\tau}$. Fix any $x \in \mathcal{A}_{\infty,\tau}$. Let $\mathcal{D}_*$ be the linear span of operators of the form $(1 + \tilde{R}^*\tilde{R})^{-1}\sigma(1 + \tilde{R}^*\tilde{R})^{-1}$ for $\sigma \in \mathcal{B}_1(h)$. Clearly, for $\rho \in \mathcal{D}_*$, $tr(\tilde{\mathcal{L}}(x)\rho) = tr(x\tilde{\mathcal{L}}_*(\rho)) = tr(\mathcal{L}(x)\rho)$ (using the explicit forms of $\mathcal{L}$ and $\tilde{\mathcal{L}}$), and since $\mathcal{D}_*$ is a core for $\tilde{\mathcal{L}}_*$ (see [7]), we have $tr(x\tilde{\mathcal{L}}_*(\rho)) = tr(\mathcal{L}(x)\rho)$ for all $\rho \in \mathrm{Dom}(\tilde{\mathcal{L}}_*)$. Now, for $\rho \in \mathrm{Dom}(\tilde{\mathcal{L}}_*)$, $tr(\frac{\tilde{T}_t(x)-x}{t}\rho) = tr(x(\frac{\tilde{T_{t,*}}(\rho)-\rho}{t}) = tr(x\tilde{\mathcal{L}}_*(t^{-1}\int_0^t \tilde{T_{s,*}}(\rho)ds)) = tr(\mathcal{L}(x)t^{-1}\int_0^t \tilde{T_{s,*}}(\rho)ds))$; and we extend this equality by continuity to all $\rho \in \mathcal{B}_1(h)$. Letting $t \to 0+$, we get that $x \in \mathrm{Dom}(\tilde{\mathcal{L}})$ and $tr(\tilde{\mathcal{L}}(x)\rho) = tr(\mathcal{L}(x)\rho) \forall \rho \in \mathcal{B}_1(h)$, which implies that $\tilde{\mathcal{L}}(x) = \mathcal{L}(x)$. From this, it follows by easy arguments using the fact that the resolvents of $\mathcal{L}$ leaves $\mathcal{A}_{\infty,\tau}$ invariant that $\tilde{T}_t(x) = T_t(x)\ \forall x \in \mathcal{A}_{\infty,\tau}$, and hence by the ultraweak density of $\tilde{\mathcal{A}}_{\infty,\tau}$ in $\bar{\mathcal{A}}$, T_t and $\tilde{T}_t$ agree on $\bar{\mathcal{A}}$ (where we use the same notation for the C^* semigroup T_t and its canonical normal extension on $\bar{\mathcal{A}}$). In particular $\tilde{T}_t(1) = 1$. □

We note that since the set of smooth complex-valued functions on G with compact supports is dense in $L^2(G)$ in the L^2-norm, it is clear that $k_{0\infty}$ is dense in the Hilbert space k_0, so let us choose and fix an orthonormal basis $\{e_i\}$ of k_0 from $k_{0,\infty}$. (note that k_0 can be chosen to be separable since $\bar{\mathcal{A}}$ is σ-finite von Neumann algebra and G is second countable)

Theorem 5.3. *The q.s.d.e.*

$$dU_t = U_t(a^\dagger_{\tilde{R}}(dt) - a_{\tilde{R}}(dt) - \frac{1}{2}\tilde{R}^*\tilde{R}dt); U_0 = I \tag{5.1}$$

on the space $h \otimes \Gamma(L^2(R_+) \otimes k_0)$ admits a unitary operator-valued solution which implements a HP dilation for T_t.

Proof. Since $\tilde{R}^*\tilde{R} = -2\mathcal{L}_2$, and since $h_\infty \subseteq \mathrm{Dom}(\mathcal{L}_2) \subseteq \mathrm{Dom}(\tilde{R})$, the closed Hilbert space operator $\tilde{R}$ is also continuous as a map from h_∞ to $h \otimes k_0$ with respect to the Frechet topology and the Hilbert space topology of the domain and the range respectively. Thus the relation $\tilde{R}u_g = (u_g \otimes v_g)\tilde{R}$ on $\mathcal{A}_{\infty,\tau}$ extends by continuity to h_∞. That is, $\tilde{R}$ is covariant, and by the assumptions made on $\mathcal{L}_2$ at the beginning of this section it is easy to see that the conditions of the Theorem 2.6 are satisfied, so that there are C, p such that $\|\tilde{R}w\|_{2,0} \leq C\|w\|_{2,p}$. Moreover, by Theorem 2.6, we obtain in particular that $\mathrm{Dom}(\tilde{R}^*)$(Hilbert space domain) contains $(h \otimes k_0)_\infty$. For any vector $\xi \in k_{0\infty}$, it is clear that $h_\infty \subseteq \mathrm{Dom}(\langle \xi, \tilde{R}\rangle^*)$.

We shall now apply the Theorem 3.1 to prove the existence and unitarity of solution of the q.s.d.e. (5.1). To this end, take $\mathcal{D} = \tilde{\mathcal{D}} = h_\infty$, $\mathcal{V}_0 = \tilde{\mathcal{V}_0} = k_{0\infty}$ and $Z = \begin{pmatrix} -\frac{1}{2}\tilde{R}^*\tilde{R} & -\tilde{R}^* \\ \tilde{R} & 0 \end{pmatrix}$. Let $G_n = n(n - \mathcal{L}_2)^{-1}$, and

$$Z^{(n)} := \begin{pmatrix} -\frac{1}{2}G_n\tilde{R}^*\tilde{R}G_n & -G_n\tilde{R}^* \\ \tilde{R}G_n & 0 \end{pmatrix}.$$

We shall show that the hypotheses of Theorem 3.1 are satisfied. Clearly, ${Z^{(n)}}^*$ and $Z^{(n)}$ belong to $\mathcal{Z}_c$. Furthermore, note that G_n is clearly a bounded (with $\|G_n\| \leq 1$) covariant map, hence smooth of order 0 with bound ≤ 1. In particular, it maps $\mathcal{D}$ into itself. We have that

$$\begin{aligned}\|\tilde{R}G_n w\|^2 &= \langle \tilde{R}G_n w, \tilde{R}G_n w\rangle \\ &= \langle w, G_n^*(-2\mathcal{L}_2)G_n w\rangle \\ &= \langle w, (-2\mathcal{L}_2)^{\frac{1}{2}}G_n^* G_n(-2\mathcal{L}_2)^{\frac{1}{2}}w\rangle \text{ (as } \mathcal{L}_2, G_n \text{ commute)} \\ &= \|G_n(-2\mathcal{L}_2)^{\frac{1}{2}}w\|^2 \\ &\leq \|(-2\mathcal{L}_2)^{\frac{1}{2}}w\|^2.\end{aligned}$$

From this it follows that

$$\begin{aligned}\sup_{n\geq 1}\|Z_{\hat{\xi}}^{(n)}w\|^2 &= \sup_{n\geq 1}\{\|\tilde{R}G_n w\|^2 + \|G_n\tilde{R}^*(w\xi)\|^2\} \\ &\leq \|(-2\mathcal{L}_2)^{\frac{1}{2}}w\|^2 + \|\tilde{R}^*(w\xi)\|^2 \\ &< \infty.\end{aligned}$$

Thus the condition (3.2) is verified. To verify that $\lim_{n\to\infty}\langle\hat{\eta}, Z_{\hat{\xi}}^{(n)}\rangle w = \langle\hat{\eta}, Z_{\hat{\xi}}\rangle w$ for all $w \in \mathcal{D}$, $\xi, \eta \in \mathcal{V}_0$, we first prove the following general fact:
If L is a closed linear map from h to h with h_∞ in its domain, so that $\|Lw\|_{2,0} \leq M\|w\|_{2,r}$ for some M and r, then for $w \in h_\infty$, each of the sequences G_nLw, LG_nw and G_nLG_nw converges to Lw as $n \to \infty$. To prove this fact, it suffices to observe that G_nw clearly in h_∞ and $\|G_nw - w\|_{2,r}^2 = \sum_{i_1,i_2,\ldots i_k; k\leq r}\|(G_n - I)(d_{i_1}d_{i_2}\ldots d_{i_k}w)\|_{2,0}^2$ (as G_n is covariant), which goes to 0 as $G_n \to I$ strongly. Thus we have

$$\begin{aligned}\|G_nLG_nw - Lw\|_{2,0} &\leq \|G_nL(G_nw - w)\|_{2,0} + \|(G_n - I)Lw\|_{2,0} \\ &\leq M\|G_nw - w\|_{2,r} + \|(G_n - I)Lw\|_{2,0},\end{aligned}$$

which proves that $G_nLG_nw \to Lw$. Similarly, one can show $G_nLw \to Lw$ and $LG_nw \to Lw$.

Using this fact, it is easy to see that

$$\begin{aligned}\langle\hat{\eta}, Z_{\hat{\xi}}^{(n)}\rangle w &= -\frac{1}{2}G_n\tilde{R}^*\tilde{R}G_nw - G_n\tilde{R}^*(w\xi) + \langle\eta, \tilde{R}G_nw\rangle \\ &\to -\frac{1}{2}\tilde{R}^*\tilde{R}w - \tilde{R}^*(w\xi) + \langle\eta, \tilde{R}w\rangle \\ &= \langle\hat{\eta}, Z_{\hat{\xi}}\rangle w.\end{aligned}$$

Similar facts can be proved replacing Z by Z^* and $Z^{(n)}$ by ${Z^{(n)}}^*$. The conditions (3.4) and (3.8) are also easy to verify. Moreover, We have $\mathcal{L}_0 = \tilde{\mathcal{L}_0}$ and $\beta_\lambda = \tilde{\beta}_\lambda$ in this case. Since h_∞ is a core for $\tilde{R}^*\tilde{R}$ and $\tilde{T}_t$ is conservative, it follows (see [17], [4]) that $\beta_\lambda = \{0\}$. This proves that U_t exists and is unitary for all t. That U_t implements an H-P dilation for T_t, that is, $\langle we(0)U_t(a\otimes I)U_t^*w'e(0)\rangle = \langle w, T_t(a)w'\rangle$ for all $w, w' \in h$ and $a \in \mathcal{A}$ is clear from the q.s.d.e. (5.1) satisfied by U_t. □

We conclude this article by mentioning a few natural examples of q.d.s. which satisfy the assumptions **A1-A3**.

Example 1. Let $\mathcal{A} = C_0(\mathbb{R}^n)$, $G = \mathbb{R}^n$, with the obvious action of $\mathbb{R}^n$ on $\mathcal{A}$

by translation. The trace τ is given by integration with respect to the Lebesgue measure. We take T_t to be the heat semigroup on $\mathbb{R}^n$, which is given by

$$(T_t f)(x) = \frac{1}{(\sqrt{2\pi t})^n} \int_{\mathbb{R}^n} f(y)\exp(-\frac{\sum_i (x_i - y_i)^2}{2t})dy,\ t > 0;$$

and $T_0 f = f$. It can be verified by simple calculation that T_t is indeed covariant and symmetric. Furthermore, the norm-generator $\mathcal{L}$ of T_t is nothing but the differential operator $\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, from which it is easily seen that **A2** and **A3** are satisfied.

Example 2. This is an example from noncommutative geometry (see Chapter 9 of [12]). Consider the noncommutaive $2d$-dimensional plane considered in the Chapter 9 of [12], with the notation explained there. We claim that the q.d.s. (T_t) generated by the 'Laplacian' $-\sum_{j=1}^{2d} \delta_j^2$ is covariant with respect to the action ϕ_α of $\mathbb{R}^{2d}$, and it is also symmetric with respect to the canonical trace τ on the noncommutative $2d$-plane. To verify the covariance, we observe the following:

$$\phi_\alpha(b(f)) = b(f_\alpha),$$

where $\hat{f}_\alpha(x) = e^{i\alpha x}\hat{f}(x)$. Thus,

$$T_t(b(f_\alpha)) = \int_{\mathbb{R}^{2d}} e^{-\frac{t}{2}x^2} e^{i\alpha x} \hat{f}(x) W_x dx = \phi_\alpha(T_t(b(f))).$$

Moreover, we have,

$$\tau(T_t(b(f)^*)b(u)) = \int_{\mathbb{R}^{2d}} e^{-\frac{t}{2}x^2} \overline{\hat{f}}(x)\hat{u}(x)dx = \tau(b(f)^* T_t(u)),$$

which proves symmetry. A simple computation shows that the assumptions **A2** and **A3** hold.

References

1. Accardi, L. and Kozyrev, S.: On the structure of Markov flows; in: *Irreversibility, probability and complexity (Les Treilles/Clausthal, 1999); Chaos Solitons Fractals* **12**, no. 14-15 (2001) 2639–2655.
2. Bhat, B. V. R. and Parthasarathy, K. R.: Markov dilations of noncommutative dynamical semigroups and a quantum boundary theory; *Ann. Inst. H. Poincaré, Probab. Stat.* **31**, no. 4 (1995) 601-651.
3. Christensen, E. and Evans, D. E.: Cohomology of operator algebras and quantum dynamical semigroups; *J. London Math. Soc.*(2) **20**(1979) 358-368.
4. Chebotarev, A. M. and Fagnola, F.: Sufficient conditions for conservativity of minimal quantum dynamical semigroup; *J. Funct. Anal.* **153(2)** (1998) 382-404.
5. Cipriani, F. and Sauvageot, J-L.: Derivations as Square Roots of Dirichlet Forms; *J. Funct. Anal.* **201** :1 (2003) 78-120.
6. Chakraborty, P. S., Goswami, D. and Sinha, K. B.: A covariant quantum stochastic dilation theory; in: *Stochastics in finite and infinite dimensions* Trends Math., Birkhäuser Boston, Boston, MA (2001) 89-99.
7. Davies, E. B.: Quantum dynamical semigroups and neutron diffusion equation; *Rep. Math. Phys.* **11**(1977) 169-189.
8. Davies, E. B.: *One-parameter Semigroups.* Academic Press, London, 1980.
9. Fagnola, F.: On quantum stochastic differential equations with unbounded coefficients; *Probab. Theory Related Fields* **86**(1990) 501-516.

10. Fagnola, F. and Sinha, K. B.: Quantum flows with unbounded structure maps and finite degrees of freedom; *J. London Math. Soc.* (2)**48** (no. 3) (1993) 537–551.
11. Garding, L.: Note on continuous representations of Lie groups; *Proc. Nat. Acad. Sci. U.S.A.* **33**(1947) 331-332.
12. Sinha, K. B. and Goswami, D.: *Quantum Stochastic Processes and Noncommutative Geometry.* Cambridge Tracts in Mathematics **169**, Cambridge University Press, Cambridge, U.K., 2007 (to appear).
13. Goswami, D. and Sinha, K. B.: Hilbert modules and stochastic dilation of a quantum dynamical semigroup on a von Neumann algebra; *Comm. Math. Phys.* **205** no. 2(1999) 377-403.
14. Goswami, D., Pal, A. and Sinha, K. B.: Stochastic dilation of a quantum dynamical semigroup on a separable unital C^* algebra; *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **3**, no. 1 (2000) 177-184.
15. Hudson, R. L. and Parthasarathy, K. R.: Quantum Itô's Formula and Stochastic Evolutions; *Comm. Math. Phys.* **93**(1984) 301-323.
16. Mohari, A. : Quantum stochastic differential equations with unbounded coefficients and dilations of Feller's minimal solution; *Sankhyá*, Ser. A **53** (1991) 255-287.
17. Mohari, A. and Sinha, K. B.: Stochastic dilations of minimal quantum dynamical semigroup; *Proc. Indian Acad. Sci. Math. Sci.* , **102** (3)(1992) 159-173.
18. Nelson, E.: Analytic vectors; *Ann. Math.* (2)**70** (1959) 572-615.
19. Parthasarathy, K. R.: *An Introduction to Quantum Stochastic Calculus.* Monographs in Mathematics, Birkhäuser Verlag, Bessel, 1992.
20. Sauvageot, J-L.: Tangent bimodule and locality for dissipative operators on C^*-algebras; in: *Quantum Probab. and Applications IV, Lecture Notes in Math.* **1396**(1989) 322-338.
21. Shiga, K.: Representations of a compact group on a Banach space; *J. Math. Soc. Japan* **7** (1955) 224–248.

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