Optimal main effect plans in blocks of small size

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Abstract

In this paper we first construct an universally optimal main effect plan (MEP) for an s^s experiment on $s^2(s-1)/2$ nonorthogonal blocks of size two each, s a power of 2. Next we derive another set of sufficient conditions for an MEP on nonorthogonal blocks requiring a smaller number of blocks. These conditions are used to obtain universally optimal saturated MEPs in blocks of size 2 for (i) 4^2 experiment on 6 blocks and (ii) $5^2 \times 2^2$ on 10 blocks.

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1. Introduction

Optimal fractional factorial plans are of considerable recent interest due to their wide applicability in many diverse fields, notably in the context of industrial experimentation and quality improvement work. For a review of optimal fractional factorial plans, see Dey and Mukerjee (1999). Much of the work on optimal fractional factorial plans is available either in the absence of blocks or, under an orthogonal blocking arrangement. Interesting results on optimal main effect plans (MEP) with non-orthogonal blocking are obtained in Mukerjee et al. (2002) (henceforth referred to as MDC (2002) in this paper). With an approach differing substantially from the classical ones, MDC (2002) obtained sufficient conditions for a MEP, with possibly non-orthogonal blocking, to be universally optimal. They also suggested a construction procedure for obtaining optimal block designs making use of these sufficient conditions. Continuing with this line of research, we first take up situations where the construction procedure of MDC (2002) is inapplicable and provide a new method of construction for optimal MEPs with non-orthogonal blocking. We further show that the size of the design can be considerably reduced if one is content with a design optimal under a weaker optimality criterion (say, E-optimality) rather than universal optimality. This is done in Section 2 of this paper.

In Section 3, we give a new set of sufficient conditions for optimal MEP on non-orthogonal blocking. These conditions do not require any orthogonal array and hence can be realized by a design of size smaller than those presented earlier. [See Examples 3.1 and 3.2.]

2. Optimal MEPs with blocks

Consider a factorial experiment involving m factors $F_1, \ldots, F_m$ with the ith factor having $s_i \geqslant 2$ levels, $1 \leqslant i \leqslant m$. Let $S_i = \{0, 1, \ldots, s_i - 1\}$ denote the set of levels of the ith factor, $1 \leqslant i \leqslant m$. A typical treatment combination is denoted by $x = (x_1, \ldots, x_m), x_i \in S_i, 1 \leqslant i \leqslant m$. For a symmetric experiment, $s_i = s$ and $S_i = S, 1 \leqslant i \leqslant m$. Let $v = \prod_{i=1}^{m} s_i$ and let τ_i be the $s_i \times 1$ vector of fixed effects of the levels of F_i. We assume an additive, fixed effects, main effects model with homoscedastic and uncorrelated errors. Finally, let

$$(\tau)^{\mathrm{T}} = ((\tau_1)^{\mathrm{T}}, (\tau_2)^{\mathrm{T}}, \ldots, (\tau_m)^{\mathrm{T}}). \tag{2.1}$$

For completeness, we recall the definition of an orthogonal array.

Definition 2.1. An orthogonal array $\mathrm{OA}(n, m, s_1 \times \cdots \times s_m, t)$, having n rows, $m(\geqslant 2)$ columns, $s_1, \ldots, s_m(\geqslant 2)$ symbols and strength $t(\leqslant m)$, is an $n \times m$ array, with elements in the ith column from a set of s_i distinct symbols $(1 \leqslant i \leqslant m)$, in which all t-tuples of symbols appear equally often as rows in every $n \times t$ subarray. If $s_i = s, 1 \leqslant i \leqslant m$, the OA is denoted by $\mathrm{OA}(n, m, s, t)$.

Let $\mathscr{D}(b, k, v)$ denote the class of all connected block designs with v treatments in $b \geqslant 2$ blocks, each of size $k \geqslant 2$ and let $\mathscr{D}(b, k, s_1 \times \cdots \times s_m)$ denote the class of all fractional factorial plans for an $s_1 \times \cdots \times s_m$ factorial arranged in b blocks of size k each. The $n \times 1$ unit vector will be denoted by 1_n and let $J_{m \times n} = 1_m 1_n^{\mathrm{T}}$. For a plan $d \in \mathscr{D}(b, k, s_1 \times \cdots \times s_m)$, let Y denote the response vector. Then, our model is

$$E(Y) = \mu 1_n + X\tau + Z\beta, \tag{2.2}$$

where

$$X = [X_1 \mid X_2 \mid \ldots \mid X_m], \tag{2.3}$$

μ is the general effect, τ as in (2.1) and β is the vector of block effects. Further, for $1 \leqslant l \leqslant m$, the (i, j)th entry of X_l is 1 if the ith observation contains the jth level of F_l and 0 otherwise. Similarly, the (i, j)th entry of Z is 1 if the ith observation is in the jth block.

For a plan d, let N_{id} denote the $s_i \times b$ incidence matrix of the levels of F_i versus the blocks, $1 \leqslant i \leqslant m$ and M_{ijd} be the $s_i \times s_j$ incidence matrix of the levels of F_i versus $F_j, 1 \leqslant i \neq j \leqslant m$. Also, for $1 \leqslant i \leqslant m$, let R_{id} be the diagonal matrix of order s_i with diagonal entries as the replication numbers of the levels of F_i in d. Then, we have

$$N_{id} = (X_i)^{\mathrm{T}} Z, \quad M_{ijd} = (X_i)^{\mathrm{T}} X_j, \quad R_{id} = (X_i)^{\mathrm{T}} X_i. \tag{2.4}$$

Let

$$C_{ijd} = M_{ijd} - k^{-1} N_{id} (N_{jd})^{\mathrm{T}}, \tag{2.5}$$

where M_{ijd} is to be replaced by R_{id}, when $j = i$.

For $1 \leqslant i \leqslant m$, let ϕ_i be a non-increasing optimality criterion, where ϕ_i need not be the same criterion for all i. For details on optimality, we refer to Shah and Sinha (1989). The following result can be viewed as a generalization of Theorem 1 of MDC (2002).

Theorem 2.1. *Suppose there exists a plan $d^* \in \mathscr{D}(b, k, s_1 \times \cdots \times s_m)$ satisfying the following conditions*: (ai) *the bk treatment combinations in d^* written as rows form an orthogonal array of strength two*; (aii) *for every $i \neq j, 1 \leqslant i, j \leqslant m$, $N_{id^*}(N_{jd^*})^{\mathrm{T}}$ has all elements equal*; (b) *for $1 \leqslant i \leqslant m$, N_{id^*} is the incidence matrix of a block design that is ϕ_i-optimal over $\mathscr{D}(b, k, s_i)$. Then d^* is ϕ_i-optimal for inference on τ_i, $1 \leqslant i \leqslant m$.*

Proof. For a design $d \in \mathscr{D}(b, k, s_1 \times \cdots \times s_m)$, let the coefficient matrix of the reduced normal equations for τ be given by C_d. Then, $C_d = ((C_{ijd})), 1 \leqslant i, j \leqslant m$, where the C_{ijd}'s are as in (2.5) and τ is as in (2.1).

Now, condition (ai) ensures that for every $i \neq j, 1 \leqslant i, j \leqslant m$, M_{ijd^*} has all entries equal. Thus, simple counting shows that conditions (ai) and (aii) together implies that $C_{ijd^*} = 0, i \neq j, 1 \leqslant i, j \leqslant m$. This, together with condition (b) imply the required optimality property of d^*. □

Remark 2.1. Note that unlike Theorem 1 of MDC (2002), the design d^* of Theorem 2.1 above can be optimal according to different optimality criteria for different factors. This is illustrated in Example 2.1.

Example 2.1. Consider the following design d_1^* for a $4^2 \times 2^2$ experiment, laid out in eight blocks of size 2 each:

Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7	Block 8
0000	2211	0311	2100	0101	1010	0210	1301
1111	3300	1200	3011	2310	3201	2001	3110

It is easy to verify that both $N_{3d_1^*}, N_{4d_1^*}$ are the incidence matrices of a randomized block design, while $N_{1d_1^*}$ and $N_{2d_1^*}$ are each incidence matrices of a group divisible design with two groups consisting of the pairs of treatments (0,3) and (1,2). In usual notations, this group divisible design has $\lambda_2 = \lambda_1 + 2$ and by a result in Jacroux (1983), condition (b) of Theorem 2.1 is satisfied with E-optimality as the optimality criterion for $i = 1,2$. Hence the design is universally optimal for $\tau_i, i = 3,4$ and E-optimal for $\tau_j, j = 1,2$. Of course, since universal optimality implies E-optimality, the design is E-optimal for *all* the factors.

A method of construction for obtaining universally optimal block designs for MEPs was suggested by MDC (2002). This method is based on generalized Youden designs with k rows, s_i columns and s_i symbols for each $i, 1 \leqslant i \leqslant n$. However, for a given k, such a generalized Youden design may not exist for all s_i, $1 \leqslant i \leqslant n$. For instance, such a design does not exist with $k = 2, s_i = 4$. It follows then that the method proposed in MDC (2002) cannot be applied to construct MEPs with blocks when one or more factors are at four levels and the block size $k = 2$. To fill this gap, we propose an alternative method of construction of block designs for MEPs with factors at $s = 2^p$ levels each and the block size is 2.

Let $\mathscr{F}_s$ be a finite field of order $s = 2^p, p > 1$ and $\mathscr{F}_s^* = \mathscr{F}_s \backslash \{0\}$, where 0 is the additive identity of $\mathscr{F}_s$. For each $\alpha \in \mathscr{F}_s^*$, let H_α and $\bar{H}_\alpha$ constitute a partition of $\mathscr{F}_s$ into two equal parts, such that for every $\theta \in H_\alpha$, $\theta + \alpha \in \bar{H}_\alpha$. An additive subgroup of $\mathscr{F}_s$, which does not contain α, gives one such H_α. Clearly, given $\alpha \in \mathscr{F}_s^*$, there may be more than one such partitions. For every $\alpha \in \mathscr{F}_s^*$ we choose one of these partitions and keep this fixed throughout the construction described below.

Let A be an $s \times s^2$ matrix with rows indexed by the elements of $\mathscr{F}_s$, columns by the elements of $\mathscr{F}_s \times \mathscr{F}_s$ and the entry in the zth row and (x, y)th column being the element $x.z + y$ of $\mathscr{F}_s$, $x, y, z \in \mathscr{F}_s$. It is then easy to verify that A^{T} is an OA$(s^2, s, s, 2)$. Next, for an $\alpha \in \mathscr{F}_s^*$ and a fixed H_α, consider $x \in \mathscr{F}_s$ and $y \in H_\alpha$ and let u be the (x, y)th column of A. Define $w = u + \alpha 1_s$. Form a block having the treatment combinations represented by u and w. Repeat this process for each $x \in \mathscr{F}_s$ and $y \in H_\alpha$ and call the resultant block design $d(\alpha)$. Now define the design d^* as $d^* = \bigcup_{\alpha \in \mathscr{F}_s^*} d(\alpha)$.

Before proving the optimality properties of d^*, we state a lemma without proof.

Lemma 2.1. *Consider a design $d \in \mathscr{D}(b, 2, s_1 \times \cdots \times s_m)$ and a pair of distinct factors, say F_i and F_j. Corresponding to any typical block of d, consisting, say, of the treatment combinations $u = (u_1, u_2, \ldots, u_m)$ and $w = (w_1, w_2, \ldots, w_m)$, consider the two ordered pairs (u_i, w_j) and (w_i, u_j). Let B_{ij} be the collection of the $2b$ such ordered pairs arising from the b blocks of d.*

The design d satisfies condition (aii) *of Theorem* 2.1 *if it satisfies condition* (ai) *of Theorem* 2.1 *and also the following condition*:

(aiii) *For every $i \neq j, 1 \leqslant i, j \leqslant m$, every ordered pair $(\alpha, \beta), \alpha, \beta \in S$ appears equally often in B_{ij}.*

Theorem 2.2. *The design d^* constructed above is universally optimal in $\mathscr{D}(s^2(s-1)/2, 2, s^s)$ for inference on every $\tau_i, i \in \mathscr{F}_s$.*

Proof. From the method of construction of d^*, it is clear that $S_i = \mathscr{F}_s, i \in \mathscr{F}_s$, so that $v = s^s$; further, $b = s^2(s-1)/2$ and $k = 2$.

Now, with universal optimality as the optimality criterion, to verify condition (b) of Theorem 2.1 one has to verify that N_{kl^*} is the incidence matrix of a balanced block design in $\mathscr{D}(b,k,s_i), i \in \mathscr{F}_s$.

But, from the description of $d(\alpha)$ it follows that for $u,i,j \in \mathscr{F}_s$, the (i,j)th element of $N_{ud(\alpha)}(N_{ud(\alpha)})^{\mathrm{T}}$ is equal to s when either $j=i$ or, $j=i+\alpha$ and, is zero otherwise. This fact implies the required property of $N_{kl^*}, i \in \mathscr{F}_s$.

So, it remains to verify conditions (ai) and (aii) of Theorem 2.1. But for that it is enough to verify conditions (ai) and (aiii), in view of Lemma 2.1. Towards that, we observe that the pair of treatment combinations u, w of any block of $d(\alpha)$ are really the (x,y)th and the $(x,\bar{y})$th columns of A, with $\bar{y}=y+\alpha$, for some $x \in \mathscr{F}_s$ and $y \in H_\alpha$. Condition (ai) now follows by recalling that x takes all values in $\mathscr{F}_s$ and y, all values in H_α. Next, fix $i \neq j, i,j \in F_s$ and consider a block containing the treatment combinations u and w. Then, for some $x \in \mathscr{F}_s$ and $y \in H_\alpha$, $u_t = tx+y, w_t = tx+y+\alpha, t=i,j$. Thus, $u_i - w_j = (i-j)x - \alpha = (i-j)x + \alpha = w_i - u_j = z$ say. Since i,j,α are fixed, for a fixed x, z is fixed. Hence, $(u_i, w_j) = (p, p-z)$ with $p = ix+y$ and $(w_i, u_j) = (p+\alpha, p+\alpha-z)$. It is now clear that as y varies over H_α and x over $\mathscr{F}_S$, the ordered pairs vary over $\mathscr{F}_s \times \mathscr{F}_s$ and condition (aiii) of Lemma 2.1 holds. This completes the proof of the theorem. □

Example 2.2. Putting $s=4$ in the construction procedure above, we get the following MEP d_2^* for a 4^4 experiment, laid out in 24 blocks of size 2 each. (The set of elements of the finite field of order 4 is denoted by $\{0,1,a,b\}$.) By Theorem 2.2, d_2^* is universally optimal for all factors.

Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7	Block 8
0000	*aaaa*	01*ab*	*ab*01	0*ab*1	*a*01*b*	0*b*1*a*	*a*1*b*0
1111	*bbbb*	10*ba*	*ba*10	1*ba*0	*b*10*a*	1*a*0*b*	*b*0*a*1
Block 9	Block 10	Block 11	Block 12	Block 13	Block 14	Block 15	Block 16
0000	1111	01*ab*	10*ba*	0*ab*1	1*ba*0	0*b*1*a*	1*a*0*b*
aaaa	*bbbb*	*ab*01	*ba*10	*a*01*b*	*b*10*a*	*a*1*b*0	*b*0*a*1
Block 17	Block 18	Block 19	Block 20	Block 21	Block 22	Block 23	Block 24
0000	1111	01*ab*	10*ba*	0*ab*1	1*ba*0	0*b*1*a*	1*a*0*b*
bbbb	*aaaa*	*ba*10	*ab*01	*b*10*a*	*a*01*b*	*b*0*a*1	*a*1*b*0

Note that d_2^* allows one 3-level factor to be added. This is illustrated in Example 2.3.

Example 2.3. The following design d_3^* for a $4^4 \times 3$ experiment can be obtained from d_2^* by adding a factor with levels $\{0,1,2\}$. It can be verified that d_3^* satisfies conditions (ai), (aiii) and (*b*) and hence is universally optimal for all five factors.

Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7	Block 8
00000	*aaaa*0	01*ab*0	*ab*010	0*ab*10	*a*01*b*0	0*b*1*a*0	*a*1*b*00
11111	*bbbb*2	10*ba*1	*ba*102	1*ba*01	*b*10*a*2	1*a*0*b*1	*b*0*a*12
Block 9	Block 10	Block 11	Block 12	Block 13	Block 14	Block 15	Block 16
00001	11110	01*ab*1	10*ba*0	0*ab*11	1*ba*00	0*b*1*a*1	1*a*0*b*0
*aaaa*2	*bbbb*1	*ab*012	*ba*101	*a*01*b*2	*b*10*a*1	*a*1*b*02	*b*0*a*11
Block 17	Block 18	Block 19	Block 20	Block 21	Block 22	Block 23	Block 24
00002	11112	01*ab*2	10*ba*2	0*ab*12	1*ba*02	0*b*1*a*2	1*a*0*b*2
*bbbb*0	*aaaa*1	*ba*100	*ab*011	*b*10*a*0	*a*01*b*1	*b*0*a*10	*a*1*b*01

3. Optimal MEPs on fewer blocks

In this section we look for the minimum number (b) of blocks of a given size (k) necessary for the existence of an optimal MEP in $\mathscr{D}(b,k,s^m)$.

The necessary conditions for the existence of an orthogonal array and a BIBD(b,k,v) can be used to derive the following Lemma.

Lemma 3.1. *An universally optimal MEP for an s^m experiment in b blocks of size k each can satisfy the conditions of MDC (2002), only if* (i) *bk is divisible by s^2 and* (ii) *$bk(k-1)$ is divisible by $s(s-1)$.*

In particular, if $s=4, k=2$, then b has to be divisible by 8 as well as 6 and hence b must be a multiple of 24. Thus, Example 2.2 uses the minimum number of blocks and further reduction in size is not possible. In Example 2.1, $bk=4^2$ is not a multiple of 6 and so universal optimality for the 4-level factors could not be achieved.

We give the following example of an universally optimal MEP ($\tilde{d}_1$) for a 4^2 experiment in 6 blocks of size 2 each, to illustrate that there exists optimal MEPs with fewer number of blocks than what is required by Lemma 3.1.

Example 3.1. Consider the following design $\tilde{d}_1$

Block 1	Block 2	Block 3	Block 4	Block 5	Block 6
01	10	02	20	03	30
23	32	31	13	12	21

Clearly, $C_{12\tilde{d}_1}=0$ and $N_{1\tilde{d}_1}$ and $N_{2\tilde{d}_1}$ are incidence matrices of BIBD(6,2,4). Thus, $\tilde{d}_1$ is universally optimal for inference on both τ_1 and τ_2. This example inspires us to find sufficient conditions weaker than those of MDC (2002), which we present now.

Recall the notation in Lemma 2.1. For fixed $i \neq j, 1 \leqslant i,j \leqslant m$, let U_{ij} denote the collection of all $2b$ ordered pairs (u_i,u_j), where $u=(u_1,\ldots,u_m)$ is a treatment combination appearing in the design.

Theorem 3.1. *Suppose a design $\tilde{d}$ in $\mathscr{D}(b,2,s^m)$ satisfies the following condition*:

(a0) *For every $i \neq j, 1 \leqslant i,j \leqslant m$, the multisets B_{ij} and U_{ij} are the same.* [*This means that if an ordered pair $(\alpha,\beta), \alpha,\beta \in S$ appears t times in B_{ij} then it also appears t times in U_{ij}.*]

Then, $\tilde{d}$ is an orthogonal MEP, which is possibly non-orthogonal to the blocks.

Proof. Fix $j \neq i, 1 \leqslant j \leqslant m$. it is easy to verify that condition (a0) implies that $C_{ij\tilde{d}}=0$. Hence the result. □

Corollary 3.1. *If a design $\tilde{d} \in \mathscr{D}(b,k,s^n)$ satisfies conditions* (a0) *of Theorem* 3.1 *and* (b) *of Theorem* 2.1, *then d^* is ϕ_i-optimal for inference on $\tau_i, 1 \leqslant i \leqslant m$.*

Example 3.2. The design $\tilde{d}_2$ below is an MEP for a $5^2 \times 2^2$ experiment on 10 blocks of size 2 each. It is easy to verify that $\tilde{d}_2$ satisfies the condition (a0) of Theorem 3.1. Further, $N_{1\tilde{d}_2}$ and $N_{2\tilde{d}_2}$ are incidence matrices of BIBD(10, 2, 5), while $N_{3\tilde{d}_2}$ and $N_{4\tilde{d}_2}$ are incidence matrices of the randomized block design (10, 2, 2). Hence, by Corollary 3.1, $\tilde{d}_2$ is universally optimal for inference on $\tau_i, 1 \leqslant i \leqslant 4$.

Block	Block 2	Block 3	Block 4	Block 5
0100	1200	2300	3400	4000
3211	4311	0411	1011	2111
Block 6	Block 7	Block 8	Block 9	Block 10
0301	1401	2001	3101	4201
4110	0210	1310	2410	3010

Remark 3.1. As Example 3.1 shows, universally optimal MEP do exist with bk smaller than s^2. This is because condition (a0) requires only equality between the multisets B_{ij} and U_{ij}; they need not contain all possible ordered pairs (from $S \times S$). Example 3.2 shows that even in the case of asymmetric factorials, universally optimal MEP exists with small bk. However, construction of designs with B_{ij} as an arbitrary multiset ensuring $B_{ij} = Uij$ seems to be rather difficult when there are three or more factors.

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