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A note on the algebraic reflexivity of the isometry group of $\mathcal{K}(C(K))$

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A NOTE ON THE ALGEBRAIC REFLEXIVITY OF THE ISOMETRY GROUP OF $\mathcal{K}(C(K))$

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Abstract. This short note deals with question of algebraic reflexivity of the group of isometries of the space of compact operators $\mathcal{K}(C(K))$ for a compact set K. We show that when K is a countable metric space the group of isometries is algebraically reflexive.

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1. Introduction

Let X be a complex Banach space and let $\mathcal{G}(X)$ be its group of isometries. A linear map $\Phi : X \to X$ is said to be a local surjective isometry (l.s.i for short) if for every $x \in X$ there exists a $\Phi_x \in \mathcal{G}(X)$ with $\Phi(x) = \Phi_x(x)$. $\mathcal{G}(X)$ is said to be algebraically reflexive if every l.s.i map Φ is onto, i.e., $\Phi \in \mathcal{G}(X)$. It was proved in [5] that for any compact metric space K for the space $C(K)$ and for the space $\mathcal{K}(\ell^2)$ the group of isometries is algebraically reflexive. These results were extended to the case of vector-valued continuous functions in [4]. The group of isometries of several classical Banach spaces were shown to be algebraically reflexive in [2]. We complement this circle of ideas by studying this question for the group of isometries of $\mathcal{K}(C(K))$. We show that when K is a countable metric space $\mathcal{G}(C(K))$ is algebraically reflexive. Our key idea is to replace the use of the classical Russo-Dye theorem in the proof of the scalar-valued case (Theorem 2.2 of [5]) with a vector-valued version from [6] (We are grateful to Professor Mena-Jurado for pointing out this reference).

The algebraic reflexivity of $\mathcal{G}(\mathcal{L}(C(K)))$ or more generally that of $\mathcal{G}(\mathcal{L}(X, C(K)))$ is an open problem. In [7] we have initiated the study of properties of local surjective isometries on $\mathcal{L}(X, C(K))$. Using the proof technique for showing the algebraic reflexivity of $\mathcal{G}(\mathcal{K}(C(K)))$, we show that a l. s. i map on $\mathcal{L}(c_0, C(K))$ is a $C(K)$-module map when K is an infinite countable set.

For a Banach space X we denote by X_1 its closed unit ball and by $\partial_e X_1$ the set of extreme points. Let $S(X)$ denote the unit sphere.

2. Main Result

We use the well-known identification of the space $\mathcal{K}(X, C(K))$ with the Banach space $C(K, X^*)$ of X^*-valued, norm continuous functions on K equipped with the supremum norm, via the mapping $T \to T^*|K$ where K is canonically embedded in $C(K)^*$. Thus the group

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of isometries of this space can be described by the well-known vector-valued Banach-Stone theorems, [1]. We recall below one such result (Theorem 8.11) that we will be using in sequel.

Theorem 1. *Let X be a Banach space such that its centralizer $Z(X)$ is trivial. For any $\Phi \in \mathcal{G}(C(K,X))$ there exists a homeomorphism ϕ of K and a map $\tau : K \to \mathcal{G}(X)$ that is continuous when $\mathcal{G}(X)$ is equipped with the strong operator topology, such that for any $f \in C(K,X)$, $k \in K$, $\Phi(f)(k) = \tau(k)(f(\phi(k)))$.*

From now on we assume that K is an infinite metric space. Since $\mathcal{G}(C(K))$ is algebraically reflexive, it is reasonable (though we do not know if it is necessary) to assume, in order to consider the algebraic reflexivity of $\mathcal{G}(\mathcal{K}(C(K))$ that $\mathcal{G}(C(K)^*)$ is algebraically reflexive. If K is uncountable then as K has a perfect subset, $C(K)^*$ contains the infinite dimensional Lebesgue L^1-space. It follows from Theorem 3 of [2] that $\mathcal{G}(C(K)^*)$ is not algebraically reflexive. On the other hand when K is countably infinite, it follows from Theorem 2 of [2] that $\mathcal{G}(C(K)^*)$ is algebraically reflexive.

In what follows we use the identification of $C(K,X)^*$ as the space of X^*-valued Borel measures of finite variation and the identification $\partial_e C(K,X)_1^* = \{\delta(k) \otimes x^* : k \in K, x^* \in \partial_e X_1^*\}$. We note that $(\delta(k) \otimes x^*)(f) = x^*(f(k))$ for $f \in C(K,X)$.

Theorem 2. *Let K be a countable compact space. $\mathcal{G}(\mathcal{K}(C(K)))$ is algebraically reflexive.*

Proof. We use the identification of $\mathcal{K}(C(K))$ with $C(K, C(K)^*)$. Since for any Borel set $B \subset K$, $P : C(K)^* \to C(K)^*$ defined by $P(\mu) = \mu|B$ is a projection with the property, $\|\mu\| = \|P(\mu)\| + \|\mu - P(\mu)\|$, we get from (vi) of Proposition 5.1 in [1] that $Z(C(K)^*)$ is trivial. Thus $\mathcal{G}(C(K, C(K)^*))$ is described by the above Theorem 1.

It is well-known from the structure of extreme points of the unit ball of the space of continuous functions and its dual that for $\delta(k) \in \partial_e C(K)_1^*$, $f \in \partial_e C(K)_1^{**}$, $|f(\delta(k))| = 1$. Thus $C(K)^*$ satisfies the condition a) of Theorem 8 of [4]. Now let Φ be a l. s. i of $C(K, C(K)^*)$. We next show that $\Phi^*(\partial_e C(K, C(K)_1^*) \subset \partial_e C(K, C(K)^*)_1$. It then follows from Theorem 8 of [4] that Φ is onto and thus $\mathcal{K}(C(K))$ is algebraically reflexive.

Let $k \in K$ be an isolated point and let $\gamma \in \partial_e C(K)_1^{**}$. We shall show that for any $f \in \partial_e C\left(K, C\left(K\right)^*\right)_1$, $\mid \Phi^*((\delta\left(k\right) \otimes \gamma))\left(f\right) \mid = 1$. Suppose $\Phi\left(f\right) = \iota\left(f \circ \psi\right)$ in the canonical form given by Theorem 1. Since k is an isolated point and ψ is a homeomorphism, $\psi\left(k\right)$ is an isolated point. It is well-known and easy to see that f takes extremal values at isolated points and thus $f\left(\psi\left(k\right)\right) \in \partial_e C\left(K\right)_1^*$. So $\iota(k)\left(f\left(\psi\left(k\right)\right)\right) \in \partial_e C\left(K\right)_1^*$ as $\iota(k)$ is an onto isometry. Now by the nature of extreme points of the unit ball of $C\left(K\right)$ and its dual that we described above, we have $\mid \gamma\left(\Phi\left(f\right)\left(k\right)\right) \mid = 1$. As the set of isolated points is dense in K, it is easy to see that for any $k \in K$, $\gamma \in \partial_e C(K)_1^{**}$and $f \in \partial_e C\left(K, C\left(K\right)^*\right)_1$, $\mid \Phi^*((\delta\left(k\right) \otimes \gamma))\left(f\right) \mid = 1$.

Now if $\Phi^*\left(\delta\left(k\right) \otimes \gamma\right) = \frac{F_1 + F_2}{2}$ for $F_1, F_2 \in C\left(K, C\left(K\right)^*\right)_1$, then for $f \in \partial_e C\left(K, C\left(K\right)^*\right)_1$, $\Phi^*\left(\delta\left(k\right) \otimes \gamma\right)\left(f\right) = \frac{F_1(f) + F_2(f)}{2}$ so that $\Phi^*\left(\delta\left(k\right) \otimes \gamma\right)\left(f\right) = F_1\left(f\right) = F_2\left(f\right)$. Since K is a countable metric space by Theorem 4.6 of [6] we have that $C\left(K, C\left(K\right)^*\right)_1$ is the norm closed convex hull of its extreme points. Thus $\Phi^*\left(\delta\left(k\right) \otimes \gamma\right) = F_1 = F_2$. Hence Φ^* preserves the extreme points of the dual unit ball. Thus Φ is a surjection by Theorem 8 of [4].

□

We do not know if $\mathcal{G}(\mathcal{L}(C(K)))$ is algebraically reflexive when K is countable. Main difficulty is the non availability of a description of $\mathcal{G}(\mathcal{L}(C(K)))$. We recall that for a Banach space X, $\mathcal{L}(X, C(K))$ can be identified with $W^*C(K, X^*)$, the space of X^*-valued functions that are continuous w.r.t the weak*-topology.

It was proved in [3] that when K is a metric space and weak*-norm topologies agree on $S(X^*)$ the isometries described in Theorem 1 completely describe $\mathcal{G}(\mathcal{L}(X, C(K)))$. It is still an open question if $\mathcal{G}(\mathcal{L}(X, C(K)))$ is algebraically reflexive for some infinite dimensional Banach space X and an infinite metric space K?

Thus a natural procedure is to study the properties of l.s.i maps on $\mathcal{L}(X, C(K))$. Some results of this nature for reflexive spaces X for which $\mathcal{G}(X)$ is algebraically reflexive were obtained in [7]. We next prove a similar result for some non-reflexive Banach spaces.

Definition 3. *A linear map $\Phi : W^*C(K, X^*) \to W^*C(K, X^*)$ is said to be a $C(K)$-module map if there exists a homeomorphism ϕ of K such that $\Phi(fF)(k) = f(\phi(k))\Phi(F)$ for all $k \in K$, $f \in C(K)$ and $F \in W^*C(K, X)$.*

Let c_0 denote the space of complex sequences converging to zero. It is well-known that on $S(\ell^1)$ weak* and norm sequential convergence coincide. Proof of the following theorem proceeds along the same lines as the proof of Theorem 6 in [7]. We therefore indicate only the modifications needed to make the proof work in the current setup.

Theorem 4. *Let K be an infinite countable compact set. Any l. s. i map $\Phi : W^*C(K, \ell^1) \to W^*C(K, \ell^1)$ such that Φ^* preserves extreme points of the dual unit ball is a $C(K)$-module map.*

Proof. As K is a metric space, in view of the description of $\mathcal{G}(W^*C(K, \ell^1))$ from [3] it is easy to see that any onto isometry maps $C(K, \ell^1)$ onto itself. The arguments given during the proof of Theorem 2 can be used to conclude that $\mathcal{G}(C(K, \ell^1))$ is algebraically reflexive. Thus $\Phi|C(K, \ell^1)$ is an onto isometry. Hence by Theorem 1, $\Phi = \iota\phi$ on $C(K, \ell^1)$ for a homeomorphism ϕ of K.

Now as in the proof of Theorem 6 in [7], we verify that Φ is a $C(K)$-module map for this homeomorphism ϕ. It is sufficient to check the functional equation at an isolated point $k' \in K$ and at a $\tau \in \partial_e \ell^\infty_1$.

Accordingly consider $\delta(k') \otimes \tau$. Since k' is an isolated point, it is not difficult to show that $\delta(k') \otimes \tau \in \partial_e W^*C(K, \ell^1)^*_1$. Thus by our assumption on Φ, $\Phi^*(\delta(k') \otimes \tau) \in W^*C(K, \ell^1)^*_1$.

Since $W^*C(K, \ell^1)^*_1$ is the weak*-closed convex hull of $\{\delta(k) \otimes \gamma : k \in K, \gamma \in \partial_e \ell^\infty_1\}$ applying Milman's theorem as in the proof of Theorem 6 in [7], we conclude that Φ is a $C(K)$-module map. □

Remark 5. *We do not know if for the l. s. i map considered above the adjoint always preserves extreme points of the dual unit ball? In particular we do not know if the Russo-Dye type arguments from the proof of Theorem 2 can be adapted here?*

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