Structure of a code related to Sp(4, q), q even

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Abstract. We determine the socle and the radical series of the binary code associated with a finite regular generalized quadrangle of even order, considered as a module for the commutator of each of the orthogonal subgroups in the corresponding symplectic group.

Keywords. Generalized quadrangle; symplectic group; socle layers; Steinberg module.

1. Introduction and statement of the result

Let $V(q) = \mathbb{F}_q^4$, $q = 2^n$, be the vector space of dimension four over $\mathbb{F}_q$ endowed with a nondegenerate symplectic bilinear form $\hbar$ and $W(q)$ denote the incidence system with the set P of all one-dimensional subspaces of $V(q)$ as its *point-set*, the set L of all two-dimensional subspaces of $V(q)$ which are isotropic with respect to $\hbar$ as its *line-set* and symmetrized inclusion as the *incidence*. Then, $W(q)$ is a regular generalized quadrangle of order q (p. 37 of [12]). Further, elements of the symplectic group G defined by $\hbar$ act as incidence-preserving permutations on the sets P and L.

Let k be an algebraically closed extension field of $\mathbb{F}_q$. We denote by $\mathcal{C}$ the image of the kG-module homomorphism from the permutation G-module k^L on L over k to the permutation G-module k^P on P over k, taking $l \in L$ to $\sum_{p\in l} p \in k^P$. The endomorphism of k^P taking each element of P to the '*all-one*' vector $\mathbf{1} = \sum_{p\in P} p$ in k^P is a G-module homomorphism onto the trivial G-submodule $k\mathbf{1}$ of k^P. If Y_P is the kernel of the 'augmentation map' from k^P to k taking $p \in P$ to $1 \in k$, then

$$k^P = k\mathbf{1} \oplus Y_P.$$

Since every element of P is incident with $q+1$ elements of L, it follows that $\mathbf{1} \in \mathcal{C}$ and

$$\mathcal{C} = k\mathbf{1} \oplus \overline{\mathcal{C}},$$

where $\overline{\mathcal{C}} = Y_P \cap \mathcal{C}$. The Loewy structure of the kG-module $\overline{\mathcal{C}}$ is determined in Theorem 2, p. 486 of [14]. Here, we determine the Loewy structure of $\overline{\mathcal{C}}$ as a $k\Omega(f)$-module, where $\Omega(f)$ is the commutator subgroup of the orthogonal group $O(f) \subset G$ defined by a nondegenerate quadratic form f on $V(q)$ polarizing to $\hbar$. That is, $\hbar(x, y) = f(x+y) - f(x) - f(y)$ holds for all $x, y \in V(q)$. There are two such quadratic forms on $V(q)$, up to G-equivalence (see Theorem 6, p. 214 of [3]). They are distinguished by the presence (hyperbolic case) or otherwise (elliptic case) of isotropic projective lines in the set $V_q(f)$ of zeroes of f in the projective 3-space $P(3,q)$ over $\mathbb{F}_q$; that is, they correspond to the cases when the Witt index of f is 2 or 1 (§4.1, p. 18 of [3]). The subgroups $O(f)$ and $\Omega(f)$ of G are isomorphic to $(SL(2,q)\times SL(2,q))\cdot 2$ and $SL(2,q)\times SL(2,q)$, respectively, when the Witt index of f is 2; and to $SL(2,q^2)\cdot 2$ and $SL(2,q^2)$, respectively, when the Witt index of f is 1. We also mention that Sp(4, q) contains exactly 2 conjugacy classes of subgroups of each of the types $SL(2,q^2)$ and $SL(2,q)\times SL(2,q)$ (see Corollary, p. 247 of [6]).

To state our theorem, we describe the simple kG-modules. Let $N=\{0,1,\ldots,2n-1\}$, with addition taken modulo $2n$. Let $V=V(q)\otimes k\simeq k^4$ and extend the symplectic form $\hbar$ to V. Then G is the subgroup of the algebraic group $\mathrm{Sp}(V)\simeq \mathrm{Sp}(4,k)$ fixed by the n-th power of the Frobenius map σ (which is the 'square-the-matrix-entries' map on $GL(4,k)$). It is well-known that $\mathrm{Sp}(V)$ has an endomorphism τ with $\tau^2=\sigma$ (Theorem 28, p. 146 of [19]). For any non-negative integer i, we denote by V_i the $\mathrm{Sp}(V)$-module whose vector space structure is the same as that of V and an element g of $\mathrm{Sp}(V)$ acts on V_i as $\tau^i(g)$ acts on V. For $I\subseteq N$, let V_I denote the kG-module $\otimes_{i\in I}V_i$ (with $V_\emptyset=k$). Then, by Steinberg's tensor product theorem (§11 of [18]), $\{V_I\colon I\subseteq N\}$ is a complete set of inequivalent simple kG-modules.

Let $N_1=\{0,1,\ldots,n-1\}$, with addition taken modulo n. For $I\subseteq N$, define $I_e=\{i\in N_1\colon 2i\in I\}$ and $I_o=\{i\in N_1\colon 2i-1\in I\}$. For $n>1$, we denote by $\mathcal{N}$ the set of all subsets I of N which has no consecutive elements, that is, all I such that I_e and I_0 are disjoint and if $i\in I_e$, then $i+1\notin I_o$. For each subset I of N_1, the subset $N_1\backslash(I\cup\{i+1|i\in I\})$ of N_1 will be called the *admissible complement* of I and will be denoted by I^{ac}. We observe that for $I\subseteq N_1$, the subset $K=\{2i\colon i\in I\}\cup\{2i-1\colon i\in I^{\mathrm{ac}}\}$ is the unique maximal subset of N such that $K_e=I$ and $K_o=I^{\mathrm{ac}}$. Also, $I^{\mathrm{ac}}=\emptyset$ if $|I|\geq n-1$ or if $N_1\backslash I=\{i,j\}$, $i<j$ and $j\neq i+1$. For each $m\in\{0,1,\ldots,n-2,n\}$, let $\mathcal{A}_m$ denote the set of all subsets I of N_1 such that $|I^{\mathrm{ac}}|=m$. We observe that $\mathcal{A}_n=\{\emptyset\}$ and $\mathcal{A}_{n-1}=\emptyset$. For a module M admitting a chain of submodules $M=M_0\supseteq M_1\supseteq\cdots\supseteq M_r$, we indicate the chain of factor modules $E_i=M_i/M_{i+1}$ as $E_{r-1}\backslash\ldots\backslash E_0$ or $E_0/\ldots/E_{r-1}$. For a kG-module M, we write M as $\mathrm{rad}^0(M)$, denote by $\mathrm{rad}^1(M)$ the *radical of* M (that is, the smallest submodule of M with semisimple quotient) and, for a positive integer i, define the i-th radical $\mathrm{rad}^i(M)$ of M recursively as $\mathrm{rad}^1(\mathrm{rad}^{i-1}(M))$. Dually, we write $\mathrm{soc}^0(M)=\{0\}$, denote by $\mathrm{soc}^1(M)$ the *socle* of M (that is, the largest semisimple submodule of M) and, for a positive integer i, define the i-th *socle* $\mathrm{soc}^i(M)$ of M recursively by $\mathrm{soc}^i(M)/\mathrm{soc}^{i-1}(M)=\mathrm{soc}(M/\mathrm{soc}^{i-1}(M))$. The *radical length* (respectively *socle length*) of M is the positive integer r such that $\mathrm{rad}^r(M)=0$ but $\mathrm{rad}^{r-1}(M)\neq 0$ (respectively $\mathrm{soc}^r(M)=M$ but $\mathrm{soc}^{r-1}(M)\neq M$). We refer to $M/\mathrm{rad}^1(M)$ as the *head* of M and write it as $\mathrm{hd}(M)$.

Theorem 1.

(a) *Let* $n=1$. *As* $k\Omega(f)$*-modules*,

  (i) *when* f *is of Witt index* 1, V_0 *and* V_1 *are semisimple*; $\bar{C}$ *is multiplicity-free and is isomorphic to the direct sum of* V_0 *and a* $k\Omega(f)$*-module* X *with* $\mathrm{hd}(X)\simeq V_1$ *and* $\mathrm{rad}(X)\simeq k$; *and*

  (ii) *when* f *is of Witt index* 2, $\bar{C}$ *is semisimple and each composition factor of* $\bar{C}$ *appears with multiplicity one.*

(b) *Let $n \geq 2$. As a $k\Omega(f)$-module,*

(i) V_I *is semisimple for each* $I \in \mathcal{N}$ *and* $\bar{\mathcal{C}}$ *is multiplicity-free;*
(ii) *socle length of* $\bar{\mathcal{C}}$ *is* $n+1$ *and its* j*th-socle layer* $\mathrm{soc}^j(\bar{\mathcal{C}})/\mathrm{soc}^{j-1}(\bar{\mathcal{C}}), 1 \leq j \leq n+1$, *is isomorphic to*

$$\bigoplus_{I \in \mathcal{N} \text{ and } |I_o| = j-1} V_I;$$

(iii)

$$\overline{\mathcal{C}} \simeq \bigoplus_{m \in \{0,1,\ldots,n-2,n\}} X_m \quad \text{and} \quad X_m = \bigoplus_{I \in \mathcal{A}_m} X_{m,I},$$

where $X_{m,I}$ *is the unique indecomposable* $k\Omega(f)$*-submodule of* $\overline{\mathcal{C}}$ *with head* V_K *with* K *denoting the unique element of* $\mathcal{N}$ *such that* $K_e = I$ *and* $K_o = I^{\mathrm{ac}}$*. The radical length of* $X_{m,I}$ *is* $m+1$ *and its* j*-th radical layer* $\mathrm{rad}^j(X_{m,I})/\mathrm{rad}^{j+1}(X_{m,I})$, $0 \leq j \leq m$, *is isomorphic to*

$$\bigoplus_{J \in \mathcal{N},\ J_e = I,\ J_o \subseteq I^{\mathrm{ac}} \text{ and } |J_o| = m-j} V_J$$

and

(iv) $\mathrm{soc}^j(X_{m,I}) = \mathrm{rad}^{m+1-j}(X_{m,I}), \quad 1 \leq j \leq m.$

To prove this, we use the following:

Theorem 2 (Theorem 2 of [14]). *The radical series of* $\bar{\mathcal{C}}$ *as a* kG*-module has length* $2n+1$*. The radical layers are*

$$\mathrm{rad}^j(\bar{\mathcal{C}})/\mathrm{rad}^{j+1}(\bar{\mathcal{C}}) = \bigoplus_{I \in \mathcal{N} \text{ and } |I_e| - |I_o| + n = j} V_I, \quad 0 \leq j \leq 2n.$$

Moreover,

$$\mathrm{soc}^j(\bar{\mathcal{C}}) = \mathrm{rad}^{2n+1-j}(\bar{\mathcal{C}}).$$

The crucial observation is that if E is a section of $\bar{\mathcal{C}}$ which is a nonsplit extension of V_K by V_L, then E is a nonsplit $k\Omega(f)$-module if and only if $L = K \cup \{2t-1\}$ for some $t \in N_1$ (see the first para of the proof of b(ii) in §3). Thus either each or none of the simple $k\Omega(f)$-factors of V_L descends.

In §2, we study the structure of V_I as a $k\Omega(f)$-module and prove Theorem 1 in §3. We also write the socle structure of $\bar{\mathcal{C}}$ as a $k\Omega(f)$-module for $n = 1, 2, 3, 4$. We view this work as part of the program of determining the module structure of permutation modules for finite groups of Lie type (see [15,16]) and references therein for a few other cases considered in the literature.

2. V_I as k$\Omega(f)$-modules

2.1

Throughout, we choose the nondegenerate symplectic bilinear form $\hbar$ on $\mathbb{F}_q^4 \times \mathbb{F}_q^4$ to be

$$\hbar((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = x_1y_4 + x_4y_1 + x_2y_3 + y_2x_3.$$

The quadratic forms f_e and f_h on $\mathbb{F}_q^4$ polarizing to $\hbar$ and of respective Witt indices 1 and 2 are chosen to be

$$f_e(x_1, x_2, x_3, x_4) = x_1x_4 + \alpha^{q+1}x_2^2 + x_2x_3 + \alpha^{q+1}x_3^2; \text{ and}$$

$$f_h(x_1, x_2, x_3, x_4) = x_1x_4 + x_2x_3,$$

where α is an element of $\mathbb{F}_{q^2}\backslash\mathbb{F}_q$ such that $\alpha + \alpha^q = 1$.

This choice of f_e and f_h entails no loss of generality in view of the uniqueness of a nondegenerate symplectic form $\hbar$ on $V(q)$, up to $GL(V(q))$-equivalence (see Corollary 8.2, p. 587 of [11]); and the uniqueness of a quadratic form on $V(q)$ of a given Witt index and polarizing to $\hbar$, up to G-equivalence (see Theorem 6, p. 214 of [3] and p. 39 of [4]).

2.2 *The case* f_e

The polynomial $\alpha^{q+1}x_2^2 + x_2x_3 + \alpha^{q+1}x_3^2$ is irreducible over $\mathbb{F}_q$ but factors as $(\alpha x_2 + \alpha^q x_3)(\alpha^q x_2 + \alpha x_3)$ over $\mathbb{F}_{q^2}$. So, the zero set $\mathcal{E}$ of f_e in $P(3, q)$ is an elliptic quadric and the zero set $\widehat{\mathcal{E}}$ of f_e in $P(3, q^2)$ is a hyperbolic quadric. Since f_e polarizes to $\hbar$, $\mathcal{E}$ is an ovoid in the generalized quadrangle $W(q)$ defined by $\hbar$ (p. 51 of [2]).

(A) Recall that $\Omega(f_e) \simeq SL(2, q^2)$. We now interpret the $k\Omega(f_e)$-modules V_i as $kSL(2, q^2)$-modules. For this, we construct:

- (i) an isomorphism from $SL(2, q^2)$ to $\Omega(f_e)$, following Theorems 15.3.11 and 15.3.18 of [7]; and
- (ii) a graph automorphism τ of G, following pp. 58–60 of [5] (also see Chapter 12 of [20]).

(i)

(a) *The inversive planes* $(P(1, q^2), \mathcal{L})$ *and* $(\mathcal{E}, \Phi)$. Recall that a subline of the projective line $P(1, q^2)$ over $\mathbb{F}_{q^2}$ is the set of all one-dimensional subspaces of $\mathbb{F}_{q^2}^2$ generated by the $\mathbb{F}_q$-span of a fixed basis of $\mathbb{F}_{q^2}^2$. Equivalently, it is a nonsingular Hermitian variety $V_A = V(xA(x^{(q)})^{\rm tr})$ in $P(1, q^2)$ defined by an equation of the form $a_1x_1^{q+1} + bx_1x_2^q + b^q x_1^q x_2 + a_2x_2^{q+1}$, where $x = (x_1, x_2) \in \mathbb{F}_{q^2}^2$, $x^{(q)} = (x_1^q, x_2^q)$ and $A = \left(\begin{smallmatrix} a_1 & b \\ b^q & a_2 \end{smallmatrix}\right)$ with $a_i \in \mathbb{F}_q, b \in \mathbb{F}_{q^2}, a_1a_2 \neq bb^q$ (Lemma 6.2, p. 138 of [8]). Let $\mathcal{L}$ denote the set of all sublines of $P(1, q^2)$. Then, the incidence structure $(P(1, q^2), \mathcal{L})$, with symmetrized inclusion as incidence is an inversive plane of order q; that is, a $3 - (q^2 + 1, q + 1, 1)$ design (p. 257 of [2]). This is isomorphic to the incidence system $(\mathcal{E}, \Phi)$, where $\mathcal{E}$ is as above, Φ is the set of all planes in $P(3, q)$ which are not tangent to $\mathcal{E}$ and the incidence again is symmetrized containment (p. 257 of [2]).

(b) *An isomorphism between* $(P(1, q^2), \mathcal{L})$ *and* $(\mathcal{E}, \Phi)$. To construct this, we follow Theorem 15.3.11, p. 21 of [7]. Let i be the injective map from $P(1, q^2)$ to $P(3, q^2)$ defined by

$$i(P(x_1, x_2)) = P(x_1^{q+1}, x_1x_2^q, x_1^qx_2, x_2^{q+1})$$

and ψ be the involutory projectivity of $P(3, q^2)$ defined by

$$\psi(P(x_1, x_2, x_3, x_4)) = P(x_1, \alpha x_2 + \alpha^q x_3, \alpha^q x_2 + \alpha x_3, x_4).$$

We identify $P(3,q)$ with its image in $P(3,q^2)$ under the map $\mathbb{F}_q a \mapsto \mathbb{F}_{q^2} a$; where $a = (a_1, a_2, a_3, a_4) \in \mathbb{F}_q^4$. We treat $\hbar$ chosen in (2.1) also as a nondegenerate symplectic form on $\mathbb{F}_{q^2}^4$. For x in $P(3,q^2)$, we denote by $\hbar(x,-)$ the plane in $P(3,q^2)$ consisting of all points y of $P(3,q^2)$ such that $\hbar(x,y)=0$. Since ψ stabilizes $\hbar$, $\psi(\hbar(x,-)) = \hbar(\psi(x),-)$ for each x in $P(3,q^2)$. Let j be the map from $\mathcal{L}$ to Φ taking V_A to $\hbar(\psi(P(a'),-)) \cap P(3,q)$, where $a' = (a_1, b, b^q, a_2) \in \mathbb{F}_{q^2}^4$. These planes are not tangent to $\mathcal{E}$ because f_e takes the nonzero value $a_1a_2 - bb^q$ on $\psi(P(a'))$ and so $\psi(P(a')) \notin \mathcal{E}$; it is known that the tangent planes to $\mathcal{E}$ in $P(3,q)$ are all of the form $\hbar(x,-) \cap P(3,q)$, $x \in \mathcal{E}$; and the correspondence $\alpha \longleftrightarrow \hbar(\alpha,-) \cap P(3,q)$ between the points of $P(3,q)$ and the planes of $P(3,q)$ is incidence preserving and bijective. The map $\psi \circ i$ from $P(1,q^2)$ to $\mathcal{E}$ and the map j from $\mathcal{L}$ to Φ are both bijective and form an incidence preserving pair; that is, if $p \in V_A \in \mathcal{L}$, then $\psi \circ i\ (p) \in j(V_A) \in \Phi$.

(c) *The isomorphism from $SL(2,q^2)$ to $\Omega(f_e)$.* The full automorphism group of the incidence structure $(P(1,q^2), \mathcal{L})$ is $P\Gamma L(2,q^2) \simeq SL(2,q^2)\langle\xi\rangle$, where $\xi S \xi^{-1} = S^{(2)} = (s_{ij}^2)$ for all $S = (s_{ij}) \in SL(2,q^2)$ ((1), §4.1, p. 274 of [2]). We give an explicit isomorphism δ from its (unique) subgroup K isomorphic to $SL(2,q^2)\cdot 2$ into $\mathrm{Sp}(4,q^2)$ such that $\psi(\operatorname{im}\delta)\psi^{-1}$ equals the stabilizer $O(f_e)$ of $\mathcal{E}$ in G (see Theorem 15.3.18, p. 27 of [7]).

For $S = (s_{ij}) \in SL(2,q^2)$, let P_S denote the projectivity of $P(1,q^2)$ taking $P(x)$ to $P(xS)$. Then, $\langle P_S \colon S \in SL(2,q^2)\rangle \simeq SL(2,q^2)$. If θ is the involutory projectivity of $P(1,q^2)$ taking $P(x_1,x_2) \mapsto P(x_1^q, x_2^q)$, then $\theta P_S \theta^{-1} = P_{S^{(q)}}$, where S is as above and $S^{(q)} = (s_{ij}^{(q)})$. So $\langle P_S \colon S \in SL(2,q^2)\rangle\langle\theta\rangle$ is the subgroup of $P\Gamma L(2,q^2)$ of index n.

Now P_S induces a bijection T_S on $\mathcal{L}$ which takes V_A to $V_{SA((S^{(q)})^{-1})^{\mathrm{tr}}}$. With the correspondence $A \leftrightarrow a'$, with a' as above, the map $A \mapsto SA((S^{(q)})^{-1})^{\mathrm{tr}}$ can be written as $a' \mapsto a'R_S$, where

$$R_S = \begin{pmatrix} s_{11}s_{11}^q & s_{11}s_{21}^q & s_{21}s_{11}^q & s_{21}s_{21}^q \\ s_{11}s_{12}^q & s_{11}s_{22}^q & s_{21}s_{12}^q & s_{21}s_{22}^q \\ s_{12}s_{11}^q & s_{12}s_{21}^q & s_{22}s_{11}^q & s_{22}s_{21}^q \\ s_{12}s_{12}^q & s_{12}s_{22}^q & s_{22}s_{12}^q & s_{22}s_{22}^q \end{pmatrix} = S^{\mathrm{tr}} \otimes (S^{(q)})^{\mathrm{tr}}.$$

Let $\mathcal{R}_S$ denote the projectivity of $P(3,q^2)$ taking $P(x_1,x_2,x_3,x_4)$ to $P((x_1,x_2,x_3,x_4)R_S)$. Note that ψ and $\mathcal{R}_S$ both stabilize $\hbar$; $\mathcal{R}_S$ fixes the quadratic form $g(x_1,x_2,x_3,x_4) = x_1x_4 + x_2x_3$; ψ is a bijection between the varieties $V_{q^2}(g)$ and $V_{q^2}(f_e)$; and $\psi\mathcal{R}_S\psi^{-1}$ is a projectivity of $P(3,q)$ also. So, $\psi\mathcal{R}_S\psi^{-1}$ is in the stabilizer of $\mathcal{E}$ in G. If $\mathcal{R}_S$ is identity, then so is T_S. These facts imply that the map $S \mapsto \psi\mathcal{R}_S\psi^{-1}$ is a monomorphism from $SL(2,q^2)$ to $O(f_e)$. Now, the map $A \mapsto A^{(q)}$ can be written as $a' \mapsto a'\mathcal{R}$, where $\mathcal{R}$ denotes the projectivity of $P(3,q^2)$ taking $P(x_1,x_2,x_3,x_4)$ to $P(x_1,x_3,x_2,x_4)$. Further, $\mathcal{R}$ is in the stabilizer of $\mathcal{E}$ in G, ψ and $\mathcal{R}$ commute and $\mathcal{R}\mathcal{R}_S\mathcal{R}^{-1} = \mathcal{R}_{S^{(q)}}$ for each $S \in SL(2,q^2)$. So, $\langle\psi\mathcal{R}_S\psi^{-1} \colon S \in SL(2,q^2)\rangle\ \langle\mathcal{R}\rangle$ is a subgroup of $O(f_e)$ and is isomorphic to $SL_2(q^2)\cdot 2$. Now equality holds by order considerations and the isomorphism follows.

(ii)

We now describe a graph automorphism τ of $\mathrm{Sp}(V)$, following pp. 58–60 of [5]. (The argument presented in loc. cit. constructs a graph automorphism for $G = \mathrm{Sp}(V(q))$. However the arguments are valid for $\mathrm{Sp}(V)$ also.) Let Q denote the nondegenerate quadratic form on the exterior square $\Lambda^2 V$ of V defined by

$$Q(\Sigma_{1\le i<j\le 4}\lambda_{ij}e_i \wedge e_j) = \lambda_{12}\lambda_{34} + \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23}$$

(and whose zero set in $P(\Lambda^2 V)$ is the well-known *Klein quadric*). Let β denote the polarization of Q and $\gamma = e_1 \wedge e_4 + e_2 \wedge e_3$. Then the restriction of Q to the hyperplane $U = \{x \in \Lambda^2 V\colon \beta(x,\gamma) = 0\}$ of $\Lambda^2 V$ is a nondegenerate quadratic form; and the restriction of β to U is an alternating form with radical $k\gamma$. The alternating form $\bar{\beta}$ induced by β on $\bar{U} = U/k\gamma$ is nondegenerate. So the symplectic space $(\bar{U}, \bar{\beta})$ is isometric to $(V, \hbar)$. Let $\bar{p}\colon \bar{U} \to V$ be the isometric isomorphism induced by the linear map $p\colon U \to V$ defined by

$$p(e_1 \wedge e_2) = e_1,\ p(e_1 \wedge e_3) = e_2,\ p(e_2 \wedge e_4) = e_3,$$
$$p(e_3 \wedge e_4) = e_4,\ p(\gamma) = 0.$$

Then the map taking $g \in \mathrm{Sp}(V)$ to $\bar{p}(\overline{\wedge^2(g)})\bar{p}^{-1} \in \mathrm{Sp}(V)$ is a graph automorphism τ of $\mathrm{Sp}(V)$ which, on restriction to G, gives a graph automorphism of G.

(B) Let W denote the standard two-dimensional simple $kSL(2,q^2)$-module. For a non-negative integer i, let W_i denote, as in §1, the $kSL(2,q^2)$-module whose underlying vector space structure is the same as that of W and the action of $g = (a_{l,m}) \in SL(2,q^2)$ on W_i is the usual action of $g^{(2^i)} = (a_{l,m}^{2^i})$ on W. For any subset I of N, let W_I denote the $kSL(2,q^2)$-module $\otimes_{i\in I} W_i$. Then, by Steinberg's tensor product theorem, $\{W_I\colon I \subseteq N\}$ is a complete set of inequivalent simple $kSL(2,q^2)$-modules. The following decomposition was suggested to the first-named author by Peter Sin.

Lemma 3. *For* $i \in N_1$,

$$V_{2i}|_{\Omega(f_e)} \simeq W_i \otimes W_{n+i}$$

and

$$V_{2i-1}|_{\Omega(f_e)} \simeq W_i \oplus W_{n+i}.$$

Proof. We prove this by using the above interpretation of V_i as $kSL(2,q^2)$-modules. The group $H = \langle R_S\colon S \in SL(2,q^2)\rangle$, where R_S is as in (2.2A.i(c)) and $\Omega(f_e)$ are both isomorphic to $SL(2,q^2)$ and are conjugate in $\mathrm{Sp}(4,k)$ (in fact in $\mathrm{Sp}(4,q^2)$) (see the last paragraph of 2.2.A.i(c)). So, for $I \subseteq N$, V_I considered as a kH-module and V_I considered as a $k\Omega(f_e)$-module are isomorphic $SL(2,q^2)$-modules. Let (e_1, e_2, e_3, e_4) be the standard ordered basis for V and (v_1, v_2) be an ordered basis for W. For $S \in SL(2,q^2)$, $S \otimes S^{(q)}$ represents not only the action of S on V with respect to the ordered basis (e_1, e_2, e_3, e_4) of V but also the action of S on $W_0 \otimes W_n$ with respect to the ordered basis $(v_2 \otimes v_2, v_2 \otimes v_1, v_1 \otimes v_2, v_1 \otimes v_1)$ of $W_0 \otimes W_n$. So, $V_0|_{\Omega(f_e)} \simeq W_0 \otimes W_n$. Now, an application of τ^{2i} to V yields the first part of the lemma.

A simple calculation shows that R_S^τ leaves the subspaces $M^1 = ke_4 + ke_1$ and $M^2 = ke_2 + ke_3$ invariant. Further, the matrix representation of R_S^τ on M^1 with respect to the ordered basis (e_4, e_1) is S and it is $S^{(q)}$ on M^2 with respect to the ordered basis (e_3, e_2). So, an application of τ^{2i-1} to V yields the second part of the lemma. ■

For $I \subseteq N_1$, let $\mathcal{N}_I$ denote the set of $2^{|I|}$ subsets J of N of size $|I|$ such that, for each $t \in I$, only one of t and $n+t$ is in J. Let $\overline{I} = I \cup (n+I)$. Then, for $K \in \mathcal{N}$, Lemma 3 yields

$$V_K|_{\Omega(f_e)} \simeq \left(\bigoplus_{L \in \mathcal{N}_{K_0}} W_L \right) \otimes W_{\overline{K_e}}. \tag{1_e}$$

Notice that each irreducible component in (1_e) determines both the sets K_e and K_o. So we have the following:

COROLLARY 4

Let $K, K' \in \mathcal{N}$ be distinct. Then V_K and $V_{K'}$ are semisimple $k\Omega(f_e)$-modules with no irreducible factors in common.

2.3 *The case f_h*

We now study V_I as a $k\Omega(f_h)$-module. The zero set of f_h in $P(3, q)$ is a quadric of Witt index 2. Since f_h polarizes to $\hbar$, the projective lines contained in this quadric are isotropic with respect to $\hbar$. Further, $\Omega(f_h) = H_1 H_2$, $[H_1, H_2] = H_1 \cap H_2 = \langle 1 \rangle$, where H_1 (respectively H_2) $\simeq SL(2, q)$ is the stabilizer of the subspace $ku_1 + ku_2$ (respectively $kv_1 + kv_2$) of V. Here $u_1 = e_1 + e_3$, $u_2 = e_2 + e_4$, $v_1 = e_1 + e_2$ and $v_2 = e_3 + e_4$. Further, the action of H_1 (respectively H_2) on $ku_1 + ku_2$ (respectively $kv_1 + kv_2$) with respect to its ordered basis (u_1, u_2) (respectively (v_1, v_2)) is the standard $SL(2, q)$-action on k^2.

Let M^1 and M^2 be two copies of the standard 2-dimensional simple $kSL(2, q)$-module. For $j \in N_1$, define the jth-standard Frobenius twist M^i_j of M^i as in 2.2.B. For $J \subseteq N_1$, define M^i_J as $\otimes_{j \in J} M^i_j$. Further, for I and $J \subseteq N_1$, denote by $M^1_I \# M^2_J$ the outer tensor product of M^1_I and M^2_J. That is, $M^1_I \# M^2_J$ is the k-vector space $M^1_I \otimes M^2_J$ with the action of $(h_1, h_2) \in SL(2, q) \times SL(2, q)$ on it given by

$$(h_1, h_2)(m_1 \otimes m_2) = h_1(m_1) \otimes h_2(m_2).$$

Then, $\{M^1_I \# M^2_J \colon I, J \subseteq N_1\}$ is a complete set of pairwise nonisomorphic simple $k(SL(2, q) \times SL(2, q))$-modules (Theorem 9.14, p. 136 of [9]).

Lemma 5. *For $i \in N_1$,*

$$V_{2i}|_{\Omega(f_h)} \simeq M^1_i \# M^2_i$$

and

$$V_{2i-1}|_{\Omega(f_h)} \simeq (M^1_i \# k) \oplus (k \# M^2_i).$$

Proof. Let $h_i \in H_i$ be represented by the matrix $A_i \in SL(2, q)$ with respect to the ordered bases mentioned above. The matrix $A_1 \otimes A_2$ represents the action of $h = h_1 h_2$ on V with respect to its ordered basis (e_1, e_2, e_3, e_4) as well as its action on $(ku_1 + ku_2)\#(ku_3 + ku_4)$ with respect to its ordered basis $(u_1 \otimes v_1, u_2 \otimes v_1, u_1 \otimes v_2, u_2 \otimes v_2)$. So, $V_0|_{\Omega(f_h)} \simeq M^1 \# M^2$. Now, an application of τ^{2i} to V yields the first part of the lemma.

A simple calculation shows that each of the subgroups H_i^τ stabilizes the subspaces $ke_2 + ke_3$ and $ke_1 + ke_4$. Further, the action of H_1^τ (respectively of H_2^τ) on $ke_2 + ke_3$ (respectively on $ke_1 + ke_4$) with respect to the ordered basis (e_2, e_3) (respectively, (e_1, e_4)) is equivalent to the standard action of $SL(2,q)$ on k^2 and the action on $ke_1 + ke_4$ (respectively on $ke_2 + ke_3$) is trivial. So, $V_1|_{\Omega(f_h)} \simeq (M_1^1 \# k) \oplus (k \# M_1^2)$. An application of $\tau^{2(i-1)}$ to V_1 now yields the second part of the lemma. ■

For any subset I of N_1, let $\mathcal{M}_I$ denote the set of $2^{|I|}$ ordered partitions (A, B) of I. By Lemma 5, if $K \in \mathcal{N}$, then

$$\begin{aligned} V_K|_{\Omega(f_h)} &\simeq (M_{K_e}^1 \# M_{K_e}^2) \otimes (\oplus_{(A,B)\in\mathcal{M}_{K_0}} M_A^1 \# M_B^2) \\ &\simeq (\oplus_{(A,B)\in\mathcal{M}_{K_0}} M_{K_e \cup A}^1 \# M_{K_e \cup B}^2). \end{aligned} \tag{1_h}$$

Note that for $K \in \mathcal{N}$, K_e and K_o are disjoint. Further K is determined by each irreducible component in (1_h). So, we have the following:

COROLLARY 6

Let $K, K' \in \mathcal{N}$ be distinct. Then V_K and $V_{K'}$ are semisimple $k\Omega(f_h)$-modules with no irreducible factors in common.

2.4

The core of the proof of the theorem is in Lemmas 7 and 8 that we now state. Let $f \in \{f_e, f_h\}$ and assume that $n \geq 2$ for Lemmas 7 and 8.

Lemma 7. *Let $J, K \in \mathcal{N}$. If the symmetric difference of J and K is not equal to $\{2t-1\}$ for some $t \in N_1$, then* $\mathrm{Ext}^1_{k\Omega(f)}(V_J, V_K) = 0$.

Now, let $J = K \cup \{2t-1\}$ and $2t-1 \notin K$. Then, $2t-2 \notin K$ and, by Theorem, p. 159 of [17], there exists a unique kG-module E, up to isomorphism, which is a nonsplit extension of V_K by V_J.

Lemma 8. $\mathrm{soc}_{k\Omega(f)}(E|_{\Omega(f)}) \simeq V_K|_{\Omega(f)}$. *In particular, as a $k\Omega(f)$-module, E is a non-split extension of $V_K|_{\Omega(f)}$ by $V_J|_{\Omega(f)}$.*

Proofs of Lemmas 7 and 8 for the cases $f = f_e$ and $f = f_h$ are given separately.

2.4.1 *The case $f = f_e$.* For easy reference, we collect some results due to Alperin and due to Sin. Consider the following condition on $I, J \subseteq N$:

$$I \cup J = (I \cap J) \cup \{r\} \text{ and neither } r \text{ nor } r-1 \text{ is in } I \cap J. \tag{$C_{I,J}$}$$

Lemma 9.

(a) *For $I, J \subseteq N$,* $\mathrm{Ext}^1_{kSL(2,q^2)}(W_I, W_J)$ *as well as* $\mathrm{Ext}^1_{kG}(V_I, V_J)$ *are both k or both zero according as whether the condition $(C_{I,J})$ holds or not.*
(b) *For $i \in N$, $W_i \otimes W_i$ is a uniserial module with composition factors $k\backslash W_{i+1}\backslash k$.*

Proof. For (a), see Theorem 3, p. 221 of [1] and Theorem, p. 159 of [17]. For (b), see Lemma 4, p. 224 of [1]. ■

A typographical error about the assumptions on I and J in the statement of part (a) in Theorem 3, p. 221 of [1] has been corrected here (see p. 229 of [1]).

Proof of Lemma 7. In view of the isomorphism $\mathrm{Ext}^1_{kG}(V_J, V_K) \simeq \mathrm{Ext}^1_{kG}(V_{J\cup K}, V_{J\cap K})$, the biadditivity of the map $(M, N) \mapsto \mathrm{Ext}^1(M, N)$ and Lemma 9(a) above, we only need to consider the case when $J = K \cup \{r\}$, $r \notin K$. In this case, the lemma follows from (1_e) and Lemma 9(a). ∎

Proof of Lemma 8. From (1_e),

$$V_K|_{\Omega(f_e)} \simeq (\oplus_{L\in\mathcal{N}_{K_o}} W_L) \otimes W_{\overline{K_e}}$$

and

$$V_J|_{\Omega(f_e)} \simeq [\oplus_{L\in\mathcal{N}_{K_o}} (W_{L\cup\{t\}} \oplus W_{L\cup\{n+t\}})] \otimes W_{\overline{K_e}}.$$

We need to show that

$$\mathrm{Hom}_{k\Omega(f_e)}(W_{L'} \otimes W_{\overline{K_e}}, E) = 0$$

for each $L' \in \mathcal{N}_{J_o}$. Let D_t denote the unique (up to isomorphism) kG-module which is a nonsplit extension of k by V_{2t-1} (see Lemma 9(a)). Then $E \simeq D_t \otimes V_K$ (Lemma 8, p. 490 of [14]). Since D_t is isomorphic to a submodule of the kG-module $V_{2t-2} \otimes V_{2t-2}$ (see Lemma 2(a), p. 161 of [17]), $D_t \otimes V_K$ embeds in $(V_{2t-2} \otimes V_{2t-2}) \otimes V_K$. Hence it is enough to prove that

$$\mathrm{Hom}_{k\Omega(f_e)}(W_{L'} \otimes W_{\overline{K_e}}, V_{2t-2} \otimes V_{2t-2} \otimes V_K) = 0. \qquad (*)$$

Let $L'' \in \mathcal{N}_{K_o}$ and $L' = L \cup \{r\}$ where $L \in \mathcal{N}_{K_o}$, $r = t$ or $n + t$. Since simple $k\Omega(f_e)$-modules are self-dual, by Lemma 3,

$$\mathrm{Hom}_{k\Omega(f_e)}(W_{L'} \otimes W_{\overline{K_e}}, V_{2t-2} \otimes V_{2t-2} \otimes W_{L''} \otimes W_{\overline{K_e}})$$
$$\simeq \mathrm{Hom}_{k\Omega(f_e)}(W_{L'} \otimes W_{\overline{K_e\cup\{t-1\}}}, W_{L''} \otimes W_{\overline{K_e\cup\{t-1\}}}) = 0.$$

So $(*)$ holds and the lemma follows. ∎

2.4.2 *The case* $f = f_h$. First we recall a useful result due to Jones (Theorem 3, p. 629 of [10]).

Lemma 10. *Let R be a Dedekind domain, G_1 and G_2 be finite groups, $G = G_1 \times G_2$, $\Gamma_i = RG_i$, $i = 1, 2$, and $\Gamma = RG$. Let M_i and M_i' be (left) Γ_i-modules. Then, as R-modules,*

$$\mathrm{Ext}^1_\Gamma(M_1 \# M_2, M_1' \# M_2') \simeq \mathrm{Hom}_{\Gamma_1}(M_1, M_1') \otimes_R \mathrm{Ext}^1_{\Gamma_2}(M_2, M_2')$$
$$\oplus\, \mathrm{Ext}^1_{\Gamma_1}(M_1, M_1') \otimes_R \mathrm{Hom}_{\Gamma_2}(M_2, M_2')$$

and

$$\mathrm{Hom}_\Gamma(M_1 \# M_2, M_1' \# M_2') \simeq \mathrm{Hom}_{\Gamma_1}(M_1, M_1') \otimes_R \mathrm{Hom}_{\Gamma_2}(M_2, M_2').$$

Proof of Lemma 7. As in the case $f = f_e$, we need only to consider the case when $J = K \cup \{r\}$, where $r \notin K$. If $r = 2t$ for some $t \in N_1$, then (1_h) and Lemma 10 imply $\mathrm{Ext}^1_{k\Omega(f_h)}(V_J, V_K) = 0$. ■

Proof of Lemma 8. From (1_h),

$$V_K|_{\Omega(f_h)} \simeq \oplus_{(A,B)\in\mathcal{M}_{K_0}}(M^1_{K_e\cup A}\# M^2_{K_e\cup B})$$

and

$$V_J|_{\Omega(f_h)} \simeq \oplus_{(A,B)\in\mathcal{M}_{K_0}}[(M^1_{K_e\cup A\cup\{t\}}\# M^2_{K_e\cup B}) \oplus (M^1_{K_e\cup A}\# M^2_{K_e\cup B\cup\{t\}})].$$

We need to show that

$$\mathrm{Hom}_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t\}}\# M^2_{K_e\cup B}) \oplus (M^1_{K_e\cup A}\# M^2_{K_e\cup B\cup\{t\}}), E) = 0$$

for each $(A, B) \in \mathcal{M}_{K_o}$. In view of the discussion regarding D_t in the $f = f_e$ case, we prove that

$$\mathrm{Hom}_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t\}}\# M^2_{K_e\cup B}) \oplus (M^1_{K_e\cup A}\# M^2_{K_e\cup B\cup\{t\}}), V_{2t-2}\otimes V_{2t-2}\otimes V_K)$$

is zero for each $(A, B) \in \mathcal{M}_{K_o}$. Now if $(A, B), (A', B') \in \mathcal{M}_{K_o}$, then

$$\begin{aligned}
&\mathrm{Hom}_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t\}}\# M^2_{K_e\cup B}) \oplus (M^1_{K_e\cup A}\# M^2_{K_e\cup B\cup\{t\}}),\\
&\qquad V_{2t-2}\otimes V_{2t-2}\otimes (M^1_{K_e\cup A'}\# M^2_{K_e\cup B'}))\\
&\simeq \mathrm{Hom}_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t\}}\# M^2_{K_e\cup B})\otimes V_{2t-2} \oplus (M^1_{K_e\cup A}\# M^2_{K_e\cup B\cup\{t\}})\otimes V_{2t-2},\\
&\qquad V_{2t-2}\otimes (M^1_{K_e\cup A'}\# M^2_{K_e\cup B'}))\\
&\simeq \mathrm{Hom}_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t\}}\# M^2_{K_e\cup B})\otimes (M^1_{t-1}\# M^2_{t-1}) \oplus (M^1_{K_e\cup A}\# M^2_{K_e\cup B\cup\{t\}})\\
&\quad \otimes (M^1_{t-1}\# M^2_{t-1}), (M^1_{t-1}\# M^2_{t-1})\otimes (M^1_{K_e\cup A'}\# M^2_{K_e\cup B'}))\\
&\simeq \mathrm{Hom}_{k\Omega(f_h)}((M^1_{K_e\cup A\cup\{t-1,t\}}\# M^2_{K_e\cup B\cup\{t-1\}}) \oplus (M^1_{K_e\cup A\cup\{t-1\}}\# M^2_{K_e\cup B\cup\{t-1,t\}}),\\
&\qquad M^1_{K_e\cup A'\cup\{t-1\}}\# M^2_{K_e\cup B'\cup\{t-1\}}) = 0
\end{aligned}$$

by Lemma 10. Here we have used: (i) the duality between the functors 'Hom' and 'Tensor', namely for any group X and kX-modules U_i, we have $\mathrm{Hom}_{kX}(U_1, U_2\otimes U_3) \simeq \mathrm{Hom}_{kX}(U_1\otimes U_2^*, U_3)$, where U^* is the dual of U; and (ii) the fact that each simple module for $G = \mathrm{Sp}(4, q)$ is self-dual. So Lemma 8 holds for $f = f_h$ also. ■

3. Proof of Theorem 1

(a) Let $n = 1$. Then $\dim_k(\overline{\mathcal{C}}) = 9$ (see, for example, p. 308 of [13]). By Lemma 4, p. 488 of [14] and Frobenius reciprocity (p. 689 of [11]) it follows that V_1, k and V_0 (in descending order) are kG-composition factors of $\overline{\mathcal{C}}$. Semisimplicity of each composition factor of $\overline{\mathcal{C}}$ and its multiplicity freeness as a $k\Omega(f)$-module are proved in Corollaries 4 and 6. For the remaining part of the proof, we treat the cases separately. Consider the case $f = f_e$. Since $V_0|_{\Omega(f)} \simeq W_0 \otimes W_1$ (Lemma 3) is the Steinberg module (and hence projective) for $\Omega(f)$, we need to prove that the kG-module E which is the unique (up to isomorphism) nonsplit extension of k by V_1 remains nonsplit as a $k\Omega(f)$-module. Further, since E is isomorphic to a submodule of the kG-module $V_0 \otimes V_0$ (see Lemma 2(a), p. 161 of [17]), $V_1|_{\Omega(f)} \simeq W_0 \oplus W_1$ (Lemma 3) and $W_0 \otimes W_1$ is the Steinberg module for $k\Omega(f)$, by Theorem 1, p. 220 of [1], we have

$$\mathrm{Hom}_{k\Omega(f)}(V_1, V_0 \otimes V_0) = 0.$$

Now we consider the case $f = f_h$. Then M^1, M^2 and $V|_{\Omega(f)} \simeq M^1 \# M^2$ (see Lemma 5) are the Steinberg modules for H_1, H_2 and $\Omega(f)$, respectively. Hence we need to prove that E as a $k\Omega(f)$-module splits. But this is clear from Lemmas 5 and 10.

(b) Let $n \geq 2$. By Theorem 2, $\{V_I\}_{I\in\mathcal{N}}$ are the kG-composition factors of $\overline{\mathcal{C}}$ and they appear with multiplicity one. So, (i) follows from Corollaries 4 and 6.

We now prove (ii). Let V_K and V_J, $K, J \in \mathcal{N}$, be in the i-th and j-th kG-socle layers of $\overline{\mathcal{C}}$, $1 \leq i < j \leq 2n$. Assume that there is a kG-module E which is a nonsplit extension of V_K by V_J. Then, by Lemma 9(a), $j = i + 1$. Further, E is a nonsplit $k\Omega(f)$-extension of V_K by V_J if and only if $J = K \cup \{2t-1\}$ for some $t \in N_1$ (Lemma 7).

Further, if this holds, $\mathrm{soc}_{k\Omega(f)}(E) = V_K$ (Lemma 8). That is, either all or none, of the simple $k\Omega(f)$-summands of V_J descend. This observation together with the kG-socle structure of $\overline{\mathcal{C}}$ yields (ii).

Now, let $m \in \{0, 1, \ldots, n-2, n\}$ and $I \in \mathcal{A}_m$. In what follows, all modules considered are over $k\Omega(f)$. Since $\overline{\mathcal{C}}$ is multiplicity free as a $k\Omega(f)$-module, it has a unique submodule $X_{m,I}$ with head V_K, where K is the unique element of $\mathcal{N}$ with $K_e = I$ and $K_o = I^{\mathrm{ac}}$. First $X_{0,I} = V_{\{2t:\, t\in I\}}$ (see (1_e) and (1_h)) and is semisimple. Let $m > 0$ and write $X_{m,I}$ as L for brevity. Since $\mathrm{hd}(L)$ is contained in the m-th socle layer of $\overline{\mathcal{C}}$, L is a submodule of $\mathrm{soc}^m(\overline{\mathcal{C}})$. For each summand of $\mathrm{rad}^1(L)/\mathrm{rad}^2(L)$ of the form V_J, a nonsplit extension of V_J by $\mathrm{hd}(L)$ appears as a section of L. Further, $\mathrm{hd}(L) = V_K$. So, $J = K\backslash\{2t-1\}$ for some $t \in I^{\mathrm{ac}}$ (Lemma 8). This proves that $\mathrm{rad}^1(L)$ is contained in the unique submodule L' of $\mathrm{soc}^{m-1}(\overline{\mathcal{C}})$ whose head is isomorphic to

$$\bigoplus_{J\in\mathcal{N},\ J_e=I,\ J_o\subseteq I^{\mathrm{ac}} \text{ and } |J_o|=m-1} V_J. \tag{3.1}$$

The multiplicity freeness of $\overline{\mathcal{C}}$ (as a $k\Omega(f)$-module) implies that $L \cap L' = \mathrm{rad}^1(L)$. We now show that $L' = \mathrm{rad}^1(L)$. Since $(L + L')/L' \simeq L/(L \cap L') = L/\mathrm{rad}^1(L) \simeq V_K$, by uniqueness of L, $L + L' = L$. Thus $L' \subseteq L$ and $L' = L \cap L' = \mathrm{rad}^1(L)$. Thus, $\mathrm{rad}^1(L)/\mathrm{rad}^2(L)$ is the module in (3.1). Now successive application of this argument to $\mathrm{rad}^i(L)/\mathrm{rad}^{i-1}(L)$ yields the statement about the radical structure of $X_{m,I}$ in (iii). Further, (iv) is also clear.

Since $\mathrm{soc}(X_{m,I}) = V_{\{2t:\, t\in I\}}$ is simple, $X_{m,I}$ is indecomposable. For $I \neq J \in \mathcal{A}_m$, $X_{m,I} \cap X_{m,J}$ is trivial because their socles are distinct and simple. (In fact, $X_{m,I}$ and $X_{m,J}$ have no composition factors in common.) Let

$$X_m = \bigoplus_{I\in\mathcal{A}_m} X_{m,I}.$$

For distinct $m, m' \in \{0, 1, \ldots, n-2, n\}$, X_m and $X_{m'}$ have no composition factors in common and every composition factor of $\overline{\mathcal{C}}$ occurs in some X_m. Thus sum of X_m's equals $\overline{\mathcal{C}}$, completing the proof of all parts of (iii) and of Theorem 1. ■

Remark 11.

(i) If $n = 1$, as a $k\Omega^{\tau}(f)$-module, V_0 and V_1 are semisimple and $\overline{\mathcal{C}}$ is multiplicity free. Further, if f is of Witt index 1, then $\overline{\mathcal{C}}$ is isomorphic to the direct sum of V_0 and a $k\Omega^{\tau}(f)$-module U with $\mathrm{hd}(U) \simeq k$ and $\mathrm{rad}(U) \simeq V_1$. If f is of Witt index 2, then $\overline{\mathcal{C}}$ is semisimple.

(ii) If $n \geq 2$, as a $k\Omega^{\tau}(f)$-module, then V_I is semisimple for each $I \in \mathcal{N}$ and $\overline{\mathcal{C}}$ is multiplicity free and is isomorphic to

$$\bigoplus_{m \in \{0,1,\ldots,n-2,n\}} \left(\bigoplus_{I \in \mathcal{A}_m} U_{m,I} \right),$$

where $U_{m,I}$ is a unique indecomposable module of radical length $m+1$. Its j-th radical layer $\mathrm{rad}^j(U_{m,I})/\mathrm{rad}^{j+1}(U_{m,I})$ $(0 \leq j \leq m)$ is isomorphic to

$$\bigoplus_{J \in \mathcal{N},\ J_e = I,\ J_o \subseteq I^{ac} \text{ and } |J_o| = j} V_J.$$

Moreover, for each $m \in \{0, 1, \ldots, n-2, n\}$,

$$\mathrm{Soc}^j(U_{m,I}) = \mathrm{rad}^{m+1-j}(U_{m,I}).$$

Thus (i) and (ii) follow from an argument similar to the proof of Theorem 1, using Theorem 2 and Lemmas 7 and 8.

Examples. We illustrate the descent of the composition factors V_J in Theorem 2 when considered as a $k\Omega(f)$-module for the cases $n = 1, 2, 3, 4$. For $J = \{i_1, i_2, \ldots\} \subseteq N$, we write V_J as $\underline{J}$ or $\underline{i_1, i_2, \ldots}$.

If $n = 1$, then

$$\overline{\mathcal{C}} = \underline{0} \oplus (\underline{1}/k).$$

If $n = 2$, then

$$\overline{\mathcal{C}} = X_0 \oplus X_2 \equiv [\underline{0} \oplus \underline{2} \oplus \underline{0,2}] \oplus [\underline{1,3}/(\underline{1} \oplus \underline{3})/k].$$

If $n = 3$, then

$$\overline{\mathcal{C}} = X_0 \oplus X_1 \oplus X_3 \equiv [\underline{0,2,4} \oplus \underline{0,2} \oplus \underline{0,4} \oplus \underline{2,4}] \oplus [(\underline{0,3}/\underline{0}) \oplus (\underline{2,5}/\underline{2}) \oplus (\underline{1,4}/\underline{4})] \oplus [\underline{1,3,5}/(\underline{1,3} \oplus \underline{1,5} \oplus \underline{3,5})/(\underline{1} \oplus \underline{3} \oplus \underline{5})/k].$$

If $n = 4$, then

$$\overline{\mathcal{C}} = X_0 \oplus X_1 \oplus X_2 \oplus X_4,$$

where

$$X_0 = \bigoplus_{J \subseteq N_1, |J|=3 \text{ or } 4} \underline{2J} \oplus (\underline{0,4} \oplus \underline{2,6}),$$

$$X_1 = \bigoplus_{i=0}^{3} (\underline{2i, 2i+2, 2i+5})/(\underline{2i, 2i+2}),$$

$$X_2 = \begin{array}{l} (\underline{0,3,5}/(\underline{0,3} \oplus \underline{0,5})/\underline{0}) \oplus (\underline{2,5,7}/(\underline{2,5} \oplus \underline{2,7})/\underline{2}) \\ \oplus (\underline{1,4,7}/(\underline{1,4} \oplus \underline{4,7})/\underline{4}) \oplus (\underline{6,1,3}/(\underline{6,1} \oplus \underline{6,3})/\underline{6}), \end{array}$$

$$X_4 = A_4/A_3/A_2/A_1/A_0,$$

where A_i is the direct sum of $\underline{J}$ as J runs over the subsets of $\{1, 3, 5, 7\}$ of size i.

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