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An Efficient and Fast Algorithm for Estimating the Parameters of Sinusoidal Signals

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Abstract

A computationally efficient algorithm is proposed for estimating the parameters of sinusoidal signals in presence of stationary errors. The proposed estimators are consistent, and they are asymptotically equivalent to the least squares estimators. Monte Carlo simulations are performed to compare the proposed one with the other existing comparable methods. It is observed that the proposed estimator works quite well in terms of biases and mean squared errors. The main advantage of the proposed method of estimation is that the estimators can be obtained using only fixed number of iterations. Some real data sets have been analyzed for illustration purposes.

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1 Introduction

We consider the following model:

$$y(t) = \sum_{j=1}^{p} [A_j \cos(\omega_j t) + B_j \sin(\omega_j t)] + X(t); \qquad t = 1, \ldots, N. \quad (1.1)$$

Here A_js and B_js are unknown amplitudes and none of them is identically equal to zero. The ω_js are unknown frequencies lying strictly between 0 and π, and they are distinct. The error random variables $X(t)$s are stationary linear processes, and they satisfy the following assumption.

ASSUMPTION 1: The error random variables $X(t)$s have the following structure.

$$X(t) = \sum_{j=-\infty}^{\infty} a(j)e(t-j), \tag{1.2}$$

where $e(t)$s are independent and identically distributed (i.i.d.) random variables with mean zero and finite variance σ^2 and $a(j)$s are real numbers such that

$$\sum_{j=-\infty}^{\infty} |a(j)| < \infty. \tag{1.3}$$

The number of components p is assumed to be known. The problem is to estimate the unknown parameters A_js, B_js and ω_js, given a sample of size N, namely $y(1), \ldots, y(N)$. In this paper, we mainly consider efficient estimation of ω_js, even though the estimation of the linear parameters is also very important.

Estimation of frequencies in presence of additive noise is a very important problem in the time series analysis and in the area of statistical signal processing. Starting with the work of Fisher (1929), this problem has received considerable attention. Brillinger (1987) discussed some of the very important real life applications of this particular problem, see Kay and Marple (1981) also. Stoica (1993) provided an extensive list of references up to that time and see Kundu (2002) for some of the recent references.

The optimum rate of convergence can be obtained by the least squares estimators, and their convergence rate is $O_p(N^{-3/2})$. Here $O_p(N^{-\delta})$ means $N^{\delta}O_p(N^{-\delta})$ is bounded in probability. The periodogram estimators (without the constraint of Fourier frequency) also provide the best possible rate, and they are asymptotically equivalent to the least squares estimators. Finding the least squares estimators tends to be computationally intensive as the functions to be optimized are highly non-linear in parameters. Thus, very good (close enough to the true value) initial estimates are needed. Several techniques are available in the literature, for example Pisarenko (1973), Chan, Lavoie and Plant (1981), which attempt to find computationally efficient frequency estimators. However, these procedures produce estimators having convergence rate $O_p(N^{-1/2})$. The periodogram maximizer over Fourier frequencies does not generally provide good initial estimates with the convergence rate is $O_p(N^{-1})$, see for example Rice and Rosenblatt (1988), whereas an initial estimate of the convergence rate $O_p(N^{-1-\delta})$ ($\delta > 0$) is needed for most of the iterative techniques to work.

In this paper, we propose a new iterative procedure similar to the procedure of Bai et al. (2003). The method uses a correction term based on $P_N(j)$ and $Q_N(j)$ to be defined in Theorem 3.1, which are functions of the data vector as well as the available frequency estimator corresponding to the j-th component. The forms of the functions $P_N(j)$ and $Q_N(j)$ are motivated by the least squares method. It is observed that if the initial guess is accurate up to the order $O(N^{-1})$, then our three step iterative procedure produces fully efficient frequency estimator, which has the same rate of convergence as the least squares estimators. In the proposed method, we do not use the fixed sample size available for estimation at each step. At first step, we use a fraction of it and at the last step, we use the whole data set by gradually increasing the effective sample sizes.

The rest of the paper is organized as follows. Two different estimators and their properties are discussed in Section 2. The proposed algorithm is presented in Section 3. Simulation results and analysis based on some real data are presented in Section 4, and finally the conclusion appears in Section 5. All the proofs are provided in the Appendix.

2 Estimation Procedures

There are mainly two different methods of estimation of the unknown parameters. We will briefly discuss both of them below.

2.1. Least Squares Estimators. The least squares estimators (LSEs) of the unknown parameters can be obtained by minimizing the residual sum of squares, namely,

$$R(\mathbf{A},\mathbf{B},\boldsymbol{\omega}) = \sum_{t=1}^{N}\left(y(t) - \sum_{j=1}^{p}\left[A_j\cos(\omega_j t) + B_j\sin(\omega_j t)\right]\right)^2, \tag{2.1}$$

with respect to $\mathbf{A} = (A_1,\ldots,A_p)$, $\mathbf{B} = (B_1,\ldots,B_p)$ and $\boldsymbol{\omega} = (\omega_1,\ldots,\omega_p)$. Note that $R(\mathbf{A},\mathbf{B},\boldsymbol{\omega})$ can be written as follows.

$$R(\mathbf{A},\mathbf{B},\boldsymbol{\omega}) = R(\boldsymbol{\alpha},\boldsymbol{\omega}) = \left[\mathbf{Y} - \mathbf{X}(\boldsymbol{\omega})\boldsymbol{\alpha}\right]^T\left[\mathbf{Y} - \mathbf{X}(\boldsymbol{\omega})\boldsymbol{\alpha}\right], \tag{2.2}$$

where $\mathbf{Y} = [y(1), \ldots, y(N)]^T$, $\boldsymbol{\alpha} = [A_1, \ldots, A_p, B_1, \ldots, B_p]^T$, $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_p)^T$ and $\mathbf{X}(\boldsymbol{\omega})$ is an $N \times 2p$ matrix of the form

$$\mathbf{X}(\boldsymbol{\omega}) = \begin{bmatrix} \cos(\omega_1) & \ldots & \cos(\omega_p) & \sin(\omega_1) & \ldots & \sin(\omega_p) \\ \cos(2\omega_1) & \ldots & \cos(2\omega_p) & \sin(2\omega_1) & \ldots & \sin(2\omega_p) \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ \cos(N\omega_1) & \ldots & \cos(N\omega_p) & \sin(N\omega_1) & \ldots & \sin(N\omega_p) \end{bmatrix}. \qquad (2.3)$$

From (2.2), it is clear that $\boldsymbol{\alpha}$ can be separated from $\boldsymbol{\omega}$. Therefore, by using the separable regression techniques of Richards (1961), the LSE of $\boldsymbol{\alpha}$ can be obtained in terms of $\boldsymbol{\omega}$. For fixed $\boldsymbol{\omega}$, the LSE of $\boldsymbol{\alpha}$ can be obtained as

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\omega}) = \left[\mathbf{X}^T(\boldsymbol{\omega})\mathbf{X}(\boldsymbol{\omega})\right]^{-1} \mathbf{X}^T(\boldsymbol{\omega})\mathbf{Y}. \qquad (2.4)$$

If we substitute $\hat{\boldsymbol{\alpha}}(\boldsymbol{\omega})$ in (2.2), we obtain

$$Q(\boldsymbol{\omega}) = R(\hat{\boldsymbol{\alpha}}(\boldsymbol{\omega}), \boldsymbol{\omega}) = \mathbf{Y}^T \left[\mathbf{I} - \mathbf{P}_{X(\boldsymbol{\omega})}\right] \mathbf{Y}, \qquad (2.5)$$

where

$$\mathbf{P}_{X(\boldsymbol{\omega})} = \mathbf{X}(\boldsymbol{\omega}) \left[\mathbf{X}^T(\boldsymbol{\omega})\mathbf{X}(\boldsymbol{\omega})\right]^{-1} \mathbf{X}^T(\boldsymbol{\omega})$$

is the projection matrix on the column space spanned by the matrix $\mathbf{X}(\boldsymbol{\omega})$ for a given $\boldsymbol{\omega}$. Therefore, the LSE of $\boldsymbol{\omega}$ can be obtained by minimizing (2.5) with respect to $\boldsymbol{\omega}$. If $\hat{\boldsymbol{\omega}}$ minimizes (2.5), then the corresponding estimator of the linear parameter $\boldsymbol{\alpha}$ can be obtained as

$$\hat{\boldsymbol{\alpha}}(\hat{\boldsymbol{\omega}}) = \left[\mathbf{X}^T(\hat{\boldsymbol{\omega}})\mathbf{X}(\hat{\boldsymbol{\omega}})\right]^{-1} \mathbf{X}^T(\hat{\boldsymbol{\omega}})\mathbf{Y}. \qquad (2.6)$$

Most of the special purpose algorithms, for example the methods proposed by Bresler and Macovski (1988), Kumaresan, Scharf and Shaw (1986), Kundu and Kannan (1994) and Smyth (2000) attempt to minimize (2.5), which naturally saves computational time.

2.2. Approximate Least Squares Estimators. An alternative way to estimate the frequencies is to maximize the periodogram function. The periodogram function $I(\boldsymbol{\omega})$ can be defined as follows.

$$I(\boldsymbol{\omega}) = \left| \frac{1}{N} \sum_{t=1}^{N} y(t) e^{-i\omega t} \right|^2. \qquad (2.7)$$

The periodogram estimators obtained under the condition that frequencies are Fourier frequencies, provide estimators with convergence rate $O_p(N^{-1})$. When this condition is dropped, the estimators obtained by finding p local maxima of $I(\boldsymbol{\omega})$ achieve the best possible rate and are asymptotically equivalent to the LSEs. So it is called approximate least squares estimators (ALSEs) in the literature. A frequency λ is a Fourier frequency if it is of the form $\lambda = 2\pi k/N$, for some integer $1 \leq k \leq N/2$. Let us denote the ALSE of $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_p)$ as $\hat{\hat{\boldsymbol{\omega}}} = (\hat{\hat{\omega}}_1, \ldots, \hat{\hat{\omega}}_p)$. It is known that $\hat{\boldsymbol{\omega}}$ and $\hat{\hat{\boldsymbol{\omega}}}$ are both consistent estimators of $\boldsymbol{\omega}$ with the following asymptotic distribution:

$$N^{3/2}(\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}) \rightarrow \mathcal{N}_p\left(\mathbf{0}, 24\sigma^2\boldsymbol{\Sigma}\right), \tag{2.8}$$

$$N^{3/2}(\hat{\hat{\boldsymbol{\omega}}} - \boldsymbol{\omega}) \rightarrow \mathcal{N}_p\left(\mathbf{0}, 24\sigma^2\boldsymbol{\Sigma}\right), \tag{2.9}$$

where $\boldsymbol{\Sigma}$ is $p \times p$ diagonal matrix as follows.

$$\boldsymbol{\Sigma} = diag\left[\frac{c_1}{\rho_1^2}, \ldots, \frac{c_p}{\rho_p^2}\right]. \tag{2.10}$$

Here

$$c_j = \left|\sum_{k=-\infty}^{\infty} a(k) e^{ik\omega_j}\right|^2 \qquad \text{and} \qquad \rho_j^2 = A_j^2 + B_j^2.$$

The LSEs of A_j and B_j are estimated using (2.6) and ALSEs, say $\hat{\hat{A}}_j$ and $\hat{\hat{B}}_j$ that are obtained as

$$\hat{\hat{A}}_j = \frac{2}{N}\sum_{t=1}^{N} y(t)\cos(\hat{\hat{\omega}}_j t), \quad \hat{\hat{B}}_j = \frac{2}{N}\sum_{t=1}^{N} y(t)\sin(\hat{\hat{\omega}}_j t). \tag{2.11}$$

These expressions are used in Section 4.2 to estimate the linear parameters. Thus, according to (2.8) and (2.9), both the LSEs and the ALSEs produce frequency estimators which have convergence rate $O_p(N^{-3/2})$. In this paper, we have considered the efficient estimation of non-linear frequency parameters, and we do not discuss anything related to the rate of convergence of the linear parameters. In the next section, we describe a method which produces frequency estimators and which have the same convergence rate as the LSEs or the ALSEs.

3 Proposed Algorithm

Given a consistent estimator $\tilde{\omega}_j$, we compute $\hat{\omega}_j$ using (3.1) for $j = 1, \ldots, p$ as follows.

$$\hat{\omega}_j = \tilde{\omega}_j + \frac{12}{N^2} \operatorname{Im}\left[\frac{P_N(j)}{Q_N(j)}\right], \tag{3.1}$$

where

$$P_N(j) = \sum_{t=1}^{N} y(t)\left(t - \frac{N}{2}\right) e^{-i\tilde{\omega}_j t}, \tag{3.2}$$

$$Q_N(j) = \sum_{t=1}^{N} y(t) e^{-i\tilde{\omega}_j t}, \tag{3.3}$$

and Im[.] denotes the imaginary part of a complex number. We can start with any consistent estimator $\tilde{\omega}_j$ and improve it step by step using (3.1). The motivation of the algorithm is based on the following theorem.

THEOREM 3.1. *If for* $j = 1, \ldots, p$, $\tilde{\omega}_j - \omega_j = O_p(N^{-1-\delta})$, *where* $\delta \in (0, 1/2]$, *then*

$$(a) \quad \hat{\omega}_j - \omega_j = O_p\left(N^{-1-2\delta}\right) \quad \text{if} \quad \delta \le 1/4, \text{ and}$$

$$(b) \quad N^{3/2}(\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}) \rightarrow \mathcal{N}_p\left(\mathbf{0}, 24\sigma^2 \boldsymbol{\Sigma}\right) \quad \text{if} \quad \delta > 1/4.$$

PROOF. See in the Appendix. □

We start with the maximizer of the periodogram over Fourier frequencies and improve it step by step by the above recursive algorithm. The m^{th} step estimator $\hat{\omega}_j^{(m)}$ is computed from the $(m-1)^{th}$ step estimator $\hat{\omega}_j^{(m-1)}$ by the formula

$$\hat{\omega}_j^{(m)} = \hat{\omega}_j^{(m-1)} + \frac{12}{N_m^2} \operatorname{Im}\left[\frac{P_{N_m}(j)}{Q_{N_m}(j)}\right], \tag{3.4}$$

where $P_{N_m}(j)$ and $Q_{N_m}(j)$ can be obtained from (3.2) and (3.3) by replacing N and $\tilde{\omega}_j$ with N_m and $\hat{\omega}_j^{(m-1)}$ respectively. We repeatedly choose N_m suitably at each step as follows.

- Step 1: With $m = 1$, choose $N_1 = N^{0.8}$ and $\hat{\omega}_j^{(0)} = \tilde{\omega}_j$, the maximizer of the periodogram estimator at the Fourier frequencies. Note that

$\tilde{\omega}_j - \omega_j = O_p(N^{-1}) = O_p(N_1^{-1-1/4})$. Substituting $N_1^{0.8}$ and $\hat{\omega}_j^{(0)} = \tilde{\omega}_j$ in (3.4), and applying part (a) of Theorem 3.1, we obtain

$$\hat{\omega}_j^{(1)} - \omega_j = O_p(N_1^{-1-1/2}) = O_p(N^{-1-1/5}).$$

- Step 2: With $m = 2$, choose $N_2 = N^{0.9}$. Compute $\hat{\omega}_j^{(2)}$ from $\hat{\omega}_j^{(1)}$. Since $\hat{\omega}_j^{(1)} - \omega_j = O_p(N^{-1-1/5}) = O_p(N_2^{-1-1/3})$ and $1/3 > 1/4$, therefore using part (b) of Theorem 3.1 , we have

$$\hat{\omega}_j^{(2)} - \omega_j = O_p(N_2^{-3/2}) = O_p(N^{-1-7/20}).$$

- Step 3: With $m = 3$, choose $N_3 = N$ and compute $\hat{\omega}_j^{(3)}$ from $\hat{\omega}_j^{(2)}$ and apply part (b) of Theorem 3.1 again, we have

$$N^{3/2}\left(\hat{\boldsymbol{\omega}}^{(3)} - \boldsymbol{\omega}\right) \to \mathcal{N}_p(\mathbf{0}, 24\sigma^2\boldsymbol{\Sigma}).$$

Therefore, it is observed that if at any step, the estimator is of the order $O_p(N^{-1-\delta})$, then the method provides an estimator which improves the order to $O_p(N^{-1-2\delta})$ if $\delta \le 1/4$, and if $1/4 < \delta \le 1/2$, then it provides the efficient estimator. We obtain the initial estimator by maximizing the perodogram function, defined in Section 2, under the condition that the frequencies are Fourier frequencies. This way, using varying sample sizes, we get an estimator with rate of convergence $O_p(N^{-1-\delta})$ for some $\delta \in (0, 1/2]$. This can then be used as an initial estimator because the Theorem 3.1 needs a starting value of order $O_p(N^{-1-\delta})$ to work. With the increasing number of iteration, more and more data points are used to obtain an efficient estimator. The method provides an efficient frequency estimator from the relatively poor initial estimate of the periodogram maximizer. We would like to mention it again that the initial estimator used here is not the ALSE (ALSE is obtained without any constraint on the frequencies) and so is not asymptotically equivalent to the LSE.

For multiple sinusoidal model ($p \ge 2$), the LSEs of ω_j and ω_k for $j \ne k$ are asymptotically independent. We observe the same for the proposed estimators also, and the proposed algorithm does not involve joint estimation of the frequencies in case $p > 1$. This is due to the fact that the correction factor due to the frequency ω_j is $\frac{12}{N^2} Im\left[\frac{P_N(j)}{Q_N(j)}\right]$, which does not depend on ω_j.

There are several iterative procedures available in the literature to estimate the parameters of sinusoidal frequency model. In the proposed algorithm, the novelty lies in its implementation. Basically, it is an iterative scheme, which needs starting estimates of order larger than $O_p(N^{-1})$ to work. But the existing non-iterative procedures and the periodogram maximizer at Fourier frequencies give estimates of orders $O_p(N^{-1/2})$ and $O_p(N^{-1})$ respectively. So, following the theory, one cannot use them directly as starting estimates. To overcome this problem, we increase the sample size step by step, in each iteration, starting from a sub series of the observed time series to N, the available data points. We would like to mention here that this varying sample size technique may be used in some other existing algorithms, and thus one may have a bound in the number of iterations required.

4 Numerical Experiments and Data Analysis

4.1. Numerical Experiments. In this section we present some numerical results to observe how the proposed method works for different sample sizes and for different error variances. We use the random deviate generator RAN2 of Press et al. (1993). All the programs are written in FORTRAN and they can be obtained from the corresponding author on request. For comparison purposes, we consider two different models.

- Model 1: $y(t) = 2\cos(0.5t + \pi/4) + X(t)$.
- Model 2: $y(t) = 2\cos(0.5t + \pi/4) + 2\cos(1.5t + \pi/3) + X(t)$.

In both cases, the error random variable $X(t)$ is of the form

$$X(t) = \epsilon(t) + 0.75\epsilon(t-1),$$

where $\{\epsilon(t)\}$ is a sequence of i.i.d. Gaussian random variables with mean zero and variance σ^2. Note that Models 1 and 2 are equivalent to (1.1).

We consider N = 100, 200, 300, 400, 500 and σ = 0.25, 0.50, 0.75, 1.00. For each sample, we estimate the frequency/frequencies based on the proposed method and using the optimization algorithm described in the Numerical Recipes (Press et al., 1993), and from now on, we will name this algorithm as the NR algorithm. In all cases, we consider the initial guesses as the periodogram maximizer at the Fourier frequencies. For NR algorithm, we also use the true parameter values as the initial guesses. In Tables 1 and 2, NR-1 and NR-2 represent the NR algorithms when the initial guesses are the periodogram maximizers at the Fourier frequencies and the true values respectively.

TABLE 1. THE AVERAGE ESTIMATES AND THE CORRESPONDING SQUARE ROOT OF THE MEAN SQUARED ERRORS OF THE FREQUENCY BASED ON 1000 REPLICATIONS ARE REPORTED WITHIN BRACKETS BELOW FOR MODEL 1. THE TRUE PARAMETER VALUES AND THE SQUARE ROOT OF THE ASYMPTOTIC STANDARD DEVIATIONS ARE REPORTED WITHIN BRACKETS BELOW.

σ	Methods	N = 100	N = 200	N = 300	N = 400	N = 500
	Proposed	0.499859	0.500032	0.500047	0.500025	0.500015
		(1.0202e-3)	(3.6663e-4)	(2.0306e-4)	(1.3188e-4)	(9.4561e-5)
	NR-1	0.499982	0.499985	0.500002	0.500023	0.500011
0.25		(1.0209e-3)	(3.6691e-4)	(1.9582e-4)	(2.6448e-4)	(1.7661e-4)
	NR-2	0.499982	0.499985	0.500002	0.499995	0.499999
		(1.0209e-3)	(3.6640e-4)	(1.9561e-4)	(1.2559e-4)	(8.9925e-5)
	Parameter	0.50000	0.50000	0.50000	0.50000	0.50000
		(1.0390e-3)	(3.6735e-4)	(1.9996e-4)	(1.2988e-4)	(9.2933e-5)
	Proposed	0.500004	0.500056	0.500067	0.500028	0.500020
		(2.0303e-3)	(7.3551e-4)	(3.9971e-4)	(2.6045e-4)	(1.8794e-4)
	NR-1	0.499963	0.499974	0.500008	0.500041	0.500026
0.50		(2.0557e-3)	(7.3874e-4)	(4.0039e-4)	(4.1028e-4)	(3.2738e-4)
	NR-2	0.499962	0.499968	0.500005	0.499986	0.499999
		(2.0540e-3)	(7.3523e-4)	(3.9090e-4)	(2.4881e-4)	(1.7869e-4)
	Parameter	0.50000	0.50000	0.50000	0.50000	0.50000
		(2.0781e-3)	(7.3470e-4)	(3.9992e-4)	(2.5976e-4)	(1.8587e-4)
	Proposed	0.500144	0.500081	0.500087	0.500032	0.500024
		(3.0846e-3)	(1.1119e-3)	(5.9713e-4)	(3.8959e-4)	(2.8199e-4)
	NR-1	0.499944	0.499968	0.500031	0.500080	0.500110
0.75		(3.1072e-3)	(1.1176e-3)	(6.1652e-4)	(5.7175e-4)	(5.6810e-4)
	NR-2	0.499941	0.499955	0.500012	0.499982	0.500000
		(3.1073e-3)	(1.1039e-3)	(5.8358e-4)	(3.7506e-4)	(2.6569e-4)
	Parameter	0.50000	0.50000	0.50000	0.50000	0.50000
		(3.1171e-3)	(1.1021e-3)	(5.9988e-4)	(3.8964e-4)	(2.7880e-4)
	Proposed	0.500274	0.500103	0.500107	0.500034	0.500028
		(4.2552e-3)	(1.5036e-3)	(7.9654e-4)	(5.1962e-4)	(3.7739e-4)
	NR-1	0.499924	0.499954	0.500042	0.500092	0.500170
1.00		(4.1922e-3)	(1.4924e-3)	(8.2129e-4)	(7.3619e-4)	(8.8899e-4)
	NR-2	0.499915	0.499950	0.500009	0.499984	0.499992
		(4.1932e-3)	(1.4732e-3)	(7.7078e-4)	(5.0181e-4)	(3.4860e-4)
	Parameter	0.50000	0.50000	0.50000	0.50000	0.50000
		(4.1561e-3)	(1.4694e-3)	(7.9984e-4)	(5.1951e-4)	(3.7173e-4)

We replicate the procedure 1000 times and report the average estimates of the frequencies and the corresponding square root of the mean squared errors. For Model 1, the results are reported in Table 1. For Model 2, the results are reported in Tables 2 and 3 for the frequencies 1 and 2 respectively.

Table 2. The results for Model 2 and Frequency 1. The average estimates and the corresponding square root of the mean squared errors based on 1000 replications are reported within brackets below for different methods. The true parameter values and the square root of the asymptotic standard deviations are reported within brackets below.

σ	Methods	N = 100	N = 200	N = 300	N = 400	N = 500
0.25	Proposed	0.498501 (1.8158e-3)	0.499772 (4.3695e-4)	0.499956 (2.0758e-4)	0.499974 (1.3586e-4)	0.499984 (9.7147e-5)
	NR-1	0.500752 (2.0860e-3)	0.501329 (2.1688e-3)	0.501600 (1.7190e-3)	0.501760 (1.8096e-3)	0.501521 (1.6729e-3)
	NR-2	0.500004 (8.3956e-4)	0.500006 (2.3704e-4)	0.500003 (1.1027e-4)	0.499999 (6.7066e-5)	0.500001 (4.6175e-5)
	Parameter	0.50000 (1.0390e-3)	0.50000 (3.6735e-4)	0.50000 (1.9996e-4)	0.50000 (1.2988e-4)	0.50000 (9.2933e-5)
0.50	Proposed	0.498758 (2.4060e-3)	0.499829 (7.6718e-4)	0.499992 (4.0428e-4)	0.499988 (2.6662e-4)	0.499995 (1.9149e-4)
	NR-1	0.500482 (2.2384e-3)	0.501004 (1.9270e-3)	0.501658 (1.7658e-3)	0.501770 (1.8255e-3)	0.501533 (1.7268e-3)
	NR-2	0.499987 (1.6752e-3)	0.499993 (4.6894e-4)	0.500003 (2.2325e-4)	0.499998 (1.4013e-4)	0.500002 (9.3547e-5)
	Parameter	0.50000 (2.0781e-3)	0.50000 (7.3470e-4)	0.50000 (3.9992e-4)	0.50000 (2.5976e-4)	0.50000 (1.8587e-4)
0.75	Proposed	0.499025 (3.2762e-3)	0.499884 (1.1363e-3)	0.500029 (6.0565e-4)	0.500001 (3.9899e-4)	0.500005 (2.8746e-4)
	NR-1	0.500334 (2.9161e-3)	0.500938 (1.9315e-3)	0.501680 (1.7624e-3)	0.501781 (1.8509e-3)	0.501343 (1.7870e-3)
	NR-2	0.499989 (2.5281e-3)	0.499997 (7.2823e-4)	0.500007 (3.4500e-4)	0.499999 (2.1034e-4)	0.500001 (1.4190e-4)
	Parameter	0.50000 (3.1171e-3)	0.50000 (1.1021e-3)	0.50000 (5.9988e-4)	0.50000 (3.8964e-4)	0.50000 (2.7880e-4)
1.00	Proposed	0.499299 (4.3669e-3)	0.499936 (1.5323e-3)	0.500066 (8.0922e-4)	0.500013 (5.3199e-4)	0.500015 (3.8481e-4)
	NR-1	0.500227 (3.6166e-3)	0.500950 (1.9464e-3)	0.501689 (1.7830e-3)	0.501776 (1.9094e-3)	0.501115 (1.9016e-3)
	NR-2	0.499914 (3.3998e-3)	0.500001 (9.7823e-4)	0.500019 (4.9387e-4)	0.499998 (2.7539e-4)	0.500002 (1.9273e-4)
	Parameter	0.50000 (4.1561e-3)	0.50000 (1.4694e-3)	0.50000 (7.9984e-4)	0.50000 (5.1951e-4)	0.50000 (3.7173e-4)

We also report the true parameter values and the corresponding asymptotic standard deviations of the LSEs for comparison purposes.

TABLE 3. THE RESULTS FOR MODEL 2 AND FREQUENCY 2.
THE AVERAGE ESTIMATES AND THE CORRESPONDING SQUARE ROOT OF THE MEAN SQUARED ERRORS BASED ON 1000 REPLICATIONS ARE REPORTED WITHIN BRACKETS BELOW FOR DIFFERENT METHODS. THE TRUE PARAMETER VALUES AND THE SQUARE ROOT OF THE ASYMPTOTIC STANDARD DEVIATIONS ARE REPORTED WITHIN BRACKETS BELOW.

σ	Methods	N = 100	N = 200	N = 300	N = 400	N = 500
	Proposed	1.500400	1.500302	1.500098	1.499993	1.499978
		(8.9044e-4)	(4.1304e-4)	(1.8984e-4)	(1.0151e-4)	(7.6678e-5)
	NR-1	1.502041	1.501075	1.498575	1.500006	1.500966
0.25		(2.9637e-3)	(3.2403e-3)	(1.5957e-3)	(2.7695e-4)	(1.0921e-3)
	NR-2	1.499981	1.500011	1.500005	1.500000	1.499999
		(6.2126e-4)	(1.8723e-4)	(8.3067e-5)	(5.1740e-5)	(3.5315e-5)
	Parameter	1.50000	1.50000	1.50000	1.50000	1.50000
		(7.9103e-4)	(2.7967e-4)	(1.5223e-4)	(9.8879e-5)	(7.07518e-5)
	Proposed	1.500557	1.500356	1.500116	1.500001	1.499987
		(1.7092e-3)	(6.7421e-4)	(3.4650e-4)	(2.0325e-4)	(1.4821e-4)
	NR-1	1.501921	1.501033	1.498587	1.500020	1.501004
0.50		(4.3563e-3)	(4.8358e-3)	(2.2669e-3)	(3.8784e-4)	(1.5696e-3)
	NR-2	1.500025	1.500028	1.500005	1.500001	1.499998
		(1.4955e-3)	(4.4035e-4)	(1.9885e-4)	(1.2016e-4)	(7.9629e-5)
	Parameter	1.50000	1.50000	1.50000	1.50000	1.50000
		(1.5821e-3)	(5.5934e-4)	(3.0447e-4)	(1.9776e-4)	(1.4150e-4)
	Proposed	1.500722	1.500417	1.500133	1.500008	1.499995
		(2.5747e-3)	(9.6144e-4)	(5.1238e-4)	(3.0629e-4)	(2.2244e-4)
	NR-1	1.502171	1.500854	1.498706	1.500039	1.501042
0.75		(5.6615e-3)	(6.2034e-3)	(2.7460e-3)	(4.6992e-4)	(1.9444e-3)
	NR-2	1.499998	1.500037	1.500007	1.499997	1.499998
		(2.5762e-3)	(7.4791e-4)	(3.4162e-4)	(2.0314e-4)	(1.3635e-4)
	Parameter	1.50000	1.50000	1.50000	1.50000	1.50000
		(2.3731e-3)	(8.3901e-4)	(4.5670e-4)	(2.9664e-4)	(2.1226e-4)
	Proposed	1.500887	1.500475	1.500153	1.500014	1.500004
		(3.4886e-3)	(1.2648e-3)	(6.8424e-4)	(4.1088e-4)	(2.9817e-4)
	NR-1	1.502087	1.500857	1.498709	1.500022	1.501047
1.00		(7.0993e-3)	(7.5094e-3)	(3.1600e-3)	(5.4387e-4)	(2.2575e-3)
	NR-2	1.499982	1.500044	1.500005	1.499997	1.499998
		(3.8856e-3)	(1.1272e-3)	(5.2067e-4)	(3.0229e-4)	(2.0151e-4)
	Parameter	1.50000	1.50000	1.50000	1.50000	1.50000
		(3.1641e-3)	(1.1187e-3)	(6.0894e-4)	(3.9552e-4)	(2.8301e-4)

The following points are observed from this experiment. In most of the cases, for Model 1, all methods work almost in an identical manner. The square root of the mean squared errors in most of the cases are quite close to the asymptotic standard deviations. Interestingly, for sample sizes 400 and 500, the square root of the mean squared errors are quite higher than the corresponding asymptotic standard deviations for the NR-1 algorithm. Since it is known (see Rice and Rosenblatt, 1988) that for large sample sizes the least square surface has several local minima, it seems that there are local minima close to the periodogram maximizers and the conventional algorithm converges to some local minimums rather than the global minimum. On the other hand, it is clear from the mean squared errors that starting from the true values, NR algorithm converges to the global minima. For Model 2, the performances of the proposed and NR-1 algorithms are quite different. Although the initial guesses are same for both of them, the performance of the proposed algorithm is much better than that of the NR-1 algorithm in terms of the lower mean squared errors. The performance of the NR-2 algorithm is very good, having slightly lower mean squared errors than that of the proposed one in several cases. It is not surprising because it uses the true values as the initial guesses. Moreover, the performance of NR-2 algorithm indicates that NR algorithm works very well provided the starting estimates are quite close to the true parameter values. The proposed algorithm produces fast and efficient frequency estimators quite effectively from the periodogram estimators even when multiple sinusoids are present.

4.2. Data Analysis. In this subsection, we illustrate the proposed algorithm with three real data examples. We analyze the widely used variable star data and two short duration voiced speech signals. We estimate the frequencies using the algorithm described in Section 3, and the linear parameters are then estimated by approximate least square technique using the frequency estimates (as given in (2.11)). Two voiced speech data, namely "eee" and "uuu" are analyzed. In each data set, 512 signal values sampled at 10 kHz frequency are available. We have used the mean corrected data in each case for this analysis. The data sets "eee" and "uuu" are displayed in Figures 1 and 2, and their periodogram functions in Figures 3 and 4.

The plots in Figures 1 and 2 suggest that the signals are non-stationary and there exists strong periodicity. The number of components are estimated from the peridogram plots. For both data sets, we estimate p as 4. As described in Section 3, the periodogram maximizer at the Fourier frequencies $2\pi j/N$, $j = 1, \ldots, N$ are used as initial estimates for the frequencies. Then, using the three-step algorithm, we first estimate the frequencies and then the linear parameters. The estimated parameters $(\hat{A}_k, \hat{B}_k, \hat{\omega}_k), k = 1, \ldots, 4$ for both "eee" and "uuu" data sets are listed in Table 4. Using these point

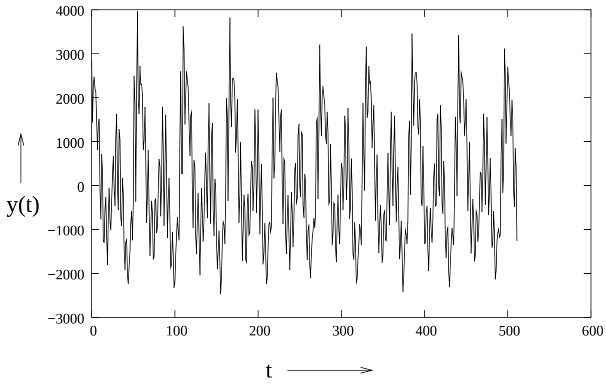

Figure 1. The plot of the observed "eee" sound.

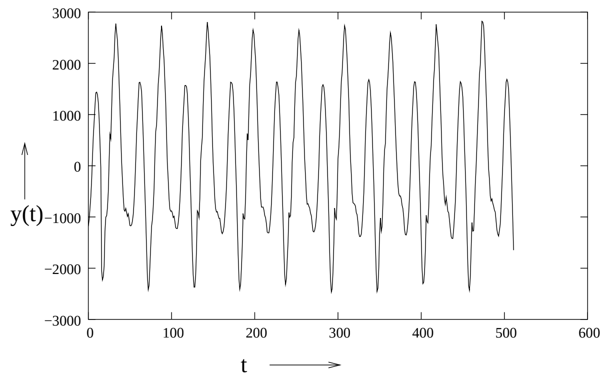

Figure 2. The plot of the observed "uuu" sound.

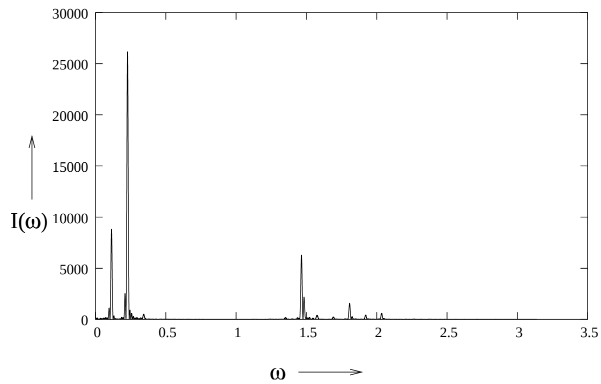

Figure 3. The periodogram function of "eee" sound.

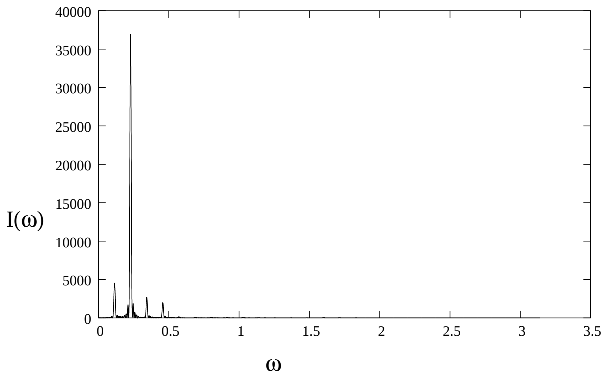

Figure 4. The periodogram function of “uuu” sound.

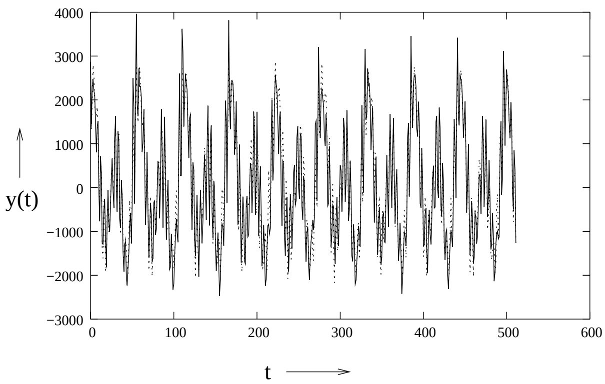

Figure 5. The observed (solid line) and estimated values (dotted line) of “eee” data.

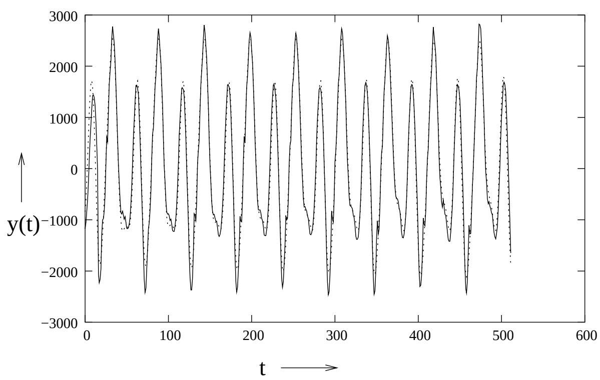

Figure 6. The observed “uuu” data (solid line) and its estimated values (dotted line).

TABLE 4. THE PARAMETER ESTIMATES FOR DIFFERENT DATA SETS.

		DATA SET: "EEE"			
A_1	893.430237	B_1	1115.15039	ω_1	0.227799729
A_2	611.317688	B_2	511.464569	ω_2	0.114047945
A_3	575.300537	B_3	401.475098	ω_3	1.46539402
A_4	347.595306	B_4	-51.168354	ω_4	1.80787051
		DATA SET: "UUU"			
A_1	8.92728806	B_1	1698.65247	ω_1	0.228173435
A_2	-584.679871	B_2	-263.790344	ω_2	0.112697937
A_3	-341.408905	B_3	-282.075409	ω_3	0.343263
A_4	-193.936096	B_4	-300.509613	ω_4	0.4577021
		DATA SET: "STAR.DAT"			
A_1	7.48262215	B_1	7.46288395	ω_1	0.216232359
A_2	-1.85116708	B_2	6.75062132	ω_2	0.261817634
A_3	-0.807285547	B_3	0.0688061789	ω_2	0.213608354

estimates, we obtain the predicted signal for both data sets. The predicted signals (solid line) along with observed (dotted line) data sets are plotted in Figures 5 and 6 for "eee" and "uuu" data sets respectively. The fitted values match quite well with the observed series.

Next, we consider an astronomical data set, which represents the daily brightness of a variable star on 600 successive midnights. The data is collected from Time Series Library of StatLib (http://www.stat.cmu.edu; Source: Rob J. Hyndman). The observed data is plotted in Figure 7, and its periodogram function in Figure 8. This is a well-known data set used in the study of multiple frequency model (1.1). From the periodogram plot, it seems that number of components $p = 2$. However, with $\hat{p} = 2$, we see that the periodogram plot of the residual series gives evidence of another significant component (plot is not provided here). This third component is not visible in the periodogram plot of the original series (Figure 8), as the first two frequencies are dominant due to the large absolute values of the associated amplitudes. Also, the first one is quite close to the third one as compared to the available data points to distinguish them. So we have considered $\hat{p} = 3$ and then estimated the other parameters. The parameter estimates of this data set, $(\hat{A}_k, \hat{B}_k, \hat{\omega}_k), k = 1, \ldots, 3$ are also given in Table 4. Similarly as before, the observed (solid line) and the estimated values (dotted line) are plotted in Figure 9. In Figure 9, it is not possible to distinguish the estimated one from the observed series. So the performance of the developed algorithm is quite good for the data sets considered here for the analysis.

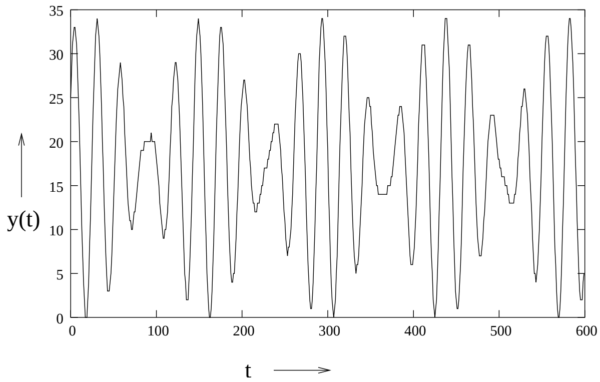

Figure 7. The plot of variable star data.

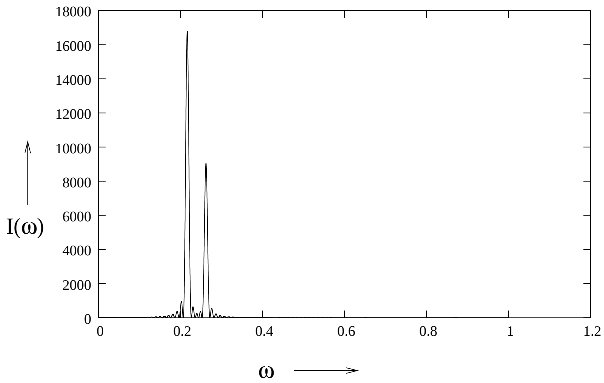

Figure 8. The periodogram function of variable star data.

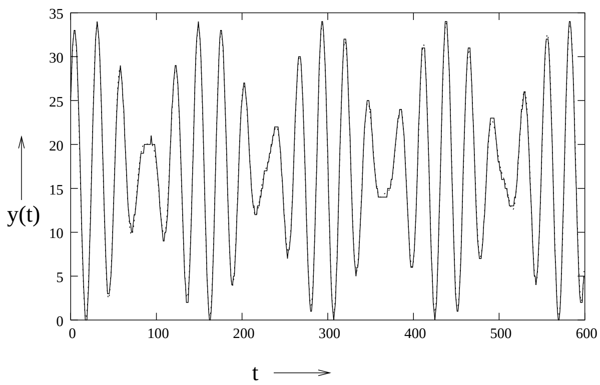

Figure 9. The observed variable star data (solid line) and its estimated values (dotted line).

5 Conclusions

In this paper, we propose a new 3-step iterative procedure to estimate the frequencies of sinusoidal signals from an initial estimator, which has the rate of convergence $O_p(N^{-1})$. There are several algorithms available in the literature for estimating the parameters of sinusoidal frequencies. It is known that all the non-iterative algorithms produce estimators which have convergence rate $O_p(N^{-1/2})$ and most of the iterative algorithms have the convergence rate $O_p(N^{-3/2})$. The performance of any iterative algorithm heavily depends on the stopping criterion as well as on the maximum number of iterations. The estimators, which can be obtained by the proposed 3-step procedure, do not have those deficiencies. The performance of the estimators is also quite good and for multiple sinusoids the proposed one is better than some of the existing estimators. The algorithm is basically a iterative algorithm, but at the same time since the number of steps is fixed, it can be implemented like non-iterative procedures, and so it can easily be used for on-line implementation purposes.

Finally, we should mention that our proposed method works, unlike any existing methods, even if the initial estimators have the convergence rate $O_p(N^{-1/2})$. In this case, the proposed method takes seven steps to converge to the estimator which has the optimum convergence rate $O_p(N^{-3/2})$.

Appendix

PROOF OF THEOREM 3.1.

$$
\begin{aligned}
Q_N(j) &= \sum_{j=1}^{N} y(t) e^{-i\tilde{\omega}_j t} \\
&= \sum_{t=1}^{N}\left[\sum_{k=1}^{p}\{A_k\cos(\omega_k t) + B_k\sin(\omega_k t)\} + X(t)\right] e^{-i\tilde{\omega}_j t} \\
&= \sum_{t=1}^{N}\left[\sum_{k=1}^{p}\left\{\frac{A_k}{2}\left(e^{i\omega_k t} + e^{-i\omega_k t}\right) + \frac{B_k}{2i}\left(e^{i\omega_k t} - e^{-i\omega_k t}\right)\right\} + X(t)\right] e^{-i\tilde{\omega}_j t} \\
&= \left[\sum_{k=1}^{p}\left(\frac{A_k}{2} + \frac{B_k}{2i}\right)\sum_{t=1}^{N} e^{i(\omega_k - \tilde{\omega}_j)t}\right] + \left[\sum_{k=1}^{p}\left(\frac{A_k}{2} - \frac{B_k}{2i}\right)\sum_{t=1}^{N} e^{-i(\omega_k + \tilde{\omega}_j)t}\right] \\
&\quad + \sum_{t=1}^{N} X(t) e^{-i\tilde{\omega}_j t}.
\end{aligned}
$$

Now we would like to study the behavior of $\sum_{t=1}^{N} e^{-i(\tilde{\omega}_j \pm \omega_k)t}$ for different k and j, when $\tilde{\omega}_j - \omega_j = O_p(N^{-1-\delta})$. Note that

$$\begin{aligned}\sum_{t=1}^{N} e^{-i(\omega_k + \tilde{\omega}_j)t} &= O_p(1) \qquad \text{for all } k, j = 1, \dots p,\\ \sum_{t=1}^{N} e^{i(\omega_k - \tilde{\omega}_j)t} &= O_p(1) \qquad \text{for all } k \neq j = 1, \dots p,\end{aligned}$$

and

$$\begin{aligned}\sum_{t=1}^{N} e^{i(\omega_j - \tilde{\omega}_j)t} &= N + i(\omega_j - \tilde{\omega}_j)\sum_{t=1}^{N} e^{i(\omega_j - \omega_j^*)t}\\ &= N + O_p(N^{-1-\delta})O_p(N^2)\\ &= N + O_p(N^{1-\delta}),\end{aligned}$$

where ω_j^* is a point between ω_j and $\tilde{\omega}_j$. Choose L large enough such that $L\delta > 1$. Therefore, using Taylor series approximation of $e^{-i\tilde{\omega}_j t}$ up to the L-th order terms

$$\begin{aligned}\sum_{t=1}^{N} X(t)e^{-i\tilde{\omega}_j t} &= \sum_{k=-\infty}^{\infty} a(k)\sum_{t=1}^{N} e(t-k)e^{-i\tilde{\omega}_j t}\\ &= \sum_{k=-\infty}^{\infty} a(k)\sum_{t=1}^{N} e(t-k)e^{-i\omega_j t}\\ &\quad + \sum_{k=-\infty}^{\infty} a(k)\sum_{l=1}^{L-1}\frac{(-i(\tilde{\omega}_j - \omega_j))^l}{l!}\sum_{t=1}^{N} e(t-k)t^l e^{-i\omega_j t}\\ &\quad + \sum_{k=-\infty}^{\infty} a(k)\frac{\theta(N(\tilde{\omega}_j - \omega_j))^L}{L!}\sum_{t=1}^{N} |e(t-k)|,\end{aligned}$$

where $|\theta| < 1$. Since $\sum_{k=-\infty}^{\infty} |a(k)| < \infty$,

$$\begin{aligned}\sum_{t=1}^{N} X(t)e^{-i\tilde{\omega}_j t} &= O_p(N^{1/2}) + \sum_{l=1}^{L-1}\frac{O_p(N^{-(1+\delta)l})}{l!}O_p(N^{l+1/2})\\ &\quad + O_p\left((N.N^{-1-\delta})^L.N\right)\\ &= O_p(N^{1/2}) + O_p(N^{1/2+\delta-L\delta}) + O_p(N^{1-L\delta}) = O_p(N^{1/2}).\end{aligned}$$

Therefore,

$$\begin{aligned} Q_N(j) &= \left(\frac{A_j}{2} + \frac{B_j}{2i}\right)\left(N + O_p(N^{1-\delta})\right) + O_p(1) + O_p(N^{1/2}) \\ &= \frac{N}{2}\left[(A_j - iB_j) + O_p(N^{-\delta})\right] \end{aligned}$$

as $\delta \in (0, 1/2]$.

$$\begin{aligned} P_N(j) &= \sum_{t=1}^{N} y(t)\left(t - \frac{N}{2}\right) e^{-i\tilde{\omega}_j t} \\ &= \frac{1}{2}\sum_{k=1}^{p} A_k \sum_{t=1}^{N}\left(t - \frac{N}{2}\right)\left[e^{i(\omega_k - \tilde{\omega}_j)t} + e^{-i(\omega_k + \tilde{\omega}_j)t}\right] \\ &\quad + \frac{1}{2i}\sum_{k=1}^{p} B_k \sum_{t=1}^{N}\left(t - \frac{N}{2}\right)\left[e^{i(\omega_k - \tilde{\omega}_j)t} - e^{-i(\omega_k + \tilde{\omega}_j)t}\right] \\ &\quad + \sum_{t=1}^{N} X(t)\left(t - \frac{N}{2}\right) e^{-i\tilde{\omega}_j t} \\ &= \sum_{k=1}^{p}\left[\left(\frac{A_k}{2} + \frac{B_k}{2i}\right)\sum_{t=1}^{N}\left(t - \frac{N}{2}\right) e^{i(\omega_k - \tilde{\omega}_j)t}\right] \\ &\quad + \sum_{k=1}^{p}\left[\left(\frac{A_k}{2} - \frac{B_k}{2i}\right)\sum_{t=1}^{N}\left(t - \frac{N}{2}\right) e^{-i(\omega_k + \tilde{\omega}_j)t}\right] \\ &\quad + \sum_{t=1}^{N} X(t)\left(t - \frac{N}{2}\right) e^{-i\tilde{\omega}_j t}. \end{aligned}$$

Since for $k, j = 1, \ldots, p$

$$\begin{aligned} \sum_{t=1}^{N}\left(t - \frac{N}{2}\right) e^{-i(\omega_k + \tilde{\omega}_j)t} &= O_p(N) \quad \text{for all} \quad k, j, \\ \sum_{t=1}^{N}\left(t - \frac{N}{2}\right) e^{i(\omega_k - \tilde{\omega}_j)t} &= O_p(N) \quad \text{for} \quad k \neq j, \end{aligned}$$

and

$$\begin{aligned}
&\sum_{t=1}^{N}\left(t-\frac{N}{2}\right)e^{i(\omega_j-\tilde{\omega}_j)t} \\
&= \sum_{t=1}^{N}\left(t-\frac{N}{2}\right)+i(\omega_j-\tilde{\omega}_j)\sum_{t=1}^{N}\left(t-\frac{N}{2}\right)t \\
&\quad -\frac{(\omega_j-\tilde{\omega}_j)^2}{2!}\sum_{t=1}^{N}\left(t-\frac{N}{2}\right)t^2 \\
&\quad -\frac{i(\omega_j-\tilde{\omega}_j)^3}{3!}\sum_{t=1}^{N}\left(t-\frac{N}{2}\right)t^3e^{i(\omega_j-\omega_j^*)t} \\
&= \frac{N}{2}+i(\omega_j-\tilde{\omega}_j)\frac{N(N+1)(N+2)}{12} \\
&\quad -\frac{(\omega_j-\tilde{\omega}_j)}{24}N^2(N+1)(N+2)O_p(N^{-1-\delta}) \\
&\quad -\frac{i}{6}(\omega_j-\tilde{\omega}_j)O_p(N^{-2-2\delta})O_p(N^5) \\
&= O(N)+i(\omega_j-\tilde{\omega}_j)O_p(N^3)+(\omega_j-\tilde{\omega}_j)O_p(N^{3-\delta}) \\
&\quad +i(\omega_j-\tilde{\omega}_j)O_p(N^{3-2\delta}) \\
&= O(N)+i(\omega_j-\tilde{\omega}_j)N^3\left[O_p(1)+O_p(N^{-\delta})+O_p(N^{-2\delta})\right],
\end{aligned}$$

where, ω_j^* is a point between ω_j and $\tilde{\omega}_j$. Moreover,

$$\begin{aligned}
&\sum_{t=1}^{N}X(t)\left(t-\frac{N}{2}\right)e^{-i\tilde{\omega}_jt} \\
&= \sum_{k=-\infty}^{\infty}a(k)\sum_{t=1}^{N}e(t-k)\left(t-\frac{N}{2}\right)e^{-i\tilde{\omega}_jt} \\
&= \sum_{k=-\infty}^{\infty}a(k)\sum_{t=1}^{N}e(t-k)\left(t-\frac{N}{2}\right)e^{-i\omega_jt} \\
&\quad +\sum_{k=-\infty}^{\infty}a(k)\sum_{l=1}^{L-1}\frac{(-i(\tilde{\omega}_j-\omega_j))^l}{l!}\sum_{t=1}^{N}t^l\left(t-\frac{N}{2}\right)e(t-k)e^{-i\omega_jt} \\
&\quad +\sum_{k=-\infty}^{\infty}a(k)\frac{\theta(N(\tilde{\omega}_j-\omega_j))^L}{L!}\sum_{t=1}^{N}\left(t-\frac{N}{2}\right)|e(t-k)| \\
&\qquad \text{(here } |\theta|\le 1)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{\infty} a(k) \sum_{t=1}^{N} e(t-k)\left(t-\frac{N}{2}\right) e^{-i\omega_j t} \\
&\quad + \sum_{k=-\infty}^{\infty} a(k) \sum_{l=1}^{L-1} \frac{O_p(N^{-(1+\delta)l})}{l!} O_p(N^{l+3/2}) \\
&\quad + \sum_{k=-\infty}^{\infty} a(k) O_p(N^{-L\delta}) O_p(N^{5/2}) \\
&= \sum_{k=-\infty}^{\infty} a(k) \sum_{t=1}^{N} e(t-k)\left(t-\frac{N}{2}\right) e^{-i\omega_j t} \\
&\quad + \sum_{k=-\infty}^{\infty} a(k) O_p(N^{5/2-L\delta}).
\end{aligned}
$$

$$
\begin{aligned}
P_N(j) &= \sum_{k=-\infty}^{\infty} a(k) \sum_{t=1}^{N} e(t-k)\left(t-\frac{N}{2}\right) e^{-i\omega_j t} \\
&\quad + \sum_{k=-\infty}^{\infty} a(k) O_p(N^{5/2-L\delta}) \\
&\quad - i\frac{N^3}{24}(\tilde{\omega}_j - \omega_j)(A_j - iB_j)(1 + O_p(N^{-\delta})).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\hat{\omega}_j &= \tilde{\omega}_j + \frac{12}{N^2}\ \mathrm{Im}\left(\frac{P_N(j)}{Q_N(j)}\right) \\
&= \tilde{\omega}_j + \frac{12}{N^2}\ \mathrm{Im}\left[\frac{\sum_{k=-\infty}^{\infty} a(k) \sum_{t=1}^{N} e(t-k)\left(t-\frac{N}{2}\right) e^{-i\omega_j t} + O_p(N^{5/2-L\delta})}{\frac{N}{2}\left[(A_j - iB_j) + O_p(N^{-\delta})\right]}\right. \\
&\qquad \left. \frac{-i\frac{N^3}{24}(\tilde{\omega}_j - \omega_j)(A_j - iB_j)(1 + O_p(N^{-\delta}))}{\frac{N}{2}\left[(A_j - iB_j) + O_p(N^{-\delta})\right]}\right] \\
&= \omega_j + O_p(N^{-\delta})(\tilde{\omega}_j - \omega_j) + \\
&\qquad \frac{24}{N^3}\ \mathrm{Im}\left[\frac{\sum_{k=-\infty}^{\infty} a(k) \sum_{t=1}^{N} e(t-k)\left(t-\frac{N}{2}\right) e^{-i\omega_j t}}{A_j - iB_j}\right]. \quad (5.1)
\end{aligned}
$$

Now

$$
\begin{aligned}
&\operatorname{Im}\left[\frac{\sum_{k=-\infty}^{\infty} a(k) \sum_{t=1}^{N} e(t-k)\left(t-\frac{N}{2}\right) e^{-i\omega_j t}}{A_j - iB_j}\right] \\
= \quad & \frac{1}{A_j^2+B_j^2}\left[-A_j \sum_{k=-\infty}^{\infty} a(k) \sum_{t=1}^{N} e(t-k)\left(t-\frac{N}{2}\right)\sin(\omega_j t)\right] \\
& + \frac{1}{A_j^2+B_j^2}\left[B_j \sum_{k=-\infty}^{\infty} a(k) \sum_{t=1}^{N} e(t-k)\left(t-\frac{N}{2}\right)\cos(\omega_j t)\right] \\
= \quad & \frac{1}{A_j^2+B_j^2} \times \left[-A_j \sum_{k=-\infty}^{\infty} a(k) \sum_{t=1}^{N} e(t-k)\left(t-\frac{N}{2}\right)\right. \\
& \qquad \{\sin(\omega_j(t-k))\cos(k\omega_j) + \cos(\omega_j(t-k)\sin(k\omega_j)\} \\
& \qquad + B_j \sum_{k=-\infty}^{\infty} a(k) \sum_{t=1}^{N} e(t-k)\left(t-\frac{N}{2}\right) \\
& \qquad \left.\{\cos(\omega_j(t-k))\cos(k\omega_j) - \sin(\omega_j(t-k)\sin(k\omega_j)\}\right] \\
= \quad & X_j \qquad \text{(say)}.
\end{aligned}
$$

Note that

$$
\operatorname{Var}\left(\frac{24}{N^{3/2}} X_j\right) = 24\sigma^2 \frac{|\sum_{k=-\infty}^{\infty} a(k) e^{-ik\omega_j}|^2}{(A_j^2+B_j^2)} = 24\sigma^2 \frac{c_j}{\rho_j^2}; \quad j = 1, \ldots, p, \tag{5.2}
$$

and for $j \neq k$,

$$
\operatorname{Cov}\left[\frac{24}{N^{3/2}} X_j, \frac{24}{N^{3/2}} X_k\right] \to 0. \tag{5.3}
$$

Therefore, if $\tilde{\omega}_j - \omega_j = O_p(N^{-1-\delta})$ and $\delta \leq 1/4$, then from (5.1), $\hat{\omega}_j - \omega_j = O_p(N^{-1-2\delta})$. If $\delta \in (1/4, 1/2]$, then from (5.1)-(5.3) and using the Central Limit Theorem of the linear process (Fuller, 1976, page 251, see also Hannan, 1971 and Kundu, 1997), it follows that $N^{3/2}(\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}) \to \mathcal{N}_p(\mathbf{0}, 24\sigma^2\boldsymbol{\Sigma})$.

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