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Discrete Life Distributions with Decreasing Reversed Hazard

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Abstract

This paper considers the class of discrete distributions for which the distribution function is a log-concave sequence. It is shown that such distributions arise from a wide variety of circumstances in Reliability, and that these have a decreasing reversed hazard rate. After examining the closures of the class under certain key operations, sharp upper and lower bounds on the reliability function for the member distributions are given. Some useful inequalities for maintained systems are provided. Some results for the related class of discrete concave distributions are also given.

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1 Introduction

Discrete life distributions arise in several common situations in reliability theory where clock time is not the best scale to describe lifetime. A discrete life distribution is a natural choice where failure occurs only due to incoming shocks. For example, in weapons reliability, the number of rounds fired until failure is more important than age at failure (Shaked et al., 1995). Discrete lifetimes also occur through grouping or finite precision measurement of continuous time phenomena. Since there is a limit on the precision of any measurement, it can be arguably said that samples from a continuous distribution exist only in theory. However, there has been relatively less work on discrete distributions, particularly in the area of Reliability. Some parametric models for discrete life distributions have been discussed, viz. Bain (1991), Adams and Watson (1989), Xekalaki (1983) and the references therein. Roy and Gupta (1992) and Sengupta et al. (1995) considered nonparametric classes of such distributions. These distributions are applicable not only to discrete life data, but also to count of events such as repeated failures of a maintained unit.

Most of the nonparametric classes of distributions (continuous or discrete) that are commonly found in the reliability literature are based on some notion of aging. Sengupta and Nanda (1999) showed that the class of log-concave life distributions have some interesting properties, even though no aging interpretation is available for these classes. They also obtained reliability bounds and other inequalities for log-concave distributions.

DEFINITION 1.1 A distribution function F with support $[0, \infty)$ is said to be log-concave if

$$F\left(\alpha x + (1-\alpha)y\right) \geq F^{\alpha}(x).F^{1-\alpha}(y)$$

for $\alpha \in (0,1)$ and $0 \leq x, y < \infty$.

The reversed hazard rate (see Shaked and Shanthikumar, 1994) of a distribution F at the point t is defined as $(d/dt) \ln F(t)$, provided the derivative exists. If the reversed hazard rate exists, F is log-concave if and only if the reversed hazard rate is non-increasing in t. In the case of a discrete distribution F with support included in $\mathbb{N} = \{0, 1, 2, ...\}$, the reversed hazard rate $\mu_F(k)$ is defined as $(F(k) - F(k-1))/F(k)$.

DEFINITION 1.2 A discrete distribution F with support contained in $\mathbb{N}$ is said to be discrete decreasing reversed hazard (d-DRH) if the reversed hazard rate μ_F is a non-increasing sequence.

The purpose of the present paper is to explore the properties of the d-DRH class of discrete distributions. A few results obtained here happen to be similar to those obtained for continuous log-concave distributions by Sengupta and Nanda (1999) but there are many others which are qualitatively different.

The following are a few instances of d-DRH distributions arising in practice.

1. Many common discrete distributions are d-DRH for all values of the parameters. These include binomial, Poisson, geometric, hypergeometric, negative binomial, logarithmic series, hyper-Poisson, Zeta and Yule distributions.

2. Grouped data arising from samples of any log-concave life distribution are found to have d-DRH distribution (see Section 3). As noted by Sengupta and Nanda (1999), most of the parameric models of distributions in common use in reliability are log-concave.

3. In a stress-strength model where the independent and exponentially distributed stresses accumulate to cause failure and the strength has a log-concave distribution, the number of shocks causing the eventual failure has a d-DRH distribution (see Section 3).

4. Consider a maintained system where a failure is treated with instantaneous and perfect repair/replacement. Suppose the (continuous) inter-replacement times are independent with distribution F. If F is IFR, then the number of replacements till any fixed time has a d-DRH distribution (see Sengupta and Nanda, 1999).

5. Given a collection of n independent events with various probabilities, the distribution of the number of events actually taking place is d-DRH (see Sathe and Bendre, 1991).

The scope of the d-DRH class and its characterizations are given in Section 2. The results concerning relationships between d-DRH and (continuous) log-concave distributions are studied in Section 3. Section 4 deals with the closure properties of the d-DRH class under different reliability operations. The sharp reliability bound for a d-DRH distribution is given in Section 5, while some inequalities for maintained systems are given in Section 6. The sub-class of discrete concave distributions is briefly considered in Section 7.

Throughout the paper, we refer to discrete distributions with support contained in $\mathbb{N}$ as 'life distributions'. The words 'increasing' and 'decreasing' would mean 'non-decreasing' and 'non-increasing', respectively. For a distribution function F, we denote the corresponding survival function by $\bar{F}$, that is, $\bar{F}(k) = 1 - F(k-1)$.

2 Characterization and scope

We begin with two characterizations of d-DRH distributions which are easy to prove.

THEOREM 2.1 *Let K be a discrete random variable having life distribution F. The following three statements are equivalent.*

(a) *F is d-DRH.*

(b) *$F(n+k)/F(k)$ is decreasing in k for all $n \in \mathbb{N}$.*

(c) *The distribution function G_k of $k-K$ given $K \leq k$, defined as $G_k(n) = P[k-K \leq n | K \leq k]$ is stochastically increasing in k.*

Note that G_k in part (c) of Theorem 2.1 is the distribution of the *time since failure*, at time k. This is analogous to the remaining life at age k.The time since failure are of interest, for instance, in problems involving calculation of premiums for insurance which are payable at the end of the year of death.

The following lemma, stated without proof, would be useful in proving the next theorem which shows how d-DRH distributions may arise in a variety of ways.

LEMMA 2.2 *Suppose a_k and b_k for $k = 1, 2, \ldots$ are positive sequences with a_k/b_k increasing in k. Then*

$$\left(\sum_{i=1}^{k} a_i\right) \Big/ \left(\sum_{i=1}^{k} b_i\right) \quad \text{is increasing in } k.$$

THEOREM 2.3 *Let F be a discrete life distribution and the corresponding probability mass function be denoted by the sequence f_k, $k \geq 0$.*

(a) *If f_k is decreasing in k, then F is d-DRH.*

(b) *If F has decreasing failure rate (d-DFR), then it is d-DRH.*

(c) *If f_{n+k}/f_k is monotone in k for $n \in \mathbb{N}$ and $k = 1, 2, \ldots$, then F is d-DRH.*

(d) *If F has a finite number of modes, then there is an age k_0 which is less than or equal to the rightmost mode such that the distribution of the 'remaining life' at any age greater than k_0 is d-DRH.*

PROOF. Part (a) is easy to prove. Part (b) follows from part (a) by checking that f_k is decreasing in k whenever F is d-DFR.

To prove (c), let f_{k-1}/f_k be increasing in k, so that

$$\delta/f_0 < f_0/f_1 < f_1/f_2 < \ldots < f_{k-1}/f_k,$$

where δ is a small positive quantity. Hence, on using Lemma 2.2 and taking limit as δ approaches zero, we have $F(k-1)/F(k)$ increasing in k. If f_{k-1}/f_k is decreasing in k, that is, f_{n+k}/f_n is increasing in n, we have

$$f_{n+k}/f_n \le f_{m+k}/f_m, \qquad \text{for all } k > 0 \text{ and all } n \le m.$$

This means

$$f_{n+k}f_m \le f_{m+k}f_n \qquad \text{for all } k > 0 \text{ and all } n \le m.$$

Taking sum over all m ranging from n to ∞ on both sides of the above inequality, we have

$$\bar{F}(n)f_{n+k} \le f_n\bar{F}(n+k) \qquad \text{for all } k \text{ and } n,$$

that is, F is d-DFR. It follows from Part (b) above that F is d-DRH.

To prove (d), let f_k be decreasing for all $k \ge k_0$. The probability mass function of the remaining life at any age greater than k is decreasing, and the stated result follows from part (a). □

Remark 2.1 Using part (c) of the above theorem, one can easily see that the binomial, negative binomial, Poisson, hyper-Poisson, geometric, hypergeometric, logarithmic series, zeta and Yule distributions belong to the d-DRH class for all values of the parameters.

Theorem 2.3 shows how the number of certain events may have a d-DRH distribution. Yet other instances of emergence of d-DRH distributions will be given in the next section. We end this section with the observation that there is no discrete distribution with support on the entire domain $\mathbb{N}$ which has an increasing reversed hazard rate.

3 Relationship Between d-DRH and Log-concave Distributions

We begin with a stronger version of a result stated in Section 1.

Theorem 3.1 *A discrete distribution is d-DRH if and only if it is the discretized version of a continuous log-concave distribution.*

Proof. Let X be a random variable with continuous log-concave distribution function F. Let K be the integer part of X. Then

$$\frac{P(K \le n+k)}{P(K \le k)} = \frac{F(n+k+1)}{F(k+1)},$$

which is decreasing in k, since $\log F$ is concave. The reverse implication is proved by observing that for a d-DRH distribution G, the continuous distribution obtained by linearly interpolating $\log G$ in between integer points is log-concave. □

Some preliminaries are needed for the next characterization. Define, for $\lambda > 0$ and $n = 1, 2, \ldots$, the functional

$$\Gamma(\lambda, g, n) = \int_0^\infty \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} g(x) dx,$$

for any function g defined on $[0, \infty)$, for which the above integral exists. Let $\Gamma(\lambda, g, 0) = 0$ for all λ and g. The functional Γ has the property given below.

LEMMA 3.2 *If f is any density and F is the corresponding distribution function, then*

(a) $\sum_{k=1}^n \Gamma(\lambda, f, k) = \lambda \Gamma(\lambda, F, n)$.

(b) $\Gamma(\lambda, F, \cdot)$ *is a discrete distribution.*

PROOF. The proof of part (a) follows from the fact that

$$\Gamma(\lambda, f, n) = \lambda \Gamma(\lambda, F, n) - \lambda \Gamma(\lambda, F, n-1), \quad n = 1, 2, \ldots,$$

which can be proved by integration by parts.

In order to prove part (b), note that $\Gamma(\lambda, F, n)$ is increasing in n and it is bounded above by $\Gamma(\lambda, 1, n) = 1$. Therefore, it is enough to show that $\lim_{n\to\infty} \Gamma(\lambda, F, n) > 1 - \delta$ for any small and positive δ. Choose a δ, choose x_0 such that $F(x_0) > 1 - \delta/2$ and choose n_0 such that

$$\int_0^{x_0} \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} dx < \frac{\delta}{2} \qquad \text{for all } n > n_0.$$

Note that such an n_0 exists because for every fixed λ and x_0 the integral in the left hand side decreases monotonically to 0 as n goes to infinity. Define the function g by $g(x) = (1-\delta/2) I(x > x_0)$, where $I(\cdot)$ is the usual indicator function. It follows that for $n > n_0$,

$$\begin{aligned}\Gamma(\lambda, F, n) &\geq \Gamma(\lambda, g, n) \\ &= \Gamma(\lambda, 1-\delta/2, n) - (1-\delta/2) \int_0^{x_0} \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!} dx \geq (1-\delta/2)^2 > 1-\delta.\end{aligned}$$

This completes the proof. □

We are now ready for a characterization of the distribution $\Gamma(\lambda, F, \cdot)$.

THEOREM 3.3 *An absolutely continuous life distribution F is log-concave if and only if the discrete distribution $\Gamma(\lambda, F, \cdot)$ is d-DRH for all $\lambda > 0$.*

PROOF. Let the distribution function F having density f be log-concave. This means $f(x)/F(x)$ is decreasing in x, and that $f - \theta F$ for any $\theta > 0$ changes sign at most once (from positive to negative as x goes from 0 to ∞). Since the kernel

$$K(n, x) = exp(-\lambda x)\frac{(\lambda x)^n}{n!}$$

is TP_2 over $\mathbb{N} \times [0, \infty)$, its variation diminishing property (see Karlin, 1968, Chapter 5 or Barlow and Proschan, 1975, p.93) implies that $\Gamma(\lambda, f, n) - \theta\Gamma(\lambda, F, n)$ changes sign at most once (from positive to negative as n goes from 0 to ∞). Therefore, $\Gamma(\lambda, f, n)/\Gamma(\lambda, F, n)$ is decreasing in n. Lemma 3.2 indicates that this sequence is λ times the reversed hazard rate of the discrete distribution $\Gamma(\lambda, F, \cdot)$. This completes the necessity part.

To prove the sufficiency, note that the d-DRH property of $\Gamma(\lambda, F, n)$, by Lemma 3.2 reduces to

$$\Gamma(\lambda, f, n) \cdot \Gamma(\lambda, F, n+k) \geq \Gamma(\lambda, F, n) \cdot \Gamma(\lambda, f, n+k) \quad \text{for } n, k \in \mathbb{N}. \quad (3.1)$$

It is clear that $\Gamma(\lambda, g, n) = E(g(X))$ where X has a gamma distribution. Block and Savits (1980, page 468) used the argument that when x is a continuity point of g,

$$\lim_{n\to\infty} \Gamma(n/x, g, n) = g(x),$$

since the gamma distribution tends to the degenerate distribution at x. In the present case, if x (> 0) and $x + y$ (> 0) are two continuity points of F, and if we choose k in (3.1) as the integer part of ny/x and λ as n/x, then the four terms of (3.1) tend to $f(x)$, $F(x + y)$, $F(x)$ and $f(x + y)$, respectively, as $n \to \infty$. It follows that F has a decreasing reversed hazard rate. □

REMARK 3.1 Let Y, $X_1, X_2, \ldots$ be independent, with Y having distribution function F and the X_i's following the $\exp(\lambda)$ distribution. Let N be such that

$$X_1 + X_2 + \ldots + X_{N-1} < Y \leq X_1 + X_2 + \ldots + X_N.$$

Then the above theorem suggests that the distribution of N is d-DRH for all λ if and only if F is log-concave.

The random variables mentioned in Remark 3.1 may be interpreted in any one of the following ways.

1. In a stress-strength model, Y may be the strength and $X_1, X_2, \ldots$ be independent stresses (shocks) which accumulate. The device fails as soon as the cumulative stress exceeds strength. Then, N is the number of stresses (shocks) causing failure of the device.

2. $X_1, X_2, \ldots$ may be a sequence of inter-failure times of a maintained unit, while Y may be the time of a catastrophic failure. In such a case, N would be the number of repairs during the lifetime of the unit, which is terminated by the catastrophic failure.

3. Y may be the lifetime of a single server whose clients come as per a Poisson process and are instantly served. Then N would be the number of clients served in the entire life of the server.

We end this section by mentioning a result due to Sengupta and Nanda (1999) that links d-DRH distributions with continuous log-concave distributions through a Poisson shock model.

Theorem 3.4 *Suppose that random shocks arrive in continuous time according to a homogeneous Poisson process, and that the probability that a unit fails to survive k shocks is $P(k)$. If the distribution $P(k)$ has the d-DRH property, then the continuous life distribution of the unit is log-concave.*

4 Closures and Non-closures

We start this section by proving closure property under convolution of discrete IFR (d-IFR) class, which will be used in proving the corresponding closure of the d-DRH class.

Lemma 4.1 *Let X and Y be two independent d-IFR random variables, not necessarily nonnegative. Then their sum $X+Y$ is also discrete IFR.*

Proof. Let us write $P(X=k)=f_k, P(X \geq k)=\bar{F}(k), P(Y=k)=g_k, P(Y \geq k)=\bar{G}(k)$ and assume that F and G are discrete IFR (i.e. $\bar{F}(k+n)/\bar{F}(k)$ and $\bar{G}(k+n)/\bar{G}(k)$ are both decreasing in k for $n \in \mathbb{N}$). We have to show that

$$D = \begin{vmatrix} \sum_k \bar{F}(m_1-k)g_{k-n_1} & \sum_k \bar{F}(m_1-k)g_{k-n_2} \\ \sum_k \bar{F}(m_2-k)g_{k-n_1} & \sum_k \bar{F}(m_2-k)g_{k-n_2} \end{vmatrix} \geq 0$$

for all indices $n_1 \le n_2$ and $m_1 \le m_2$. Note that

$$\begin{aligned} D &= \sum_{k_1}\sum_{k_2} \bar{F}(m_1-k_1)\bar{F}(m_2-k_2)[g_{k_1-n_1}g_{k_2-n_2} - g_{k_1-n_2}g_{k_2-n_1}] \\ &= \sum\sum_{k_1 \le k_2} \left[\bar{F}(m_1-k_1)\bar{F}(m_2-k_2) - \bar{F}(m_1-k_2)\bar{F}(m_2-k_1)\right] \\ &\quad \times[g_{k_1-n_1}g_{k_2-n_2} - g_{k_1-n_2}g_{k_2-n_1}]. \end{aligned}$$

The first of the four product terms can be rewritten as

$$\begin{aligned} &\sum\sum_{k_1 \le k_2} \bar{F}(m_1-k_1)\bar{F}(m_2-k_2)g_{k_1-n_1}g_{k_2-n_2} \\ &= \sum_{k_2} \bar{F}(m_2-k_2)g_{k_2-n_2} \sum_{k_1 \le k_2}\sum_{j \ge m_1-k_1} f_j g_{k_1-n_1} \\ &= \sum_{k_2} \bar{F}(m_2-k_2)g_{k_2-n_2} \sum_{k_1 \le k_2}\sum_{i \le k_1} f_{m_1-i} g_{k_1-n_1} \\ &= \sum_{k_2} \bar{F}(m_2-k_2)g_{k_2-n_2} \sum_{i \le k_2} f_{m_1-i} \sum_{k_1=i}^{k_2} g_{k_1-n_1} \\ &= \sum_{k_2} \bar{F}(m_2-k_2)g_{k_2-n_2} \sum_{k_1 \le k_2} f_{m_1-k_1}[\bar{G}(k_1-n_1) - \bar{G}(k_2-n_1+1)] \\ &= \sum\sum_{k_1 \le k_2} \bar{F}(m_2-k_2)g_{k_2-n_2}f_{m_1-k_1}\bar{G}(k_1-n_1) \\ &\quad -\sum_{k} \bar{F}(m_2-k)g_{k-n_2}\bar{F}(m_1-k)\bar{G}(k-n_1+1). \end{aligned}$$

The other three products of sums in the expression of D can be rewritten as

$$\begin{aligned} &\sum\sum_{k_1 \le k_2} \bar{F}(m_1-k_1)\bar{F}(m_2-k_2)g_{k_2-n_1}g_{k_1-n_2} \\ &\quad= \sum\sum_{k_1 \le k_2} \bar{F}(m_2-k_2)g_{k_2-n_1}f_{m_1-k_1}\bar{G}(k_1-n_2) \\ &\quad\quad - \sum_{k} \bar{F}(m_2-k)g_{k-n_1}\bar{F}(m_1-k)\bar{G}(k-n_2+1); \\ &\sum\sum_{k_1 \le k_2} \bar{F}(m_1-k_2)\bar{F}(m_2-k_1)g_{k_2-n_1}g_{k_1-n_2} \\ &\quad= \sum\sum_{k_1 \le k_2} \bar{F}(m_1-k_2)g_{k_2-n_1}f_{m_2-k_1}\bar{G}(k_1-n_2) \\ &\quad\quad - \sum_{k} \bar{F}(m_1-k)g_{k-n_1}\bar{F}(m_2-k)\bar{G}(k-n_2+1); \end{aligned}$$

$$\begin{aligned}
&\sum\sum_{k_1 \le k_2} \bar{F}(m_1 - k_2)\bar{F}(m_2 - k_1) g_{k_1 - n_1} g_{k_2 - n_2} \\
&\quad = \sum\sum_{k_1 \le k_2} \bar{F}(m_1 - k_2) g_{k_2 - n_2} f_{m_2 - k_1} \bar{G}(k_1 - n_1) \\
&\quad - \sum_{k} \bar{F}(m_1 - k) g_{k - n_2} \bar{F}(m_2 - k) \bar{G}(k - n_1 + 1).
\end{aligned}$$

Once the four terms are put back together, the subtracted terms from the above expressions would cancel. The combination of the double summations can be conveniently factored as

$$\begin{aligned}
D &= \sum\sum_{k_1 \ge k_2} [\bar{F}(m_1 - k_1) f_{m_2 - k_2} - \bar{F}(m_2 - k_1) f_{m_1 - k_2}] \\
&\quad \times [g_{k_1 - n_1} \bar{G}(k_2 - n_2) - g_{k_1 - n_2} \bar{G}(k_2 - n_1)].
\end{aligned}$$

The first factor of the summand can be written as

$$\begin{aligned}
&\bar{F}(m_1 - k_1) f_{m_2 - k_2} - \bar{F}(m_2 - k_1) f_{m_1 - k_2} \\
&\quad = \left(\frac{\bar{F}(m_2 - k_2)}{\bar{F}(m_2 - k_1)} \cdot \frac{f_{m_2 - k_2}}{\bar{F}(m_2 - k_2)} - \frac{\bar{F}(m_1 - k_2)}{\bar{F}(m_1 - k_1)} \cdot \frac{f_{m_1 - k_2}}{\bar{F}(m_1 - k_2)} \right) \\
&\qquad \times \bar{F}(m_1 - k_1) F(m_2 - k_1).
\end{aligned}$$

The two ratios contained in the first term within parantheses are greater than the corresponding ratios of the second term, since F is discrete IFR. Therefore, the above expression is nonnegative. Similarly,

$$\begin{aligned}
&g_{k_1 - n_1} \bar{G}(k_2 - n_2) - g_{k_1 - n_2} \bar{G}(k_2 - n_1) \\
&\quad = \left(\frac{\bar{G}(k_1 - n_1)}{\bar{G}(k_2 - n_1)} \cdot \frac{g_{k_1 - n_1}}{\bar{G}(k_1 - n_1)} - \frac{\bar{G}(k_1 - n_2)}{\bar{G}(k_2 - n_2)} \cdot \frac{g_{k_1 - n_2}}{\bar{G}(k_1 - n_2)} \right) \\
&\qquad \times \bar{G}(k_2 - n_2) \bar{G}(k_2 - n_1) \ge 0.
\end{aligned}$$

It follows that $D \ge 0$ and hence the distribution of $X + Y$ is IFR. □

The ensuing theorem gives the closure properties of the d-DRH distributions.

THEOREM 4.2 *The d-DRH class of distributions has the following closure properties.*

(a) *If a sequence of d-DRH distributions converges to a limiting distribution, the limiting distribution is d-DRH.*

(b) *If the components of a parallel system have independent lifetimes with d-DRH distributions, then the system life distribution is d-DRH.*

(c) *If the components of a k-out-of-n system have independent lifetimes with identical d-DRH distributions, then the system life distribution is d-DRH.*

(d) *Convolution of two d-DRH distributions produces a d-DRH distribution.*

Proof. Parts (a) and (b) are easy to prove from the definition of the d-DRH class.

In order to prove part (c), let F be a d-DRH distribution and G be a continuous distribution obtained by linearly interpolating $\log F$ at the non-integral points. Let B_{kn} be the function defined in Barlow and Proschan (1975, p.107), so that the composition $B_{kn} \circ F$ is the life distribution of a k-out-of-n system of independent components having life distribution F. Since the continuous distribution G is log-concave, Theorem 2(g) of Sengupta and Nanda (1999) implies that $B_{kn} \circ G$ is log-concave. Therefore, the sequence $\log(B_{kn} \circ G(j)/B_{kn} \circ G(j-1))$ is decreasing in j. It follows that the sequence $1 - B_{kn} \circ G(j-1)/B_{kn} \circ G(j)$ is decreasing in j. However,

$$1 - \frac{B_{kn} \circ G(j-1)}{B_{kn} \circ G(j)} = 1 - \frac{B_{kn} \circ F(j-1)}{B_{kn} \circ F(j)},$$

and the expression in the right hand side is the reversed hazard rate of the distribution $B_{kn} \circ F$.

Part (d) can be proved by using Lemma 4.1 along with the fact that X is d-DRH if and only if $-X$ is discrete IFR. □

Remark 4.1 It is to be mentioned here that the proof of Theorem 2(f) of Sengupta and Nanda (1999) is erroneous. The result is true and it can be proved along the lines of the proof of Barlow and Proschan (1975) by considering $-X$ in place of X.

Remark 4.2 The life distribution of a series system with independent d-DRH distributed component lifetimes need not be d-DRH. Consider $F(k) = P(X \le k) = (1/2)^{4-k}$ and $G(k) = P(Y \le k) = (1/4)^{4-k}$, $k = 0, 1, 2, 3, 4$. Then F and G are d-DRH, but $1 - (1-F)(1-G)$ is not d-DRH.

Remark 4.3 Mixtures of d-DRH distributions is not necessarily d-DRH. Consider the distribution F having masses 0.3, 0.3 and 0.4 at 0, 1 and 2, respectively, and the distribution G having masses 0.99 and 0.01 at 0 and 1, respectively. F and G are d-DRH, but $(F+G)/2$ is not.

5 Reliability Bounds

Consider a discrete distribution F with support on $\mathbb{N}$ and a specified mean μ. The following theorem gives the sharp upper bound on $\bar{F}(k)$ when F is known to be d-DRH.

THEOREM 5.1 *If X is a discrete random variable having d-DRH distribution with mean μ, then*

$$P(X > k) \leq \begin{cases} 1 & if \quad k < [\mu] \\ \max\limits_{i=k,k+1,\ldots} (1 - r_i^{i-k}) & if \quad k \geq [\mu], \end{cases}$$

where $[\mu]$ is the integer part of μ and r_i is the unique solution to the equation

$$(1 - r_i^{i+1})/(1 - r_i) = i + 1 - \mu.$$

The bound is sharp.

PROOF. In order to prove that the trivial upper bound in the case $k < [\mu]$ is sharp, consider the distribution having masses $1-\mu+[\mu]$ and $\mu-[\mu]$ at the points $[\mu]$ and $[\mu]+1$, respectively. This distribution is d-DRH, has mean μ and $P(X > k)$ is equal to 1 for all $k < [\mu]$. Now let $k \geq [\mu]$ and define the class of distributions $\mathcal{G} = \{F : F$ is d-DRH with mean $\mu\}$. The task is to find the infimum of $F(k)$ when $F \in \mathcal{G}$. We shall show that this task is equivalent to finding the infimum of $F(k)$ when F is in the sub-class

$$\mathcal{G}_0 = \left\{F : \ F \in \mathcal{G}, \ F(l) = \min\{e^{-a(t-l)}, 1\} \text{ for some } t > \mu \text{ and } a > 0\right\}. \quad (5.1)$$

To see this, let us choose a distribution F from $\mathcal{G}$ and define $M(l) = -\ln F(l)$. The distribution F is d-DRH if and only if $M(l) - M(l+1)$ is decreasing in l. For the chosen integer $k \geq [\mu]$ set

$$M_1(l) = \begin{cases} a(t-l) & \text{if} \quad l < t, \\ 0 & \text{if} \quad l \geq t, \end{cases}$$

where $a = M(k) - M(k+1)$ and $t = k + 1 + M(k+1)/a$. (We ignore the trivial case $a = 0$, when $M(k) = M(k+1) = M(k+2) = \cdots = 0$ so that $F(k)$ is larger than any distribution in $\mathcal{G}_0$.) By construction $M_1(l) = M(l)$ for $l = k, k+1$. It follows from the convexity of the sequence M that $M_1(l) \leq M(l)$ for all l, with equality holding for $l = k, k+1$. Since

$$\sum_{l \geq 0} \left(1 - e^{-M(l)}\right) = \mu,$$

we have

$$\sum_{l\geq 0}\left(1-e^{-M_1(l)}\right)\leq \mu. \tag{5.2}$$

Note that even though the distribution e^{-M_1} has the requisite shape, it is not necessarily a member of $\mathcal{G}_0$ as its mean may be too small. However, the LHS of (5.2) would increase if the value of a is increased from $M(k)-M(k+1)$, keeping t fixed, and would exceed μ when a goes to infinity. Therefore, there exists a number a_0 such that the distribution e^{-M_2} defined by

$$M_2(l)=\begin{cases}a_0(t-l) & \text{if } l<t,\\ 0 & \text{if } l\geq t,\end{cases}$$

where $t=k+1+\frac{M(k+1)}{M(k)-M(k+1)}$, has mean μ. Specifically, a_0 is defined by the equation

$$\mu=\sum_{l\geq 0}\left(1-e^{-M_2(l)}\right)=[t]+1-\frac{(e^{-a_0(t-[t])}-e^{-a_0(t+1)})}{(1-e^{-a_0})}.$$

Clearly, e^{-M_2} is a member of $\mathcal{G}_0$ and $a_0\geq M(k)-M(k+1)$, implying $e^{-M_2(k)}\leq e^{-M_1(k)}=e^{-M(k)}$.

Thus, for every distribution in $\mathcal{G}$ there is a distribution in $\mathcal{G}_0$ which has a smaller value of distribution function at k. Therefore, instead of seeking the supremum of $P(X>k)$ over $\mathcal{G}$, it is enough to look for the supremum of $P(X>k)$ over $\mathcal{G}_0$. It follows from (5.1) that the sharp upper bound of $P(X>k)$ in $\mathcal{G}_0$ is

$$\sup_{\substack{t\,:\,t\geq k,\\ a\,:\,[t]+1-(e^{-a(t-[t])}-e^{-a(t+1)})/(1-e^{-a})=\mu}}(1-e^{-a(t-k)}).$$

In order to simplify the task of maximization, define the variables $i=[t]$, $\langle t\rangle=t-[t]$ and $r=e^{-a}$. Then the upper bound of $P(X>k)$ in $\mathcal{G}_0$ is

$$\sup_{\substack{i=k,k+1,\ldots\\ 0<\langle t\rangle<1\\ 0<r<1\\ r^{\langle t\rangle}(1-r^{i+1})/(1-r)=i+1-\mu}} 1-r^{i-k+\langle t\rangle}.$$

Rewrite the constraint as

$$r^{\langle t\rangle}(1+r+\cdots+r^i)=i+1-\mu. \tag{5.3}$$

It is clear that the left hand side of (5.3) is an increasing function of r and a decreasing function of $\langle t\rangle$. Therefore, for given μ and fixed i, the constraint dictates that the variables $\langle t\rangle$ and r be increasing functions of one another.

After substituting $(i+1-\mu)(1-r)/(1-r^{i+1})$ for $r^{\langle t\rangle}$, the objective function becomes

$$1-r^{i-k}(i+1-\mu)\cdot\frac{1-r}{1-r^{i+1}}=1-(i+1-\mu)\frac{1}{r^{-(i-k)}+r^{-(i-k)+1}+\cdots+r^{k}}.$$

It is easy to see that $r^{-(i-k)}+r^{-(i-k)+1}+\cdots+r^{k}$ is a convex function of r with a unique minimum. Therefore, for fixed i the maximum of the objective function occurs at one of the end-points of the range of r permitted by the constraint. As r is an increasing function of $\langle t\rangle$, these end-points correspond to $\langle t\rangle=0$ and $\langle t\rangle=1$, respectively, in (5.3). Consequently for fixed i, the objective function is maximized by choosing either $r=r_i$ or $r=s_i$, where r_i and s_i are unique solutions to the equations

$$\begin{aligned} 1+r+\cdots+r^{i} &= i+1-\mu, \\ \text{and } r(1+r+\cdots+r^{i}) &= i+1-\mu, \end{aligned}$$

respectively. It is easy to see that $s_i=r_{i+1}$. Therefore, for $k\le[\mu]$ the upper bound of $P(X>k)$ is

$$\max_{i=k,k+1,\ldots}\max\left\{1-r_i^{i-k},1-s_i^{i-k+1}\right\}=\max_{i=k,k+1,\ldots}(1-r_i^{i-k}),$$

with r_i defined by $(1-r_i^{i+1})/(1-r_i)=i+1-\mu$. This leads to the required result. Sharpness of the bound in the case $k\ge[\mu]$ follows from the fact that for every $i>k$, the value $1-r_i^{i-k}$ is attained by a particular distribution in $\mathcal{G}_0$. □

REMARK 5.1 The upper bound in the range $k\ge[\mu]$ is qualitatively different from the continuous case. Further, the bound in the continuous case is a function of time divided by mean life, whereas the discrete case bound cannot be computed solely from the knowledge of k/μ or $(k+1)/\mu$.

REMARK 5.2 Since the d-DRH class includes the discrete DFR class, and the sharp lower bound on the reliability for the latter class is 0 (see Theorem 5 of Sengupta et al., 1995), the sharp lower bound on $\bar{F}(k)$, when F is d-DRH with mean μ, is 0.

6 Inequalities for maintained units

Consider a maintained unit where each failure is followed by instant replacement by a new or perfectly repaired unit. Let the life distribution of the units be independent with distribution F. In the continuous time case, Sengupta and Nanda (1999) showed that whenever F is IFR, the number of failures till a fixed time t_0 has a d-DRH distribution. If the mean of this distribution is known, then one can give an upper bound on this distribution using Theorem 5.1. If one does not know the mean but has a good idea about the parent distribution F, then one can use the upper and lower bounds given by Barlow and Proschan (1975, p.162)

Now consider the above set-up in discrete time, as in Shaked et al. (1993). Let F be a known distribution. If an item fails at time k, then it is immediately replaced with a new unit at the same time (which means that there is a possibility of two or more failures at a single time index k). Let $N(k)$ represent the number of replacements till time k (that is, the $N(k)$th failure occurs at or before time k and the $(N(k)+1)$th failure occurs strictly after k). The distribution of $N(k)$ for fixed k is not easy to compute in general. The following theorem provides an approximation from the lower side.

Theorem 6.1 *In the above set-up, let F be d-DRH. Then*

$$P(N(k) < l) \geq 1 - e^{-la_{kl}} \sum_{i=0}^{l-1} \frac{(ka_{kl})^i}{i!},$$

where $a_{kl} = -\log F\left(\left[\frac{k}{l}\right]+1\right)$, $\left[\frac{k}{l}\right]$ being the integer part of $\frac{k}{l}$.

Proof. Let $S = -\log F$ and T be a piecewise linear and continuous function that coincides with S at the integer points. [If S is not defined at any point, T can be assigned any finite value so that the function is convex.] Let $G(x) = e^{-T(x)}$. Note that if K is a sample from F, then for any $x > 0$,

$$\begin{aligned} P(T(K) \geq x) &= P(K \leq T^{-1}(x)) = F\big([T^{-1}(x)]\big) \\ &= G\big([T^{-1}(x)]\big) \leq G(T^{-1}(x)) = e^{-x}. \end{aligned}$$

In the above simplification the notation $[\cdot]$ has been used to indicate the 'greatest integer' function. The above result shows that the random variable $T(K)$ is stochastically smaller than the unit exponential distribution.

Let $K_1, K_2, \ldots$ be the successive lifetimes of the replacement units. We can write

$$\begin{aligned}
P(N(k)<l) &= P(K_1+K_2+\cdots+K_l > k) \\
&= P\left(T\left(\frac{K_1+K_2+\cdots+K_l}{l}\right) < T\left(\frac{k}{l}\right)\right) \\
&\geq P\left(T\left(\frac{K_1+K_2+\cdots+K_l}{l}\right) \leq S\left(\frac{k}{l}+1\right)\right) \\
&\geq P\left(\frac{T(K_1)+T(K_2)+\cdots+T(K_l)}{l} \leq S\left(\frac{k}{l}+1\right)\right) \\
&\qquad\qquad \text{(since } T \text{ is convex)} \\
&\geq P\left(E_1+E_2+\cdots+E_l \leq l\cdot S\left(\frac{k}{l}+1\right)\right) \\
&= P(E_1+E_2+\cdots+E_l \leq l\cdot a_{kl}), \quad \text{where } a_{kl} = S\left(\left[\frac{k}{l}\right]+1\right),
\end{aligned}$$

$E_1, E_2, \ldots$ being samples from the unit exponential distribution. The results stated in the theorem follows from the form of the gamma distribution with shape parameter l and scale parameter 1. □

7 Discrete concave distributions

Note from the proof of Part (b) of Theorem 2.3 that there is a class of distributions which lie somewhere in between the class of d-DFR and d-DRH classes. These are distributions with decreasing probability mass function. We define the class of such distributions as discrete concave or d-concave class.

It can be shown with some algebraic manipulation that the 'remaining life' (at any age) corresponding to any d-concave distribution (at any age) is d-concave. Further, the class of d-concave distributions is closed under limits of distributions, formation of arbitrary series systems and arbitrary mixtures. The class is not closed under convolution and formation of coherent systems. A simple counterexample that works for both the situations is one where the first distribution has masses 0.5, 0.3, 0.1 and 0.1 at 0, 1, 2 and 3, respectively and the second distribution has masses 0.4, 0.3, 0.2 and 0.1 at the same points, respectively.

Sengupta and Nanda (1999) had shown that d-concave distributions give rise to continuous concave distributions through a Poisson shock model, along the lines of Theorem 3.4.

When F is d-concave, one can use the decreasing property of $F(k+1)-$

$F(k)$ to obtain a sharp upper bound of the reliability function $\bar{F}$ for a given mean μ.

THEOREM 7.1 *Let X be a discrete random variable having d-concave distribution with mean μ. Then the sharp lower and upper bounds on the survival function are given by*

$$0 \le P(X > k) \le \begin{cases} 1 - \frac{2k([2\mu]-\mu)}{[2\mu]\cdot[2\mu-1]} & \textit{if} \quad k < \mu - 1, \\ \frac{\mu}{2k+1} & \textit{if} \quad k \ge \mu - 1, \end{cases}$$

where, $[\cdot]$ indicates the integer part.

PROOF. Following an argument similar to the one used in the proof of Theorem 5.1 it can be shown that the upper bound can only be attained by a distribution of the form

$$P(X \le k) = \begin{cases} \alpha + (1-\alpha)k/\tau & \text{if} \quad 0 \le k \le \tau, \\ 1 & \text{if} \quad k > \tau, \end{cases} \tag{7.4}$$

where, τ is a real number ($\tau \ge k$) and α is a fraction such that the mean of the above distribution is μ. The latter condition implies that

$$\begin{aligned} \mu &= \sum_{k=0}^{[\tau]} P(X > k) = (1-\alpha)\sum_{k=0}^{[\tau]}(1 - k/\tau) = (1-\alpha)\{[\tau]+1-[\tau]([\tau]+1)/(2\tau)\} \\ &= (1-\alpha)([\tau]+1)\{1-[\tau]/(2\tau)\} = (1-\alpha)([\tau]+1)\{1 - [\tau]/(2[\tau] + 2\langle t\rangle)\}, \end{aligned}$$

where $\langle t\rangle = \tau - [\tau]$, the fractional part of τ. Note from the last equation that

$$1 - \alpha = \frac{\mu}{([\tau]+1)\{1 - [\tau]/(2[\tau] + \langle t\rangle)\}}. \tag{7.5}$$

Consider three cases regarding the existence of an α satisfying (7.5).

Case I: $[\tau] > 2\mu - 1$. The right hand side of (7.5) has a fractional value for every $\langle t\rangle \in [0, 1)$.

Case II: $[\tau] \le 2\mu - 2$. The right hand side of (7.5) does not have a fractional value (that is, the value is bigger than 1) for any $\langle t\rangle \in [0, 1)$.

Case III: $2\mu - 2 < [\tau] \le 2\mu - 1$ (that is, $[\tau] = [2\mu - 1]$). The right hand side of (7.5) has a fractional value whenever $\langle t\rangle \ge \dfrac{(2\mu - [2\mu])([2\mu] - 1)}{2([2\mu] - \mu)}$.

In summary, an α satisfying (7.5) exists if and only if $\tau \geq \tau_0$, where

$$\tau_0 = [2\mu - 1]\left\{1 + \frac{2\mu - [2\mu]}{2([2\mu] - \mu)}\right\}. \tag{7.6}$$

Note that $[2\mu - 1] \leq \tau_0 \leq 2\mu - 1$.

For the distribution described in (7.4), we can conclude from (7.5) that

$$\begin{aligned} P(X > k) &= (1-\alpha)(1 - k/\tau) = \frac{\mu\{1 - k/([\tau] + \langle t\rangle)\}}{([\tau]+1)\{1 - [\tau]/(2([\tau] + \langle t\rangle))\}} \\ &= \frac{2\mu([\tau] - k + \langle t\rangle)}{([\tau]+1)([\tau] + 2\langle t\rangle)}. \end{aligned} \tag{7.7}$$

Therefore, the sharp upper bound on $P(X > k)$ is

$$\begin{aligned} &\sup_{\substack{\tau \geq \max\{k,\tau_0\} \\ \alpha \in [0,1] \\ \mu = (1-\alpha)([\tau]+1)\{1 - [\tau]/(2[\tau] + \langle t\rangle)\}}} (1-\alpha)\cdot\left(1 - \frac{k}{[\tau] + \langle t\rangle}\right) \\ &= \sup_{\tau \geq \max\{k,\tau_0\}} \frac{2\mu([\tau] - k + \langle t\rangle)}{([\tau]+1)([\tau] + 2\langle t\rangle)}. \end{aligned}$$

Note that $2\mu([\tau] - k + \langle t\rangle)/\{([\tau]+1)([\tau] + 2\langle t\rangle)\}$ can also be written as $2\mu(\tau - k)/(\tau^2 + \tau + \langle t\rangle - \langle t\rangle^2)$ which is a continuous function of τ. The function can be differentiated at non-integer values of τ. The derivative is

$$\begin{aligned} &2\mu \cdot \frac{-\tau^2 - \tau + 2\tau\langle t\rangle + \langle t\rangle - \langle t\rangle^2 + 2k(\tau + 1 - \langle t\rangle)}{(\tau^2 + \tau + \langle t\rangle - \langle t\rangle^2)^2} \\ &= \frac{2\mu(2k - \tau + \langle t\rangle)(\tau - \langle t\rangle + 1)}{(\tau^2 + \tau + \langle t\rangle - \langle t\rangle^2)^2} = \frac{2\mu(2k - [\tau])([\tau] + 1)}{(\tau^2 + \tau + \langle t\rangle - \langle t\rangle^2)^2}. \end{aligned}$$

and it has the same sign as $2k - [\tau]$. Clearly, the objective function is maximized when $2k \leq \tau \leq 2k + 1$ and the maximum value is $\mu/(2k+1)$. This maximum is attained when the range $[2k, 2k+1]$ has an intersection with the feasible range of τ, which is $\tau \geq \tau_0$. In other words, the above maximum value is attained when $\tau_0 \leq 2k + 1$. As $[2\mu - 1] \leq \tau_0 \leq 2\mu - 1$, the condition $\tau_0 \leq 2k + 1$ is equivalent to $2\mu - 1 \leq 2k + 1$ or $2\mu - 2 \leq 2k$ or $k \geq \mu - 1$.

In the other case ($k < \mu - 1$), we have $2k + 1 < \tau_0 \leq \tau$. Thus, the objective function is decreasing in τ over its entire feasible range. Therefore, the maximum occurs at the smallest feasible value, τ_0, and the maximum value is

$$\begin{aligned}\frac{2\mu([\tau_0]-k+\langle\tau_0\rangle)}{([\tau_0]+1)([\tau_0]+2\langle\tau_0\rangle)} &= \frac{2\mu\left([2\mu-1]-k+\frac{[2\mu-1](2\mu-[2\mu])}{2([2\mu]-\mu)}\right)}{[2\mu]\cdot[2\mu-1]\left(1+\frac{2\mu-[2\mu]}{2([2\mu]-\mu)}\right)} \\ &= \frac{\mu\{[2\mu-1](2[2\mu]-2\mu+2\mu-[2\mu])-2k([2\mu]-\mu)\}}{[2\mu]\cdot[2\mu-1]([2\mu]-\mu+2\mu-[2\mu])} \\ &= \frac{[2\mu-1]\cdot[2\mu]-2k([2\mu]-\mu)}{[2\mu]\cdot[2\mu-1]} = 1-\frac{2k([2\mu]-\mu)}{[2\mu]\cdot[2\mu-1]}.\end{aligned}$$

This completes the proof. □

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