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The Borel–Cantelli lemma under dependence conditions

Tapas Kumar Chandra

BIRU, Indian Statistical Institute, 203 B.T. Road, Kolkata 700108, India

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Abstract

Generalizations of the second Borel–Cantelli lemma are obtained under very weak dependence conditions, subsuming several earlier results as special cases.

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1. Introduction

In a recent note, Petrov (2004) proved using clever arguments an interesting extension of the (second) Borel–Cantelli lemma; the theorem in Section 2 of Petrov (2004) contains several earlier extensions of the Borel–Cantelli lemma as special cases. This note extends Petrov's result further (see Theorems 1 and 1′); it is expected that the present version will be more widely applicable. A related result is also included in the final section.

To motivate the extension, we state the following results. If there are non-negative reals c_1, c_2, c_3 and an integer $N \geqslant 1$ such that

$$P(A_i \cap A_j) \leqslant (c_1 P(A_i) + c_2 P(A_j))P(A_{j-i}) + c_3 P(A_i)P(A_j) \tag{1}$$

whenever $N \leqslant i < j$, and

$$\sum_{n=1}^{\infty} P(A_n) = \infty, \tag{2}$$

then $c \geqslant 1$ and

$$P(\limsup A_n) \geqslant \frac{1}{c}, \tag{3}$$

where

$$c := c_3 + 2(c_1 + c_2). \tag{4}$$

E-mail address: tkchandra@gmail.com

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The result of Petrov (2002) is a special case where $c_1 = c_2 = 0$. For a simple proof, see Chandra (1999). Regarding condition (1), see Kochen and Stone (1964, Example 2) and Lamperti (1963, p. 62). On the other hand, the extended Rényi–Lamperti lemma states that if

$$\liminf_{n\to\infty} \frac{\sum_{1\leqslant j,k\leqslant n} P(A_j \cap A_k)}{(\sum_{1\leqslant j\leqslant n} P(A_j))^2} = c \tag{5}$$

and (2) holds, then $c \geqslant 1$ and (3) holds. (The Rényi–Lamperti lemma states that

$$P(\limsup A_n) \geqslant 2 - c;$$

see Billingsley, 1979, p. 76.) For a simple proof, see Spitzer (1964, p. 319). Erdös and Rényi (1959) considers the special case where (5) holds with $c = 1$, and contains the case where (1) holds with $c_1 = c_2 = 0$ and $c_3 = 1$ and (2) obtains; in particular, the second Borel–Cantelli lemma holds if the independence of $\{A_n\}$ is replaced by pairwise independence or pairwise negative quadrant dependence of $\{A_n\}$ (this fact can also be obtained from the result that (1) and (2) together imply (3)). In his paper, Petrov (2004) combined conditions (1), with $c_1 = c_2 = 0$, and (5) in a clever manner and obtained a *common generalization* of Petrov (2002) and the extended Rényi–Lamperti lemma; see, also, Ortega and Wschebor (1983). He used the following inequality of Chung and Erdös (1952):

$$P\left(\bigcup_{k=1}^{n} A_k\right) \geqslant \frac{(\sum_{k=1}^{n} P(A_k))^2}{\sum_{j,k=1}^{n} P(A_j \cap A_k)}, \tag{6}$$

which is, in turn, a special case of Spitzer's inequality (see Spitzer, 1964, p 319): If $E(X) > 0$ and $0 \leqslant \varepsilon < 1$, then

$$P(X > \varepsilon E(X)) \geqslant (1-\varepsilon)^2 \frac{(E(X))^2}{E(X^2)}. \tag{7}$$

(Take $\varepsilon = 0$ and $X = \sum_{k=1}^{n} I_{A_k}$.) We shall prove the following extension of Petrov (2004) using suitable modifications of his arguments.

Theorem 1. *Let* $\{A_n\}_{n\geqslant 1}$ *satisfy* (2) *and assume that*

$$\liminf_{n\to\infty} \frac{\sum_{1\leqslant j<k\leqslant n} (P(A_j \cap A_k) - a_{jk})}{(\sum_{1\leqslant k\leqslant n} P(A_k))^2} = L,$$

where $a_{ij} = (c_1 P(A_i) + c_2 P(A_j))P(A_{j-i}) + c_3 P(A_i)P(A_j)$ *for* $1 \leqslant i < j$, $c_1 \geqslant 0$, $c_2 \geqslant 0$, c_3 *being constants* (L *may depend on* c_1, c_2 *and* c_3). *Assume that* L *is finite. Then* $c + 2L \geqslant 1$ *and*

$$P(\limsup A_n) \geqslant \frac{1}{(c+2L)},$$

where c *is given by* (4).

Petrov's result follows when $c_1 = c_2 = 0$. Furthermore, it follows that (1) and (2) imply (3), since then $L \leqslant 0$ and so $(c+2L)^{-1} \geqslant c^{-1}$.

The proof of Theorem 1 shows that the following more general result is also true; the details are omitted.

Theorem 1′. *Let* $\{A_n\}_{n\geqslant 1}$ *satisfy* (2) *and*

$$\liminf_{n\to\infty} \frac{\sum_{1\leqslant j<k\leqslant n} (P(A_j \cap A_k) - a_{jk})}{(\sum_{1\leqslant k\leqslant n} P(A_k))^2} = L,$$

where the a_{jk} *satisfy*

$$\sum_{1\leqslant j<m\leqslant k\leqslant n} |a_{jk}| = o\left(\left(\sum_{m\leqslant k\leqslant n} P(A_k)\right)^2\right) \quad \forall m \geqslant 1$$

and

$$\liminf_{m\to\infty} \limsup_{n\to\infty} \frac{\sum_{m\leqslant j<k\leqslant n} a_{jk}}{(\sum_{m\leqslant k\leqslant n} P(A_k))^2} \leqslant d.$$

Assume that L *and* d *are finite. Then* $d + L \geqslant \frac{1}{2}$ *and* $P(\limsup A_n) \geqslant (2d + 2L)^{-1}$.

2. Proof of Theorem 1

We need the following lemma.

Lemma 1. *Let* (2) *hold and* a_{ij} *be as in Theorem* 1. *Put*

$$\beta_m = \liminf_{n\to\infty} \frac{\sum_{m\leqslant j<k\leqslant n}(P(A_j \cap A_k) - a_{jk})}{(\sum_{k=m}^{n} P(A_k))^2}, \quad m \geqslant 1.$$

Then $\beta_1 = \beta_m$, $\forall m \geqslant 1$.

Proof. Fix an integer $m > 1$. As (2) holds, we have

$$\begin{aligned}\beta_1 &= \liminf_{n\to\infty} \frac{\sum_{1\leqslant j<k\leqslant n}(P(A_j \cap A_k) - a_{jk})}{(\sum_{k=m}^{n} P(A_k))^2} \\ &= \liminf_{n\to\infty} (I_{1,n} + I_{2,n} + I_{3,n}),\end{aligned}$$

where, for $n \geqslant m$,

$$s_{m,n} = \sum_{k=m}^{n} P(A_k), \quad b_{j,k} = P(A_j \cap A_k) - a_{jk},$$

$$I_{1,n} = \frac{\sum_{1\leqslant j<k\leqslant m} b_{j,k}}{s_{m,n}^2}, \quad I_{2,n} = \frac{\sum_{m\leqslant j<k\leqslant n} b_{j,k}}{s_{m,n}^2},$$

$$I_{3,n} = \frac{\sum_{1\leqslant j<m\leqslant k\leqslant n} b_{j,k}}{s_{m,n}^2}.$$

Condition (2) implies that $I_{1,n} \to 0$ as $n \to \infty$. Also

$$\begin{aligned}|I_{3,n}| &\leqslant \frac{\sum_{1\leqslant j<m\leqslant k\leqslant n}(P(A_j \cap A_k) + |a_{jk}|)}{s_{m,n}^2} \\ &\leqslant \frac{m(1 + |c_1| + |c_2| + |c_3|)}{s_{m,n}} + \frac{m s_{1,m} |c_1|}{s_{m,n}^2}\end{aligned}$$

so that $I_{3,n} \to 0$ as $n \to \infty$. Therefore,

$$\beta_1 = \beta_m + \lim I_{1,n} + \lim I_{3,n} = \beta_m. \qquad \square$$

We next prove Theorem 1. Let $m \geqslant 1$ be an integer; let $N \geqslant m$ be such that $\sum_{k=m}^{N} P(A_k) > 0$. Then, by (6),

$$P\left(\bigcup_{k=m}^{\infty} A_k\right) \geqslant \frac{(\sum_{k=m}^{n} P(A_k))^2}{\sum_{j,k=m}^{n} P(A_j \cap A_k)} \quad \forall n \geqslant N.$$

Now,

$$\sum_{j,k=m}^{n} P(A_j \cap A_k) = s_{m,n} + T_1 + T_2,$$

where the $s_{m,n}$ are as in the proof of Lemma 1 and

$$T_1 = 2 \sum_{m \leqslant j < k \leqslant n} (P(A_j \cap A_k) - a_{jk}),$$

$$T_2 = 2 \sum_{m \leqslant j < k \leqslant n} a_{jk},$$

where the a_{jk} are as in the statement of Theorem 1. As

$$T_2 \leqslant 2\left(\frac{c_3}{2} + c_1 + c_2\right) s_{m,n}^2 + (c_1 + c_2) s_{1,m} s_{m,n} - c_3 \sum_{j=m}^{n} (P(A_j))^2,$$

one has

$$\limsup_{n \to \infty} \frac{T_2}{s_{m,n}^2} \leqslant c,$$

where c is given by (4). Therefore, condition (2) implies that

$$\begin{aligned} P\left(\bigcup_{k=m}^{\infty} A_k\right) &\geqslant \left\{ \liminf_{n \to \infty} \frac{T_1}{s_{m,n}^2} + \limsup_{n \to \infty} \frac{T_2}{s_{m,n}^2} \right\}^{-1} \\ &\geqslant \{2L + c\}^{-1} \quad \text{by Lemma 1.} \end{aligned}$$

3. An alternative approach

In this section we derive another version of the second Borel–Cantelli lemma under a suitable dependence condition using the Chebyshev Inequality.

Lemma 2. *Let $\{X_n\}$ be a sequence of non-negative random variables with finite $E(X_n^2)$ and put $S_n = \sum_{i=1}^{n} X_i$, $n \geqslant 1$. Assume that $E(S_n) \to \infty$ as $n \to \infty$, and*

$$\operatorname{Var}(S_n) \leqslant c_n E(S_n) \quad \forall n \geqslant 1, \quad \liminf_{n \to \infty} \left(\frac{c_n}{E(S_n)} \right) = 0. \tag{8}$$

Then

$$P\left(\sum_{n=1}^{\infty} X_n = \infty \right) = 1. \tag{9}$$

Proof. Note that

$$\begin{aligned} P\left(\sum_{n=1}^{\infty} X_n < \infty \right) &= \lim_{n \to \infty} P\left(\sum_{n=1}^{\infty} X_n \leqslant \frac{1}{2} E(S_n) \right) \\ &\leqslant \liminf_{n \to \infty} P\left(S_n \leqslant \frac{1}{2} E(S_n) \right) \\ &\leqslant \liminf_{n \to \infty} \left(\frac{4c_n}{E(S_n)} \right) = 0. \qquad \square \end{aligned}$$

Theorem 2. *Let $\{X_n\}$ be a sequence of non-negative random variables with finite $E(X_n^2)$ and put $S_n = \sum_{i=1}^{n} X_i$, $n \geqslant 1$. Assume that $E(S_n) \to \infty$ as $n \to \infty$.*

(a) *Assume that*

$$\sum_{i=1}^{n} \mathrm{Var}(X_i) \leqslant k_n E(S_n) \quad \forall n \geqslant 1, \tag{10}$$

$$\sum_{j=2}^{n} \sum_{i=1}^{j-1} \mathrm{Cov}(X_i, X_j) \leqslant c_n E(S_n) \quad \forall n \geqslant 2 \tag{11}$$

and

$$\liminf_{n\to\infty} ((k_n + 2c_n)/E(S_n)) = 0. \tag{12}$$

Then (9) *holds. If* $0 \leqslant X_n \leqslant k_n\ \forall n \geqslant 1$ *where* $\{k_n\}$ *is nondecreasing, then* (10) *holds.*

(b) *If* $0 \leqslant X_n \leqslant k_n\ \forall n \geqslant 1$ *where* $\{k_n\}$ *is nondecreasing, and* $\{q(m)\}$, $\{a_m\}$ and $\{b_m\}$ *are non-negative sequences and* α, β *are non-negative constants such that*

$$\mathrm{Cov}(X_i, X_j) \leqslant q(|i-j|)(a_i + b_j) \quad \forall i \neq j, \tag{13}$$

$$\sum_{i=1}^{n-1} a_i \leqslant \alpha E(S_n), \qquad \sum_{j=2}^{n} b_j \leqslant \beta E(S_n) \quad \forall n \geqslant 2 \tag{14}$$

and

$$\liminf_{n\to\infty} \left(\frac{\sum_{m=1}^{n-1} q(m)}{E(S_n)} \right) = 0, \quad \lim_{n\to\infty} (k_n/E(S_n)) = 0, \tag{15}$$

then (9) *holds.*

Proof. (a) follows from Lemma 2.

(b) follows from Part (a) and the following observation:

$$\begin{aligned}
\sum_{j=2}^{n} \sum_{i=1}^{j-1} \mathrm{Cov}(X_i, X_j) &\leqslant \sum_{j=2}^{n} \sum_{i=1}^{j-1} q(j-i)(a_i + b_j) \\
&= \sum_{j=2}^{n} \sum_{m=1}^{j-1} q(m)(a_{j-m} + b_j) \\
&= \sum_{m=1}^{n-1} q(m) \sum_{j=m+1}^{n} (a_{j-m} + b_j) \\
&\leqslant (\alpha + \beta) \left(\sum_{m=1}^{n-1} q(m) \right) E(S_n). \qquad \square
\end{aligned}$$

We remark that one choice of the a_m and b_m so that (14) holds with appropriate α and β is

$$a_m = E(X_m) + E(X_{m+1}), \quad b_m = E(X_m) + E(X_{m-1}).$$

Remark 1. If in the above theorem, we put $X_n = I_{A_n}$, $k_n = 1\ \forall n \geqslant 1$ so that $\limsup A_n = [\sum_{n=1}^{\infty} X_n = \infty]$ where $\{A_n\}$ is a given sequence of events satisfying $\sum_{n=1}^{\infty} P(A_n) = \infty$, we get another set of sufficient conditions for $P(\limsup A_n) = 1$. Thus, we have: If (2) holds, and for each $i < j$,

$$P(A_i \cap A_j) - P(A_i)P(A_j) \leqslant q(|i-j|)[P(A_i) + P(A_{i+1}) + P(A_j) + P(A_{j-1})],$$

$$\liminf_{n\to\infty} \left(\frac{\sum_{m=1}^{n-1} q(m)}{\sum_{m=1}^{n} P(A_m)} \right) = 0 \quad \left(a\ fortiori \sum_{m=1}^{\infty} q(m) < \infty \right),$$

then $P(\limsup A_n) = 1$. In particular, if there exists an integer $m \geqslant 0$ such that

$$P(A_i \cap A_j) \leqslant P(A_i)P(A_j) \quad \text{whenever } |i-j| > m \tag{16}$$

and (2) holds, then $P(\limsup A_n) = 1$. The last inequality definitely holds if I_{A_i} and I_{A_j} are pairwise m-dependent (*a fortiori*, pairwise independent).

Remark 2. Let $\{X_n\}$ be as in Theorem 2(b). Suppose that the conditions of Theorem 2(b) hold with $k_n = 1$, $\forall n \geqslant 1$, except that (13) and (15) are strengthened to

$$\mathrm{Cov}(X_i, X_j) \leqslant q(|j-i|)E(X_j) \quad \forall i < j, \tag{17}$$

$$\sum_{n=1}^{\infty} (q(n)/E(S_n)) < \infty. \tag{18}$$

Then (9) can be strengthened to '$S_n/E(S_n) \to 1$ a.s.' which gives an indication of the rate of growth of S_n in (9). For a proof, see Chandra and Ghosal (1998). Note that if (16) holds for some $m \geqslant 0$, then (17) and (18) hold with an appropriate q.

It is an interesting problem to derive the best possible result in the setup of either of the two remarks.

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