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CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS VIA BINARY ASSOCIATIVE OPERATION

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SUMMARY. A binary operation * over real numbers is said to be associative if $(x * y) * z = x * (y * z)$ and it is said to be reducible if $x * y = x * z$ or $y * w = z * w$ if and only if $z = y$. The operation * is said to have an identity element $\tilde{e}$ if $x * \tilde{e} = x$. We characterize different classes of probability distributions under such binary operations between random variables. Further more we characterize distributions with the almost lack of memory property or with the strong Markov property or with the periodic failure rate under such a binary operation extending the results for exponential distributions under addition operation as binary operation.

1. Introduction

Summarization of statistical data without loosing information is one of the fundamental objectives of statistical data analysis. More precisely the problem is to determine whether the knowledge of a possibly smaller set of functions of several random variables is sufficient to determine the behaviour of individual random components. For instance, if X, Y, and Z are three independent random variables, we would like to know sufficient conditions if any under which the joint distribution of $U = g(X, Y, Z)$ and $V = h(X, Y, Z)$ determine either the individual distributions of X, Y and Z, or the family to

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which they belong when $g(.)$ and $h(.)$ are specified. The functions $g(.)$ and $h(.)$ could be linear or nonlinear functions or they could be the maximum or minimum functions etc. Problems of this nature were discussed in Prakasa Rao (1992). We now study such characterization problems when the operation between the variables is a binary operation which is associative and reducible with an identity.

A binary operation * over real numbers is said to be associative if $(x * y) * z = x * (y * z)$ for all real numbers x, y, z. The binary operation * is said to be reducible if $x * y = x * z$ if and only if $y = z$ and if $y * w = z * w$ if and only if $y = z$. It is known that the general reducible continuous solution of the functional equation is $x * y = g^{-1}(g(x) + g(y))$ where $g(.)$ is a continuous and strictly monotone function provided $x, y, x * y$ belong to a fixed (possibly infinite) interval A (cf. Aczel (1966)). The function g is determined up to a multiplicative constant, that is, $g_1^{-1}(g_1(x) + g_1(y)) = g_2^{-1}(g_2(x) + g_2(y))$ for all x, y in a fixed interval A implies $g_2(x) = \alpha g_1(x)$ for all x in that interval for some $\alpha \neq 0$. We assume here after that the binary operation is reducible and associative with the function $g(.)$ continuous and strictly increasing. Further assume that there exists an identity element $\tilde{e} \in \bar{R}$ such that $x * \tilde{e} = x, x \in A$. It is also known that every continuous, reducible and associative operation defined on an interval A in the real line is commutative (cf. Aczel (1966), p.267).

Examples of such binary operations are given in Muliere and Scarsini (1987). For instance (i) if $A = (-\infty, \infty)$ and $x * y = x + y$, then $g(x) = x$, (ii) if $A = (0, \infty)$ and $x * y = xy, x > 0, y > 0$ then $g(x) = \log x$, (iii) if $A = (0, \infty)$ and $x * y = (x^\alpha + y^\alpha)^{1/\alpha}, x > 0, y > 0$ for some $\alpha > 0$, then $g(x) = x^\alpha$, (iv) if $A = (-1, \infty)$ and $x * y = x + y + xy + 1, x > -1, y > -1$, then $g(x) = \log(1 + x)$ (v)if $A = (0, \infty)$ and $x * y = xy/(x + y), x > 0$, then $g(x) = 1/x$ and (vi)if $A = (0, \infty)$ and $x * y = (x + y)/(1 + xy), x > 0, y > 0$, then $g(x) = arth\ x$.

A characterization of the multivariate normal distribution through a binary operation which is associative is given in Prakasa Rao (1974) and in Prakasa Rao (1977) for Gaussian measures on locally compact abelian groups. Muliere and Scarsini (1987) characterize a class of bivariate distributions that generalize the Marshall-Olkin bivariate exponential distribution through a functional equation involving two associative operations.

Let * be a binary operation over an interval A contained in R as described above. Suppose $X_i, 1 \leq i \leq 3$ are independent real valued random variables with probability distributions with support A. Define $\tilde{Z}_1 = X_1 * X_3$ and $\tilde{Z}_2 = X_2 * X_3$ where $Z_i = g(\tilde{Z}_i), i = 1, 2$. Suppose the joint distribution of (Z_1, Z_2) is specified. We give a characterization of the probability distributions of $X_i, 1 \leq i \leq 3$ up to changes in location with respect to the binary operation * under certain conditions in Section 2. Explicit determination of the distributions of individual components is discussed. The concept of the almost lack of memory property under a binary operation * and its relation to the periodic failure rate under the binary operation * is investigated in Section 3. A representation for distributions with periodic failure rate under the binary operation * is given. Distributions with the strong Markov property under the binary operation * are characterized in Section 4. In view of the relationship between the binary associative reducible operation * and the corresponding function $g(.)$ described earlier, most of the results are consequences of the results in Prakasa Rao (1992, 1997) and a result due to Lau and Rao (1982). Hence we omit the detailed proofs. For details, see Muliere and Prakasa Rao (2002).

2. Characterization via Binary Operations

Let * be a binary reducible associative operation over an interval A contained in R and $g(.)$ be the associated function. Suppose $X_i, 1 \leq i \leq 3$ are independent real valued random variables with probability distributions with support A contained in R. Define $\tilde{Z}_1 = X_1 * X_3$ and $\tilde{Z}_2 = X_2 * X_3$, and $Z_i = g(\tilde{Z}_i), i = 1, 2$.

THEOREM 2.1. *If the characteristic function of (Z_1, Z_2) does not vanish, then the joint distribution of (Z_1, Z_2) determines the distributions of X_1, X_2 and X_3 up to changes of locations under the binary operation *.*

This theorem is a consequence of Theorem 2.1.1 in Prakasa Rao (1992). For details, see Muliere and Prakasa Rao (2002).

REMARKS: (i)The above result was proved by Kotlarski (1967) for real valued random variables when the binary operation was the addition. It was later generalised to other types of operations such as maximum and to random elements taking values in abstract spaces by Kotlarski , Prakasa Rao and others (cf. Prakasa Rao (1992)).

(ii) The above theorem can be extended to n independent random variables $X_i, 1 \leq i \leq n$ in the following form: Define $Z_i = g(X_i * X_n), 1 \leq i \leq n-1$ where $g(.)$ is the function corresponding to the binary operation *. Suppose the characteristic function of $(Z_1, \ldots, Z_{n-1})$ does not vanish. Then the joint distribution of $(Z_1, \ldots, Z_{n-1})$ determines the distributions of $X_i, 1 \leq i \leq n$ up to changes of locations under the binary operation * .

2.1. *Explicit determination of the distributions of individual components.* Denote the characteristic function of the bivariate random vector (Z_1, Z_2) by $\phi(t_1, t_2)$ and let $\phi_X(t)$ denote the characteristic function of a random variable X. Then

$$\phi(t_1, t_2) = \phi_{g(X_1)}(t_1)\phi_{g(X_2)}(t_2)\phi_{g(X_3)}(t_1 + t_2) \tag{2.1}$$

by the independence of the random variables $X_i, 1 \leq i \leq 3$. Suppose the characteristic functions of $g(X_i), 1 \leq i \leq 3$ are nonvanishing everywhere. Then $\phi(t_1, t_2)$ is nonvanishing for $-\infty < t_1, t_2 < \infty$. Let $\gamma_i(t) = \log \phi_{g(X_i)}(t)$ be the continuous branch of the logarithm of $\phi_{g(X_i)}(.)$ with $\gamma_i(0) = 0$. Assume that $E(X_3) = m$ is finite. Then it can be shown, under some technical conditions involving interchange of limit and integration (cf. Prakasa Rao (1992)), that $\lim_{t\to 0} \frac{\gamma_3(t)}{t} = im$ and

$$\gamma_3(t) = imt + \int_0^t \frac{\partial}{\partial u}[\log[\frac{\phi(u, v)}{\phi(u, 0)\phi(0, v)}]]|_{u=0} dv. \tag{2.2}$$

Using this formula for $\gamma_3(t)$, one can compute $\phi_{g(X_3)}(t)$ and hence $\phi_{g(X_1)}(t)$ and $\phi_{g(X_2)}(t)$ by using the relations

$$\phi_{g(X_1)}(t) = \frac{\phi(t, 0)}{\phi_{g(X_3)}(t)}, \phi_{g(X_2)}(t) = \frac{\phi(0, t)}{\phi_{g(X_3)}(t)}, -\infty < t < \infty. \tag{2.3}$$

Equations (2.2) and (2.3) give the explicit formulae for computing the characteristic functions of $g(X_i), 1 \leq i \leq 3$ given the characteristic function of $(g(X_1 * X_3), g(X_2 * X_3))$. Since the function g is continuous and strictly increasing , one can obtain the distributions of $X_i, 1 \leq i \leq 3$.

2.2. *Almost lack of memory property.* Let X be a nonnegative random variable with distribution function $F(x)$. Then X is said to have the *lack of memory property* if

$$P(X > s + t|X > s) = P(X > t) \tag{2.4}$$

for all $s, t > 0$. If $P(X > s) > 0$ for all $s > 0$, then it follows that $\bar{F}(s + t) = \bar{F}(s)\bar{F}(t)$ for all $s > 0$ and $t > 0$ where $\bar{F}(x) = 1 - F(x)$. It is well known that the only continuous solution of this equation is $\bar{F}(s) = \exp\{-\lambda s\}, s > 0$ for some $\lambda > 0$. This result was generalized by Muliere and Scarsini (1987) in the following manner. Let * be a binary operation as discussed above with an identity $\tilde{e}$ and the associated function $g(.)$.. Suppose it is continuous, reducible and associative. Further suppose that X is a random variable with the distribution function F having support $(\tilde{e}, g^{-1}(\infty))$ and satisfying the relation

$$P(X > s * t | X > s) = P(X > t) \tag{2.5}$$

for all $s > \tilde{e}$ and $t > \tilde{e}$. They proved that the only continuous solution of the above equation is $\bar{F}(s) = \exp\{\alpha g(s)\}$ for some $\alpha < 0$ and for $\tilde{e} = g^{-1}(0) < s < g^{-1}(\infty)$.

REMARKS. Suppose the equation (2.5) holds. By choosing the binary operation appropriately, we can get different classes of distributions (cf. Muliere and Scarsini (1987)). For instance (i) if $x * y = x + y$, then we obtain the characterization of exponential distribution through the lack of memory property; (ii) if $x * y = xy$, then we obtain a characterization of the Pareto distribution; and (iii) if $x * y = (x^\alpha + y^\alpha)^{1/\alpha}$, then we obtain a characterization of the Weibull distribution.

A nonnegative random variable X is said to have the *almost lack of memory property* if the equation (2.4) holds for a sequence $s_n > 0, n \geq 1$ and for all $t \geq 0$. It is known that the equation (2.4) holds for a sequence $s_n > 0, n \geq 1$ and for all $t \geq 0$ if and only if there exists $d > 0$ such that $s_n = nd$ except in case when $P(X \geq d) = 0$ or $P(X \geq d) = 1$. (cf. Ramachandran and Lau (1991); Prakasa Rao (1997)).

Suppose that * is a binary associative reducible operation with an identity $\tilde{e} \in \bar{R}$ as discussed above and further suppose that the equation (2.5) holds for a random variable X, with a continuous distribution function F with support $(\tilde{e}, g^{-1}(\infty))$, for a sequence $g^{-1}(\infty) > s_n > \tilde{e}, n \geq 1$ for all $g^{-1}(\infty) > t > \tilde{e}$. Here $g(.)$ is the continuous strictly increasing function corresponding to the binary associative operation *. Equation (2.5) implies that

$$\bar{F}(s_n * t) = \bar{F}(s_n)\bar{F}(t), n \geq 1 \tag{2.6}$$

for all $t \geq 0$.

A random variable X satisfying the equation (2.6) is said to have the *almost lack of memory property under the binary associative reducible operation* *.

We now characterize the class of all such distributions.

THEOREM 2.2. *A nonnegative random variable X with a continuous distribution function has the almost lack of memory property under a binary operation * as described above if and only if its distribution function F is* *of the form* $\bar{F}(s) = p(g(s))e^{-\alpha g(s)}, \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty)$ *where* $\alpha > 0$, $g(.)$ *is the continuous strictly increasing function corresponding to the binary operation* $*$ *and* $p(.)$ *is a periodic function with period* d *for some constant* $d > 0$.

2.3. *Distributions with periodic failure rate under the binary operation* $*$. Consider a binary operation $*$ with an identity $\tilde{e}$ as described earlier. Let $g(.)$ be the corresponding continuous strictly increasing function such that $x * y = g^{-1}(g(x) + g(y))$. Let X be a random variable with a continuous distribution function of the form $\bar{F}(s) = p(g(s))e^{-\alpha g(s)}, \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty)$ where $\alpha > 0$ and $p \circ g(.)$ is periodic under the operation $*$ with period $\rho > \tilde{e}$. It is obvious that the function $p(g(s))$ is nonnegative for $\tilde{e} = g^{-1}(0) < s < g^{-1}(\infty)$ and $p(g(\tilde{e})) = p(0) = 1$. Suppose the function $p(g(.))$ is differentiable with respect to s. Then the probability density function of X is given by

$$f(s) = \begin{cases} e^{-\alpha g(s)} g'(s)(\alpha p(g(s)) - p'(g(s)), & \tilde{e}=g^{-1}(0) < s < g^{-1}(\infty), \\ 0 & \text{otherwise.} \end{cases} \quad (2.7)$$

It can be checked that the distribution function F has periodic failure rate with period ρ under the binary operation $*$ (cf. Muliere and Prakasa Rao (2002)). The following result holds.

THEOREM 2.3. *A random variable* X *with a continuous probability density function has a periodic failure rate under a binary associative reducible operation* $*$ *if and only if it has the almost lack of memory property under that operation.*

For proofs of Theorems 2.2 and 2.3, see Muliere and Prakasa Rao (2002).

2.4. *Representation for distributions with periodic failure rate under binary associative operation* $*$. Suppose X is a random variable with periodic failure rate $\lambda(.)$ with period ρ under a binary associative operation $*$ with the associated function $g(.)$. Suppose the support of X is $(\tilde{e}, g^{-1}(\infty))$ where $g(.)$ is the continuous strictly increasing function corresponding to $*$. Let F and f denote the probability distribution function and the probability density function of the random variable X respectively. It is easy to see that the random variable $g(X)$ has the distribution function $\tilde{H}(y) = 1 - H(y) = (F \circ g^{-1})(y), 0 \leq y < \infty$ and it has the periodic failure rate $(\lambda \circ g^{-1})(y)$ with period $d = g(\rho)$. For convenience, let us denote the derivative of $\tilde{H}(y)$ by $\tilde{h}(y)$. Define a new random variable Y with probability distribution function $\tilde{H}_Y(y)$ and the probability density function

$$\tilde{h}_Y(y) = \begin{cases} \tilde{h}(y)/\tilde{H}(d), & 0 \leq y \leq d \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Note that $\tilde{h}_Y(.)$ is the probability density function of the conditional distribution of the random variable $g(X)$ restricted to the interval $[0, d]$. Let $\gamma = 1 - \tilde{H}(d) = H(d)$. It follows from the arguments given in Prakasa Rao (1997) that the density function $\tilde{h}(y)$ of $g(X)$ can be represented in the form

$$\tilde{h}(y) = \tilde{h}_Y\left(y - \left[\frac{y}{d}\right] d\right)(1-\gamma)\gamma^{[\frac{y}{d}]},\ 0 \leq y < \infty \tag{2.9}$$

where $[x]$ denotes the greatest integer less than or equal to x and $\gamma = H(d)$. Further more the distrbution function of $g(X)$ is given by

$$\tilde{H}(y) = 1 - \gamma^{[\frac{y}{d}]} + (1-\gamma)\gamma^{[\frac{y}{d}]}\tilde{H}_Y\left(y - \left[\frac{y}{d}\right] d\right),\ 0 \leq y < \infty. \tag{2.10}$$

Equations (2.9) and (2.10) give representations for the density and the distribution functions of the random variable $g(X)$ with a periodic failure rate with period d and $\gamma = H(d)$. Let Y be a random variable as defined above and Z be a random variable independent of Y with $P(Z = k) = (1-\gamma)\gamma^k, k \geq 0$. It is easy to check that the the random variable $g(X)$ can be represented in the form $Y + dZ$. For details, see Prakasa Rao (1997) and Muliere and Prakasa Rao (2002). In particular the random variable X can be represented in the form $g^{-1}(Y + g(\rho)Z)$ in distribution.

3. Distributions with Strong Markov Property under a Binary Associative Operation *

Suppose X is a random variable with an exponential distribution and Y be a nonnegative random variable *independent* of X. Further suppose that $P(X > Y) > 0$. Then it is known that

$$P(X > Y + x | X > Y) = P(X > x), x \geq 0. \tag{3.1}$$

This property is known as the *strong lack of memory property* or the *strong Markov property* of the exponential distribution (cf. Ramachandran and Lau (1991); Prakasa Rao (1997)).

Suppose X is a random variable with the distribution function F and the support of X is $(\tilde{e}, g^{-1}(\infty))$ where $\tilde{e}$ is the identity corresponding to a binary operation * which is continuous, reducible and associative with an identity and $g(.)$ is the continuous strictly increasing function corresponding to the binary operation *. Let Y be a nonegative random variable *independent* of X. Further suppose that $P(X > Y) > 0$ and $P(X > s) > 0$ for all $s > \tilde{e}$ and

$$P(X > Y * s | X > Y) = P(X > s), \tilde{e} = g^{-1}(0) < s < g^{-1}(\infty). \tag{3.2}$$

Applying Theorem 2.5.1 in Ramachandran and Lau (1991), we obtain that

(i) $\bar{F}(s) = e^{-\alpha g(s)}$ for all $s \in (\tilde{e}, g^{-1}(\infty))$ with parameter $\alpha > 0$ if the support of the random variable $g(Y)$ is not contained in $A(d)$ for any $d > 0$; and

(ii)$\bar{F}(s) = p(g(s))e^{-\alpha g(s)}$ for all $s \in (\tilde{e}, g^{-1}(\infty))$ where $p(.)$ is right continuous and has period d if the support of $g(Y)$ is contained in $A(d)$ for some $d > 0$ which we take it to be the largest such d.

For details, see Muliere and Prakasa Rao (2002).

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