Sankhyā : The Indian Journal of Statistics
Special issue on Ergodic Theory and Harmonic Analysis
2000, Volume 62, Series A, Pt. 3, pp. 419–424

SOME QUESTIONS ON INTEGRAL GEOMETRY ON RIEMANNIAN MANIFOLDS*

By VISHWAMBHAR PATI
and
ALLADI SITARAM
Indian Statistical Institute, Bangalore

SUMMARY. It is a surprising but known fact that an L^1 function on $\mathbf{R}^n$ is determined once *all* its spherical averages with centres coming from certain "small" sets are known. We generalise this result to complete connected real-analytic Riemannian manifolds with a real-analytic metric (with some curvature restrictions), where superisothermal sets play the same role as balls do in the case of Euclidean space.

1. **Introduction**

Broadly speaking, integral geometry deals with the recovery of a function from the knowledge of its integrals over a class of sets, with some prescribed geometric properties. For instance, a question that has received considerable attention is the recovery of a function on $\mathbf{R}^n$ from the knowledge of its integrals on hyperplanes in $\mathbf{R}^n$. This kind of enquiry has been quite fruitful, and has led to the theory of the Radon transform. Another question is the recovery of a function from the knowledge of its integrals over spheres or balls of a *fixed* radius, with centres ranging over all of $\mathbf{R}^n$. This question is related to the Pompeiu problem.

Yet another question that has received some attention recently is : Let f be a locally integrable function on $\mathbf{R}^n$, or more generally on a (globally) symmetric space of the compact or non-compact type. If all the spherical averages of f, with the centres of the spheres coming from a "small" set Γ, are zero, then can one conclude that f is the zero function? See, for example, Agranovsky, Berenstein and Kuchment (1998), Agranovsky and Quinto (1996), Rawat and Sitaram (2000). In this paper we address this kind of question for a general complete real-analytic Riemannian manifold with sectional curvatures bounded both above and below. Note that this includes the cases of compact real-analytic manifolds, $\mathbf{R}^n$, hyperbolic spaces, symmetric spaces, and in fact all homogeneous spaces.

AMS (1991) *subject classification.* 53C65.
Key words and phrases. Integral geometry, heat kernel, Riemannian manifold.
*Dedicated to Professor M.G. Nadkarni on his 60th birthday.

If one considers measures instead of functions, one can think of the above as questions in measure theory. That is, we are asking whether two measures will agree, if they agree on a collection of specified subsets. However, for simplicity, in this paper we only consider functions.

2. The Heat Kernel

In whatever follows, M denotes a connected, complete real-analytic manifold equipped with a real analytic Riemannian metric, and with sectional curvatures bounded both above and below. Δ denotes the Laplace-Beltrami operator, which necessarily has real-analytic coefficients since M is real-analytic.

The following facts are known about the heat kernel for complete manifolds with Ricci curvature bounded below, and hence for the class of manifolds described above.

Proposition 2.1 *If M is as above, then there exists a unique heat kernel $p(t,x,y)$ with the following properties:*

(i) $p(t,x,y)$ is a smooth non-negative function defined on $(0,\infty)\times M\times M$, which is symmetric in x and y.

(ii) The integral :

$$\int_M p(t,x,y)dV(y)=1$$

for each $x\in M$ and $t>0$, where $dV(y)$ is the standard Riemannian volume element on M.

(iii) $p(t,x,y)$ satisfies the semigroup property:

$$p(s+t,x,y)=\int_M p(s,x,z)p(t,z,y)dV(z)$$

(iv) For each fixed t and y, $p(t,x,y)$ satisfies the heat-equation:

$$\frac{\partial p(t,x,y)}{\partial t}=\Delta_x p(t,x,y)$$

(v) For $\phi\in C_c^\infty(M)$, and $t>0$ define :

$$u^\phi(t,x):=\int_M p(t,x,y)\phi(y)dV(y)$$

Then $u^\phi(t,x)$ is a smooth function and satisfies:

$$\frac{\partial u^\phi}{\partial t}=\Delta_x u^\phi$$

$$\lim_{t\to 0}u^\phi(t,x)=\phi(x);\ \forall x\in M$$

For a fixed t and y, it can be easily verified that in the cases of $M = \mathbf{R}^n$, hyperbolic spaces, and compact symmetric spaces of rank 1, the set $\{x : p(t,x,y) \geq a\}$, for each fixed $t > 0$, is a ball centred at y, and as a varies in $(0, \infty)$, one recovers the collection of *all* balls centred at y. We therefore define:

DEFINITION 2.2. *For a fixed $t \in (0,\infty)$, $y \in M$ and $a \in (0,\infty)$ define the set:*

$$B_{t,y}(a) = \{x : p(t,x,y) \geq a\}$$

We also make the following definition:

DEFINITION 2.3. *A nonempty subset $\Gamma \subset M$ is said to be an NA-set if the only real-analytic function defined on an open set containing Γ which vanishes identically on Γ is the zero function.*

REMARK 2.4. Examples of NA-sets are subsets Γ such that their closures have positive volume with respect to the Riemannian measure. Here are some examples of "thin" NA-sets:

(i) Let X be the image in $\mathbf{R}^2$ of the map $t \mapsto (e^t \cos t, e^t \sin t), \quad t \in (-\infty, 0]$ which is called the *logarithmic spiral*. Its intersection with each line L through the origin is a countably infinite set Σ of points converging to $(0,0)$. If f is any real analytic function on $\mathbf{R}^2$ which is identically zero on X, its restriction to such a line L would be a real analytic function of one variable, whose zeroes would have to be isolated points. Thus its vanishing on Σ would force this restriction to be the zero function, and since L is arbitrary, f is identically zero.

(ii) Consider the set:

$$X = \{(x,y,z) \in \mathbf{R}^3 : \ z(x^2 + y^2) = x^3 a(z)\}$$

where $a(z) = \exp\left(\frac{1}{z^2-1}\right)$ for $|z| < 1$ and $= 0$ for $|z| \geq 1$. See Narasimhan, 1966, p. 106. There it is shown that X is a real analytic subspace of $\mathbf{R}^n$ (i.e., its germ at every point of $\mathbf{R}^n$ is a real analytic germ.) However, from Cartan (1957) it is known that every real analytic function vanishing on it is identically zero. In fact, in Cartan (1957) there are even examples of compact analytic subspaces of $\mathbf{R}^3$ which are NA-sets.

(iii) Consider a smooth or continuous closed Jordan curve $X \subset \mathbf{R}^2$ such that for each finite subset $F \subset X$, $X \setminus F$ is not a real analytic submanifold of $\mathbf{R}^2$. Then we claim that X is an NA set in $\mathbf{R}^2$. For, let f be a real analytic function on $\mathbf{R}^2$ vanishing identically on X, where f is not the zero function. It is known from the work of Whitney, Thom, Mather and Hironaka (see Hironaka, 1973 p. 489) that the zero set $V(f)$ of f is a Whitney stratified set. In other words, $V(f) = \cup_\alpha S_\alpha$ where the strata S_α satisfy:

(i) $S_\alpha \cap S_\beta = \phi$ for $\alpha \neq \beta$.

(ii) S_α is a connected real analytic submanifold of $\mathbf{R}^2$ for each α.

(iii) $S_\alpha \cap \overline{S}_\beta \neq \phi$ implies $S_\alpha \subset \overline{S}_\beta$.

(iv) The collection $\{S_\alpha\}$ is a locally finite collection of subsets of $\mathbf{R}^2$.

Clearly since $f \neq 0$, it cannot vanish on any non-empty open set, so the only strata of $V(f)$ are one and zero dimensional. Also the local finiteness in (iv) above forces the intersection of X with the zero dimensional strata to be a finite set F. Thus, if f vanishes on X, $X \setminus F$ is a subset of the union of all one dimensional strata, and since $X \setminus F$ is a one dimensional manifold, it is forced to be an open subset of the union of one dimensional strata. In particular $X \setminus F$ is a real analytic submanifold of $\mathbf{R}^2$, contrary to the hypothesis on X.

We can now state the main result of this paper.

Theorem 2.5. *Let M be as above, and $f \in L^1(M, dV)$. Let Γ be an NA- set, and fix a $t \in (0, \infty)$. If*

$$\int_{B_{t,y}(a)} f dV = 0$$

for all $a \in (0, \infty)$ and $y \in \Gamma$, then f is the zero function.

Remark 2.6. For simplicity we have restricted ourselves to the case of L^1-functions, but the methods of this paper should also be applicable to other L^p spaces, or even some spaces of measures, as in Rawat and Sitaram (2000).

3. Proof of the Theorem

We begin with a few lemmas:

Lemma 3.1. *If $f \in L^1(M, dV)$, the function :*

$$u^f(t, x) := \int_M p(t, x, y) f(y) dV(y)$$

exists for all $x \in M$ and $t > 0$, and

(i) u^f defines a distribution solution to the heat equation $Lu = 0$ where $L = \partial_t - \Delta_x$. Further,

(ii) $u^f(t, .) \to f$ as $t \to 0$ in the sense of distributions.

Proof. In view of the assumptions on our manifold, since the sectional curvature is bounded above, the estimate (1.1) in Davies, (1993), and the volume comparison theorem 3.101 in Gallot, Hulin and Lafontaine, (1980), it follows that for each fixed $t > 0$, the heat kernel $p(t, x, y) \in L^\infty(M \times M)$. (In fact, for fixed $t > 0$, and $y \in M$, $p(t, ., y) \to 0$ at ∞ when M is non-compact.) Hence $u^f(t, x)$ exists and is, in fact, a bounded L^1-function on M for each fixed t.

Now just take a sequence of functions $\phi_n \to f$ in $L^1(M, dV)$ with $\phi_n \in C_c^\infty(M)$, and apply the main proposition 2.1 of the last section, to conclude that $u^{\phi_n} \to u^f$ and $Lu^{\phi_n} \to Lu^f$ in the sense of distributions. Thus (i) follows.

To prove the second assertion (ii), observe that $u^\phi(t, .) - \phi$ is uniformly bounded for all $t > 0$ for ϕ a fixed function in $C_c^\infty(M)$. Moreover, for such a ϕ, $u^\phi(t, .)$ converges to ϕ pointwise as $t \to 0$. These two facts, which easily follow from the calculations on p. 191 of Chavel, (1984), imply (ii). □

Lemma 3.2. *For $f \in L^1(M, dV)$, if $u^f(t, .)$ is the zero function, then so is $u^f(\frac{t}{2}, .)$. Consequently, if for a fixed $t > 0$, $u^f(t, .)$ is the zero function, so is f.*

Proof. As noted earlier, for a fixed $t > 0$, the heat kernel $p(t, ., .)$ is a bounded function on $M \times M$. Hence, for any fixed t, $u^f(t, .)$ is a bounded L^1-function, and hence in $L^2(M, dV)$. An easy calculation, using the semigroup property of the heat kernel, shows that :

$$0 = \int_M u^f(t,x)f(x)dV(x) = \left\| u^f\left(\frac{t}{2}, .\right)\right\|_2^2$$

which implies that $u^f(\frac{t}{2}, .)$ is the zero function. Repeating this argument shows that $u^f(\frac{t}{2^n}, .) = 0$. Now use (ii) of 3.1 to conclude that f is the zero function. □

Proof of the main theorem. Fix t. Let $B_{t,y}(a)$ be as in Definition 2.2. Further let

$$S_{t,y}(a) := \{x : p(t, x, y) = a\}$$

Under the assumptions on M, $p(t, ., y) \to 0$ at ∞ in case M is non-compact. Hence $S_{t,y}(a)$ and $B_{t,y}(a)$ are compact, whether M is compact or not. Since $p(t, ., y)$ is a solution of the heat equation, which has analytic coefficients, by the analytic regularity theorem for parabolic equations (see p. 324 of Friedman, 1983), it follows that $p(t, ., y)$ is real analytic. Hence $S_{t,x}(a)$ is an analytic submanifold of M for almost all a, i.e. the regular values a of $p(t, ., y)$. Even if a is not a regular value, $S_{t,y}(a)$ will be an analytic set, and hence a manifold upon removing a negligible subset (see Hironaka, 1973). Hence, fixing y, we parametrise the manifold M by specifying a point on the level set $S_{t,y}(a)$ and a. With this parametrisation we can express the integral:

$$\int_{B_{t,y}(a)} f dV = \int_a^{\infty} \left(\int_{S_{t,y}(r)} f d\sigma_r \right) dr$$

where $d\sigma_a$ is a suitable measure on $S_{t,y}(a)$. Thus if y is such that $\int_{B_{t,y}(a)} f dV = 0$ for all a, then it is easy to see that $\int_{S_{t,y}(a)} f d\sigma_a = 0$ for almost all a. (Note that in view of the boundedness of $p(t, ., y)$, $B_{t,y}(r)$ and $S_{t,y}(r)$ are empty for sufficiently large r, and hence the inner integral is zero for sufficiently large r).

Hence, from the definition of $S_{t,y}(a)$, it follows that :

$$\int_M p(t, x, y) f(y) dV(y) = \int_0^{\infty} a \left(\int_{S_{t,y}(a)} f d\sigma_a \right) da = 0$$

Thus, for such y, $u^f(t, y) = 0$. Since $u^f(t, .)$ is a distributional solution of the heat equation, which has analytic coefficients, by the analytic regularity theorem for parabolic equations (see p. 324 of Friedman, 1983), it follows that $u^f(t, y)$ is analytic in y.

Since we have assumed that $\int_{B_{t,y}(a)} f dV = 0$ for all $y \in \Gamma$, and all a, it follows that $u^f(t, y) = 0$ for all $y \in \Gamma$. Since Γ is an NA-set, $u^f(t, .)$ is the zero function, and by Lemma 3.2, f is the zero function, and the proof of the main theorem is complete. □

4. Concluding Remarks

We could have directly formulated our results in terms of averages over $S_{t,y}(a)$, but if we want to formulate the kind of problem considered here, for measures, then it is better to consider $B_{t,y}(a)$ (see also Rawat and Sitaram (2000)).

In questions of integral geometry on symmetric spaces, one usually considers averages of functions over spheres or balls. The isothermal (resp. superisothermal) sets $S_{t,y}(a)$ (resp. $B_{t,y}(a)$) considered here seem to be the natural generalisations of the spheres (resp. balls) in symmetric spaces to Riemannian manifolds.

References

Agranovsky, M.L., Berenstein, C., and Kuchment, P., (1998). Approximation by Spherical waves in L^p-spaces, *J. Geom. Anal.* **6**, 365–383.

Agranovsky, M.L., and Quinto, E.T., (1996). Injectivity sets for the Radon transform over circles and complete systems of radial functions, *J. Funct. Anal.* **139**, 383–414.

Chavel, I. (1984). *Eigenvalues in Riemannian Geometry*, Academic Press, New York.

Davies, E.B. (1993). The state of the art for heat kernel bounds on negatively curved manifolds, *Bull. London Math. Soc.* **25**, 289–292.

Friedman, A. (1983). *Partial Differential Equations of Parabolic Type*, Robert Krieger Publishing Co., Florida.

Gallot, S., Hulin, D., and Lafontaine, J. (1980). *Riemannian Geometry*, Springer-Verlag Universitext.

Cartan, Henri, (1957). Varietes analytiques reelles et varietes analytiques complexes, *Bull. Soc. Math. France* **85**, 77–99.

Hironaka, H., (1973). Subanalytic sets, In: *Number Theory, Algebraic Geometry and Commutative Algebra*, Volume in honour of Y. Akizuki, Kinokuniya, Tokyo, 453–493.

Narasimhan, R., (1966). *Introduction to the Theory of Analytic Spaces*, Springer Lecture Notes in Mathematics **25**.

Rawat, R., and Sitaram A., (2000). Injectivity sets for Radon tranforms over spheres on $\mathbf{R}^n$ and symmetric spaces, *J. Fourier Anal. Appl.* **6**, 343–348.

Vishwambhar Pati and Alladi Sitaram
Indian Statistical Institute
R.V. College P.O., Bangalore 560 059
India
E-mail: pati@isibang.ac.in, sitaram@isibang.ac.in