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ON THE HEWITT-SAVAGE ZERO ONE LAW IN THE STRATEGIC SETUP

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SUMMARY. In the i.i.d. strategic setup it is shown that the symmetric σ field is either trivial or nonatomic. Conditions are given for these to occur.

1. **Introduction**

In the countably additive theory of probability the Hewitt Savage $0-1$ law states the following : In a product space with independent identical components, every symmetric set has probability either zero or one. This theorem has diverse applications, especially in Random Walks, Potential theory and U-statistics. In the context of finitely additive probabilities Dubins and Savage (1965) and Dubins (1974) initiated the theory of integration on countable product spaces. This was extended by Purves and Sudderth in their seminal paper (1976) and this setup is usually referred to as the strategic setup. They observed in (Purves and Sudderth, 1983) that the Hewitt Savage $0-1$ law fails in this setup. The authors have noted in (Gangopadhyay and Rao, 1998) that this failure can indeed be spectacular – the symmetric σ field could be purely nonatomic. In this paper we restrict our attention to product spaces where the component space is a countable set. Let γ be a finitely additive probability on this set. We show that the Hewitt-Savage 0-1 law holds for the infinite product measure $\gamma \times \gamma \times \gamma \times \cdots$ if and only if γ takes at most two values when restricted to subsets of the set $\{i : \gamma(i) = 0\}$. Moreover when this does not hold, then the symmetric σ field is indeed nonatomic.

The organization of the paper is as follows : In Section 2 we set up the notation and recall basic facts as well as some results needed in the sequel. In Section 3 we describe an alternative way (Theorem 3.1) to select a point in the

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infinite product space with distribution σ_γ. In Section 4 we prove the main result (Theorem 4.1) described in the earlier paragraph. We conclude with some remarks in the last section.

2. **Preliminaries**

Throughout, $\mathbf{N}$ denotes the set of nonnegative integers $\{0, 1, 2, \cdots\}$ and $\mathbf{Z}$ is the set of all integers. $H = \mathbf{N}^\infty$ is the space of histories $h = (h_1, h_2, \cdots)$ equipped with the product topology where each coordinate space has the discrete topology. The σ field of H is its Borel σ field. Let γ be a finitely additive probability on $\mathbf{N}$. σ_γ denotes the strategic measure on H induced by the i.i.d. strategy γ. It is the unique finitely additive probability σ on the Borel σ field of H characterized by the following three properties (Theorem 4.1 p.265 and Theorem 5.1 p.268 of Purves and Sudderth, 1976)

(i) For any Borel $B \subset H, \sigma(B) = \int \sigma(Bi)d\gamma(i)$ where

$$Bi = \{(h_2, h_3, \cdots) : (i, h_2, h_3, \cdots)\epsilon B\}$$

(ii) If $O \subset H$ is open then

$$\sigma(O) = \sup\{\sigma(K) : K \text{ clopen }, \ K \subset O\}$$

(iii) If $B \subset H$ is Borel and $\epsilon > 0$ then there is a closed set C and open set O such that

$$C \subset B \subset O \quad \text{ and } \quad \sigma(O - C) < \epsilon.$$

In the countably additive case this gives the usual measure. More precisely

(iv) If γ is countably additive then σ_γ coincides with the usual product probability γ^∞. Let Seq denote the set of finite sequences of elements of $\mathbf{N}$, including the empty sequence. If $B \subset H$ and $p \in Seq$ then $Bp = \{h : ph \in B\}$ where ph is the usual concatenation. If s is a stop rule and $h \in H$ then $p_s(h)$ is the finite sequence $(h_1, \ldots, h_m)$ where $m = s(h)$. In particular if $s \equiv n$ then $p_s(h)$ is denoted by $p_n(h)$. In the sequel, we need the following properties of σ_γ :

(v) (Cor.4.1 p.265 Purves and Sudderth, 1976) If B is Borel and s a stop rule then,

$$\sigma_\gamma(B) = \int \sigma_\gamma(Bp_s(h))d\sigma_\gamma(h)$$

(vi) (Lemma 5.2 p.266 of Purves and Sudderth, 1996) If $B^1, B^2, \ldots$ are Borel and $\sigma_\gamma(B^n p_n(h)) = 0$ for all n and h then $\sigma_\gamma(\cup B^n) = 0$.

(vii) (Theorem 5.2, p.269 of Purves and Sudderth, 1976) If $B^1, B^2, \ldots$ are increasing Borel sets then

$$\sigma_\gamma(\cup B^n) = \sup_s \sigma_\gamma(B^s)$$

where the supremum is taken over all stop rules s and B^s denotes the set $\{h \in H : h \in B^{s(h)}\}$.

(viii) (Theorem 7.2 p.275 of Purves and Sudderth, 1976) If $A \subset \mathbf{N}$ then

$$\sigma_\gamma\{h : \frac{1}{n}\sum_{k=1}^{n} 1_A(h_k) \to \gamma(A)\} = 1.$$

This is a strong law of large numbers.

(ix) (Theorem 3.1 p.34 of Purves and Sudderth, 1983) If B is a tail Borel set then $\sigma_\gamma(B) = 0$ or 1. Recall that a set $B \subset H$ is a tail set provided $Bp = Bq$ whenever $p, q \in Seq$ have the same length.

This is Kolmogorov $0-1$ Law.

(x) (Theorem 4.1 p.35 of Purves and Sudderth, 1983) If B is a Borel set then

$$\sigma_\gamma\{h : \sigma_\gamma(Bp_n(h)) \to 1_B(h)\} = 1$$

This is Levy $0-1$ Law.

The reader should note that most of these results are valid more generally. We stated them for an i.i.d. strategic measure σ_γ, because we will be interested only in this case.

3. An Identification of σ_γ

Given a finitely additive probability γ on $\mathbf{N}$, define the countably additive measure γ_1 on $\mathbf{N}$ by setting $\gamma_1(i) = \gamma(i)$ and set $\gamma_2 = \gamma - \gamma_1$. Then γ_2 is purely finitely additive and $\gamma = \gamma_1 + \gamma_2$. This is nothing but the Hewitt- Yosida - Kakutani decomposition (Yosida and Hewitt, 1952 and Ranga Rao, 1958). Set $\mathbf{N}_\infty = \mathbf{N} \cup \{\infty\}$. λ denotes the countably additive probability on $\mathbf{N}_\infty$ defined by $\lambda(i) = \gamma_1(i) = \gamma(i)$ for $i \in \mathbf{N}$ and $\lambda(\infty) = \gamma_2(\mathbf{N})$. μ denotes the finitely additive probability on $\mathbf{N}$ defined by $\mu(A) = \frac{\gamma_2(A)}{\gamma_2(\mathbf{N})}$. This μ is just γ_2, normalized. Throughout this section we assume that $0 < \gamma_1(\mathbf{N}) < 1$, so that λ is not point mass at ∞ and μ is well defined.

We are going to use μ and λ to select a sequence of integers with distribution σ_γ. To get started, here is the method for selecting a single integer z_1: first use μ to select x_1 and then λ to select y_1. If $y_1 \neq \infty$, set $z_1 = y_1$, and if $y_1 = \infty$, set $z_1 = x_1$. Then the probability that $z_1 \in E$ is $\gamma(E)$.

To select an infinite sequence of integers with distribution σ_γ, first use μ (repeatedly and independently) to select $x_1, x_2, \ldots$ and then use λ (again repeatedly and independently) to select $y_1, y_2, \ldots$. Let x be the first sequence and y the second one. Let $T(x, y)$ be the sequence obtained by replacing the first occurrence of ∞ in y by x_1, the second occurrence of ∞ in y by x_2, and so on, until all the infinities have been replaced. Then $T(x, y)$ has distribution σ_γ. This is the content of Theorem 1 below.

To state the Theorem, we need some notation. Set $H_\infty = \mathbf{N}_\infty \times \mathbf{N}_\infty \times \ldots$, equipped with the product topology, where $\mathbf{N}_\infty$ has the discrete topology.

Set $H^1 = H \times H_\infty$. Points in H^1 are denoted by (x, y), where $x \in H$ and $y \in H_\infty$. Let σ_μ be the finitely additive probability on H induced by the i.i.d. strategy μ. Let σ_λ be the usual countably additive product measure $\lambda \times \lambda \times \ldots$. For any Borel set C in H^1 and any $x \in H$, let C_x be the section of C at x, namely, $C_x = \{y \in H_\infty : (x, y) \in C\}$. As σ_λ is countably additive, observe that $\sigma_\lambda(C_x)$ is a measurable function of x on H. Consequently, the expression $\int \sigma_\lambda(C_x) d\sigma_\mu(x)$ is well defined. Denote this by $\sigma'(C)$. Then σ' is a finitely additive probability on H^1.

Here is the precise formulation of the selection procedure mentioned above.

THEOREM 1. *For Borel $B \subset H$, $\sigma'(T^{-1}B) = \sigma_\gamma(B)$.*

We start making a series of observations leading to the proof. For any infinite sequence $v = (v_1, v_2, \ldots)$ we let $v_{(1)} = (v_2, v_3, \ldots)$.

1^o. For any Borel set $S \subset H^1$ and $i \in \mathbf{N}$, define

$$iS = \{(x, y) \in H^1 : y_1 = i \;\;\&\;\; (x, y_{(1)}) \in S\}$$

Similarly we can define iS for any Borel set $S \subset H_\infty$ as follows: $iS = \{y \in H_\infty : y_1 = i \;\;\&\;\; y_{(1)} \in S\}$.

Claim : $\sigma'(iS) = \lambda(i)\sigma'(S)$. To see this observe that $(iS)_x = iS_x$ so that

$$\sigma'(iS) = \int \sigma_\lambda(iS)_x d\sigma_\mu(x) = \lambda(i) \int \sigma_\lambda(S_x) d\sigma_\mu(x) = \lambda(i)\sigma'(S).$$

2^o. *Claim* : $\sigma'(\cup_{i \in \mathbf{N}} iS) = \sum_{i \in \mathbf{N}} \sigma'(iS)$. To see this, note that,

$$\sigma_\lambda(\cup iS_x) = \sum_i (\lambda(i))\sigma_\lambda(S_x)$$

for any $x \in H$ — by the countable additivity of λ. Now

$$\begin{aligned} \sigma'(\cup iS) &= \textstyle\int \sigma_\lambda(\cup iS_x) d\sigma_\mu(x) \\ &= \textstyle\int (\sum \lambda(i)) \sigma_\lambda(S_x) d\sigma_\mu(x) \\ &= \textstyle(\sum \lambda(i))\sigma'(S) = \sum \sigma'(iS) \end{aligned}$$

where 1^o is used in the last equality.

More generally we have,

3^o *Claim* : For each $i \in \mathbf{N}$, let S_i be a Borel subset of H^1. Then

$$\sigma'(\cup_{i\in\mathbf{N}} iS_i) = \sum_{i\in\mathbf{N}} \sigma'(iS_i).$$

To see this fix any integer $k > 1$. Proceeding as in 2^o we get

$$\begin{aligned} \sigma'(\cup iS_i) &= \int \textstyle\sum_i \lambda(i)\sigma_\lambda((S_i)_x)d\sigma_\mu(x) \\ &\geq \int \textstyle\sum_{i=0}^k \lambda(i)\sigma_\lambda((S_i)_x)d\sigma_\mu(x) \\ &= \textstyle\sum_{i=0}^k \sigma'(iS_i) \end{aligned}$$

where we used the finite additivity of the integral and 1^o for the last equality. k being arbitrary we get

$$\sigma'(\cup iS_i) \geq \sum \sigma'(iS_i).$$

To show the reverse inequality, fix $\epsilon > 0$ and an integer k such that $\sum_{k+1<i<\infty} \lambda(i) < \epsilon$. Observe that $\sum_{k+1<i<\infty} \lambda(i)\sigma_\lambda((S_i)_x) < \epsilon$ for each x so that proceeding as above

$$\sigma'(\cup iS_i) \leq \sum_{i=0}^k \sigma'(iS_i) + \epsilon \leq \sum_{i=0}^\infty \sigma'(iS_i) + \epsilon.$$

ϵ being arbitrary the proof is complete.

4^o. For each $i \in \mathbf{N}$, $B_i \subset H$ be a Borel set and $B = \cup_{i\in\mathbf{N}}(iB_i)$. Then

$$\sigma'\{(x,y) \in T^{-1}B : y_1 = \infty\} = \lambda(\infty) \int \sigma'(T^{-1}B_i)d\mu(i).$$

To see this denote by C the set in braces on the left side. Observe that $y \in C_x$ iff $y_1 = \infty$ and $T(x_{(1)}, y_{(1)}) \in Bx_1$ (Recall that $Bx_1 = \{h \in H : x_1h \in B\}$). Thus $y \in C_x$ iff $y_1 = \infty$ and $y_{(1)} \in (T^{-1}Bx_1)_{x(1)}$. Thus

$$\begin{aligned} \sigma'(C) &= \int \sigma_\lambda(C_x)d\sigma_\mu(x) \\ &= \lambda(\infty) \int \sigma_\lambda(T^{-1}Bx_1)_{x(1)}d\sigma_\mu(x) \\ &= \lambda(\infty) \int \sigma'(T^{-1}Bx_1)d\mu(x_1) \\ &= \lambda(\infty) \int \sigma'(T^{-1}B_i)d\mu(i) \end{aligned}$$

where in the third equality we used 2(i) applied to the function (instead of sets), $f(x) = \sigma_\lambda(T^{-1}B)_x$. For the last equality note that $Bi = B_i$.

5^o. For any clopen set $\Gamma \subset H$, $\sigma'(T^{-1}\Gamma) = \sigma_\gamma(\Gamma)$.

The proof is by induction on the rank of the clopen set Γ. Of course if $\Gamma = \phi$ or H this is trivial. Suppose then that $\Gamma = \cup(i\Gamma_i)$ and $\sigma_\gamma(\Gamma_i) = \sigma'(T^{-1}\Gamma_i)$. We show that $\sigma_\gamma(\Gamma) = \sigma'(T^{-1}\Gamma)$.

$$\begin{aligned}
\sigma_\gamma(\Gamma) &= \textstyle\int \sigma_\gamma(\Gamma i) d\gamma(i) \quad \text{(by } 2(i))\\
&= \textstyle\int \sigma_\gamma(\Gamma_i) d\gamma(i) \quad \text{(since, } \Gamma i = \Gamma_i)\\
&= \textstyle\sum_i \sigma_\gamma(\Gamma_i)\lambda(i) + \lambda(\infty)\int \sigma_\gamma(\Gamma_i) d\mu(i)\\
&= \textstyle\sum_i \sigma'(T^{-1}\Gamma_i)\lambda(i) + \lambda(\infty)\int \sigma'(T^{-1}\Gamma_i) d\mu(i)\\
&\quad \text{(by induction hypothesis)}\\
&= \sigma'(\cup_i i T^{-1}\Gamma_i) + \sigma'\{(x,y) : y_1 = \infty \quad \text{and} \quad T(x,y) \in \Gamma\}\\
&\quad \text{(by } 3^o \text{ and } 4^o)\\
&= \sigma'(T^{-1}\Gamma) \quad (\sigma' \text{ being finitely additive}).
\end{aligned}$$

6^o. For any open set $U \subset H$, $\sigma'(T^{-1}U) \geq \sigma_\gamma(U)$.

Indeed fix $\epsilon > 0$ and by the regularity property of strategic probability (see 2(ii)), get clopen $\Gamma \subset U$ with $\sigma_\gamma(U) \leq \sigma_\gamma(\Gamma) + \epsilon$. Note that $T^{-1}\Gamma \subset T^{-1}U$ so that

$$\begin{aligned}
\sigma_\gamma(U) &\leq \sigma_\gamma(\Gamma) + \epsilon = \sigma'(T^{-1}\Gamma) + \epsilon \quad \text{by } 5^o\\
&\leq \sigma'(T^{-1}U) + \epsilon.
\end{aligned}$$

ϵ being arbitrary this proves the stated inequality.

7^o. If $V \subset H^1$ is open and $\epsilon > 0$ then there is a clopen set $C \subset V$ such that $\sigma'(V) \leq \sigma'(C) + \epsilon$.

To see this, fix clopen sets $L_i \subset H$ and $M_i \subset H_\infty$ such that $V = \cup_i(L_i \times M_i)$. Put $V_n = \cup_{i \leq n}(L_i \times M_i)$ so that V_n are clopen and $V_n \uparrow V$. Set

$$A_n = \{x : \sigma_\lambda((V_n)_x) > \sigma_\lambda(V_x) - \frac{\epsilon}{2}\}.$$

As σ_λ is countably additive, $A_n \uparrow H$ and hence $\sigma_\mu(\cup_n A_n) = 1$. By 2(vii) get a stoptime τ with $\sigma_\mu(A_\tau) > 1 - \epsilon/2$. Define

$$C = \{(x,y) : (x,y) \in V_{\tau(x)}\} = \cup_n[V_n \cap \{(x,y) : \tau(x) = n\}].$$

Then $C \subset V$, C is clopen. Note that if $x \in A_\tau$ then $\sigma_\lambda(C_x) \geq \sigma_\lambda(V_x) - \epsilon/2$. Moreover for any x, $\sigma_\lambda(C_x) \geq \sigma_\lambda(V_x) - 1$. We will use the last inequality for $x \notin A_\tau$. As a result

$$\begin{aligned}\sigma'(C) &= \int \sigma_\lambda(C_x)d\sigma_\mu(x)\\ &= \int_{A_\tau} \sigma_\lambda(C_x)d\sigma_\mu(x) + \int_{A_\tau^c} \sigma_\lambda(C_x)d\sigma_\mu(x)\\ &\geq \int \sigma_\lambda(V_x)d\sigma_\mu(x) - \tfrac{\epsilon}{2}\sigma_\mu(A_\tau) - \sigma_\mu(A_\tau^c) \geq \sigma'(V) - \epsilon.\end{aligned}$$

To proceed further we introduce a hypothesis, which will be relaxed later. $(\perp) : \mathbf{N} = \mathbf{N}_1 \cup \mathbf{N}_2,\ \mathbf{N}_1 \cap \mathbf{N}_2 = \phi,\ \mu(\mathbf{N}_1) = 1,\ \lambda(\mathbf{N}_2 \cup \{\infty\}) = 1.$

8^o. Assume $(\perp)$. For any open $U \subset H$; $\sigma'(T^{-1}U) \leq \sigma_\gamma(U)$.

To see this, temporarily denote by H_1, the set of sequences of points from $\mathbf{N}_1$. Denote by H_2, the set of sequences of points from $\mathbf{N}_2 \cup \{\infty\}$ with infinitely many occurrences of ∞. Set $D = H_1 \times H_2 \subset H^1$.Let R be the set of sequences of points of $\mathbf{N}$ having infinitely many occurrences of elements from $\mathbf{N}_1$. Then by the usual SLLN, $\sigma_\lambda(H_2) = 1$.

Consequently $\sigma'(D) = 1$. By strategic SLLN (see 2(viii)), $\sigma_\gamma(R) = 1$. Moreover T is a homeomorphism on D onto R. Fix $\epsilon > 0$. $T^{-1}U$ being open in H^1, use 7^o to get a clopen set $\Gamma \subset T^{-1}U$ with $\sigma'(T^{-1}U) \leq \sigma'(\Gamma) + \epsilon$.

Note that

$$\begin{aligned}\sigma'(\Gamma) &= \sigma'(\Gamma \cap D), \quad \text{as } \sigma'(D) = 1\\ &\leq \sigma'(T^{-1}T(\Gamma \cap D)).\end{aligned}$$

As Γ is closed in H^1, $\Gamma \cap D$ is closed in D so that $T(\Gamma \cap D)$ is closed in R. Say $T(\Gamma \cap D) = C \cap R$ where C is closed in H.

Thus

$$\begin{aligned}\sigma'(\Gamma) &\leq \sigma'(T^{-1}(C \cap R))\\ &\leq \sigma'(T^{-1}C)\\ &\leq \sigma_\gamma(C) \quad \text{by } 6^o\\ &= \sigma_\gamma(C \cap R) \quad \text{as } \sigma_\gamma(R) = 1\\ &= \sigma_\gamma(T(\Gamma \cap D))\\ &\leq \sigma_\gamma(U) \quad \text{as } \Gamma \subset T^{-1}U.\end{aligned}$$

Thus $\sigma'(T^{-1}U) \leq \sigma_\gamma(U) + \epsilon$. Since ϵ is arbitrary we are done.

Combining 6^o and 8^o we immediately obtain:

9^o. Assume $(\perp)$. For any open $U \subset H$, $\sigma'(T^{-1}U) = \sigma_\gamma(U)$.

This leads us to a special case of the theorem, which we state as a Lemma.

LEMMA 1. *Assume* $(\perp)$. *For any Borel* $B \subset H$, $\sigma'(T^{-1}B) = \sigma_\gamma(B)$.

PROOF. Suffices to show that $\sigma_\gamma(B) \geq \sigma'(T^{-1}B)$. To this end, fix $\epsilon > 0$. By 2(iii) , take open $U \supset B$ with $\sigma_\gamma(B) \geq \sigma_\gamma(U) - \epsilon$. Then we have,

$$\begin{aligned}\sigma_\gamma(B) &\geq \sigma_\gamma(U) - \epsilon\\ &= \sigma'(T^{-1}U) - \epsilon \quad \text{by } 9^0\\ &\geq \sigma'(T^{-1}B) - \epsilon \quad \text{as } T^{-1}U \supset T^{-1}B\end{aligned}$$

Since ϵ is arbitrary the proof is complete. □

To remove the assumption ($\perp$) we need to work a little more. Suppose ϕ is a map on $\mathbf{Z}$ onto $\mathbf{N}$. Let η be a finitely additive probability on $\mathbf{Z}$ and γ be defined on $\mathbf{N}$ by $\gamma(A) = \eta(\phi^{-1}A)$. Let σ_η and σ_γ be the strategic measures on $\mathbf{Z}^\infty$ and $\mathbf{N}^\infty = H$ respectively. Define ϕ_∞ on $\mathbf{Z}^\infty$ by $\phi_\infty(x_1, x_2, \cdots) = (\phi(x_1), \phi(x_2), \cdots)$. It is natural to expect that as in the countably additive case, $\sigma_\gamma(B) = \sigma_\eta(\phi_\infty^{-1}B)$ for all Borel sets $B \subset H$. We have not been able to establish this. We show that if ϕ is a finite-to-one map (that is, for every i, $\phi^{-1}\{i\}$ is finite), then this is indeed correct. The general proof eludes us.

Lemma 2. *Let ϕ be a finite-to-one map. Then for every Borel set $B \subset H$*

$$\sigma_\eta(\phi_\infty^{-1}(B)) = \sigma_\gamma(B)$$

Proof. Since both σ_η and σ_γ are strategic measures, it is easy to see by using induction on the rank of clopen sets, the desired equality holds when B is a clopen set in H. Proceeding as in 6^o we can establish that $\sigma_\gamma(U) \le \sigma_\eta(\phi_\infty^{-1}U)$ for all open sets $U \subset H$. Note that ϕ_∞ takes open sets to open sets. As ϕ is finite-to-one, a simple argument shows that ϕ_∞ takes closed sets to closed sets as well. Thus for any clopen set $\Gamma \subset \mathbf{Z}^\infty$, $\phi_\infty(\Gamma)$ is a clopen set in H. Now take any open set $U \subset H$ and fix $\epsilon > 0$.Then there exists a clopen set $\Gamma \subset \phi_\infty^{-1}(U)$ with $\sigma_\eta(\phi_\infty^{-1}U) \le \sigma_\eta(\Gamma) + \epsilon$. Note that $\sigma_\eta(\Gamma) \le \sigma_\eta(\phi_\infty^{-1}\phi_\infty\Gamma) = \sigma_\gamma(\phi_\infty\Gamma) \le \sigma_\gamma(U)$ where the equality is a consequence of the fact that $\phi_\infty\Gamma$ is clopen. ϵ being arbitrary we deduce that $\sigma_\eta(\phi_\infty^{-1}U) \le \sigma_\gamma(U)$. This establishes the result for B an open set. Now proceed as in Lemma 1 to complete the proof. □

Proof of theorem 1. The main idea is to shift the purely finitely additive part of γ to the negative integers so that ($\perp$) holds. Then we will apply Lemma 1. Finally we bring back the mass from negative integers. These three steps are achieved with the help of the maps ψ_∞, $\overline{T}$ and ϕ_∞ as detailed below.

Given γ on $\mathbf{N}$ define $\overline{\gamma}$ on $\mathbf{Z}$ by $\overline{\gamma}(A) = \gamma_1(A) + \gamma_2(-A)$ where $-A = \{-x : x \in A\}$. Recall that γ_1 and γ_2 are respectively, the countably additive part and finitely additive part of γ. Define $\overline{\mu}$ and $\overline{\lambda}$ for $\overline{\gamma}$ just as μ, λ were defined for γ. Define σ'' on $\mathbf{Z}^\infty \times \mathbf{Z}_\infty^\infty$ with $\sigma_{\overline{\mu}}$ and $\sigma_{\overline{\lambda}}$ just as σ' was defined on $\mathbf{N}^\infty \times \mathbf{N}_\infty^\infty$ with σ_μ and σ_λ.

Define the map $\psi_\infty : \mathbf{N}^\infty \times \mathbf{N}_\infty^\infty \to \mathbf{Z}^\infty \times \mathbf{Z}_\infty^\infty$ by $\psi_\infty(x, y) = (-x, y)$ where $-x = (-x_1, -x_2, \ldots)$ if $x = (x_1, x_2, \ldots)$. Then it is immediate that $\sigma''(B) = \sigma'(\psi_\infty^{-1}B)$ for each Borel set $B \subset \mathbf{Z}^\infty \times \mathbf{Z}_\infty^\infty$.

Define the map $\overline{T} : \mathbf{Z}^\infty \times \mathbf{Z}_\infty^\infty \to \mathbf{Z}^\infty$ just as T was defined from $\mathbf{N}^\infty \times \mathbf{N}_\infty^\infty$ to $\mathbf{N}^\infty$. Lemma 3.2.2 applies now to yield that $\sigma_{\overline{\gamma}}(B) = \sigma''(\overline{T}^{-1}B)$ for each Borel set $B \subset \mathbf{Z}^\infty$. Lemma 1, though stated for $\mathbf{N}$ applies to $\mathbf{Z}$ as well. Finally, define the map $\phi : \mathbf{Z} \to \mathbf{N}$ by $\phi(x) = \mid x \mid$. Lemma 2 now yields that for each Borel $B \subset \mathbf{N}^\infty$, $\sigma_\gamma(B) = \sigma_{\overline{\gamma}}(\phi_\infty^{-1}B)$.

Thus for each Borel $B \subset H$, $\sigma_{\underline{\gamma}}(B) = \sigma_{\overline{\gamma}}(\phi_\infty^{-1}B) = \sigma"(\overline{T}^{-1}\phi_\infty^{-1}B) = \sigma'(\psi_\infty^{-1}\overline{T}^{-1}\phi_\infty^{-1}B)$. Observe that $T = \phi_\infty \circ \overline{T} \circ \psi_\infty$ to complete the proof. □

4. Hewitt-Savage $0-1$ Law

A permutation π of $\{1, 2, \ldots\}$ is called a finite permutation if $\pi(n) = n$ for all sufficiently large n. If $h = (h_1, h_2, \ldots) \in H$ and π is a finite permutation then $h_\pi = (h_{\pi(1)}, h_{\pi(2)}, \ldots)$. A Borel set $B \subset H$ is called symmetric if $h_\pi \in B$ whenever $h \in B$ and π is a finite permutation.

Let γ be a finitely additive probability on $\mathbf{N}$ and as usual σ_γ the strategic measure on $H = \mathbf{N}^\infty$ induced by the i.i.d. strategy γ. Let $A = \{i \in \mathbf{N} : \gamma(i) = 0\}$. Suppose γ restricted to A takes more than two values. Say $A_o \subset A$ and $0 < \gamma(A_o) < \gamma(A)$. Then, generalizing a construction of (Purves and Sudderth, 1983) we can exhibit a symmetric Borel set $S \subset H$ such that $0 < \sigma_\gamma(S) < 1$ as follows : Let H_1 be the subset of H consisting of those histories in which elements of A occur at infinitely many coordinate places and they occur in increasing order of magnitude. Note that $\sigma_\gamma(H_1) = 1$. Let S_1 be the subset of H_1 consisting of those histories in which the first occurrence from A is from A_o. Define for all $n \geq 1$, $B_n = \{h \in H_1 : h_i \notin A \text{ for } i < n \text{ and } h_n \in A_o\}$. Then $S_1 = \cup_{n\geq 1} B_n$. Direct computation shows that for $n \geq 1$, $\sigma_\gamma(B_n) = [1 - \gamma(A)]^{n-1}\gamma(A_o)$ so that $\sigma_\gamma(S_1) \geq \frac{\gamma(A_o)}{\gamma(A)}$. Similar computation shows $\sigma_\gamma(H_1 \backslash S_1) \geq \frac{\gamma(A\backslash A_o)}{\gamma(A)}$. It follows that equality must hold at both the places. Thus if S is the symmetrization of S_1 then $0 < \sigma_\gamma(S) = \sigma_\gamma(S_1) < 1$. The set S can alternatively be described as the set of histories in which there are infinitely many occurences of elements of A and their minimum belongs to A_0. If $\beta = \max\{\frac{\gamma(A_o)}{\gamma(A)}, \frac{\gamma(A\backslash A_o)}{\gamma(A)}\}$ then we have a decomposition of H into two symmetric sets, each having σ_γ measure $\leq \beta$. Since $\beta < 1$ and the above construction can be extended by taking into account the first finite number of occurrences of points of A, we get the following : given $\epsilon > 0$ there is a decomposition of H into symmetric sets each having σ_γ measure $\leq \epsilon$.

Now suppose γ restricted to subsets of A is trivial – that is γ assumes at most two values on subsets of A. In this case we show that the Hewitt-Savage $0-1$ law holds. Main idea is the following : Suppose $S \subset H$ is a symmetric Borel set. Then for $x \in H$, $(T^{-1}S)_x$ is a symmetric Borel set in H_∞ so that $\sigma_\lambda(T^{-1}S)_x = 0$ or 1. Let $E = \{x : \sigma_\lambda((T^{-1}S)_x) = 1\}$. In case γ is countably additive on A^c then of course $\gamma \mid A$ is its purely finitely additive part and hence μ is $0-1$ valued, so is σ_μ. Thus $\sigma_\mu(E)$ is either 0 or 1. Accordingly $\sigma_\gamma(S)$ is 0 or 1. The problem becomes more difficult if γ is not countably additive on A^c or, equivalently, if γ_1 and γ_2 are not supported by disjoint sets. In that case μ may not be two valued even though $\mu|A$ is so and therefore the above argument is not applicable. But observe that we are interested only in the value of $\sigma_\mu(E)$ and if we can show that $\sigma_\mu(E)$ is either 0 or 1 we are done. That is what we are going to show next. If $A = \phi$, that is, if γ gives positive mass to every singleton then a simple calculation shows that E is a tail set in H so that $\sigma_\mu(E)$ is either 0 or 1 (see 2(ix)) as we want.

But in the general situation when A is not necessarily empty E may not be a tail set. However we can apply Levy 0-1 law (see 2(x)) to conclude that $\sigma_\mu(E) = 0$ or 1. In the countably additive case the martingale theoretic proof of the Hewitt-Savage $0-1$ law is well known. See Meyer (1996). Here is our main Theorem :

THEOREM 1. *Let γ be a finitely additive probability on* $\mathbf{N}$ *and* $A = \{i : \gamma(i) = 0\}$.

a) If γ restricted to subsets of A is trivial (assumes at most two values) then for any symmetric Borel set $S \subset H$, $\sigma_\gamma(S)$ is either 0 or 1

b) If γ restricted to subsets of A is nontrivial (assumes more than two values) then for any $\epsilon > 0$ there is a finite partition of H into symmetric Borel sets each having σ_γ measure $< \epsilon$.

PROOF. Part (b) was already established above. We shall prove (a). We can and shall assume that γ is not countably additive. We use the notation of §3. In particular μ, λ, σ', T are as discussed there. Let $S \subset H$ be a symmetric Borel set.

We first observe that for $x \in H$, $(T^{-1}S)_x$ is a symmetric set in H_∞. Let $y = (y_1, y_2, \ldots) \in (T^{-1}S)_x$. Let $i < j$. Let $\tilde{y}$ be obtained by permuting the coordinates y_i and y_j in y. We show that $\tilde{y} \in (T^{-1}S)_x$. If both y_i and y_j are ∞ then of course $\tilde{y} = y \in (T^{-1}S)_x$. In the other case, a simple calculation shows that, $T(x, \tilde{y})$ is obtained by a finite permutation of $T(x, y)$ so that $T(x, \tilde{y}) \in S$ and hence $\tilde{y} \in (T^{-1}S)_x$. As a consequence, for each $x \in H$, $\sigma_\lambda((T^{-1}S)_x)$ is either 0 or 1. Let now

$$E = \{x \in H : \sigma_\lambda((T^{-1}S)_x) = 1\}$$

Here are some properties of E.

1^o. If $x = (x_1, x_2, \ldots) \in E$, $k \geq 1$, $x_k \in A^c$ and $\tilde{x}$ is obtained from x by deleting x_k, then $\tilde{x} \in E$.

To show $\tilde{x} \in E$ we only need to show that $\sigma_\lambda((T^{-1}S)_{\tilde{x}}) > 0$.

Let $M = \{y \in H_\infty \ : \ y_i = \infty \ \text{ for all } \ i \leq k\}$. Then $\sigma_\lambda(M) > 0$ and hence so is $\sigma_\lambda(M_o)$ where $M_o = (T^{-1}S)_x \cap M$. Let

$$M_1 = \{y \in H_\infty : y_k = x_k \ \ \& \ \ \overline{y} \in M_o\}$$

where

$$\overline{y}_i = \begin{cases} y_i & \text{if } \ i \neq k \\ \infty & \text{if } \ i = k \end{cases}$$

Then $\sigma_\lambda(M_1) > 0$ and it is easy to verify that $M_1 \subset (T^{-1}S)_{\tilde{x}}$.

2^o. If $x = (x_1, x_2, \ldots) \in E$; $k \geq 1$; $a \in A^c$ and $\tilde{x}$ is obtained from x by inserting a just before x_k then $\tilde{x} \in E$.

Proceed as earlier and take

$$M = \{y \in H_\infty : y_i = \infty \text{ for } i \leq k-1,\ y_k = a\}$$

and

$$M_1 = \{y \in H_\infty : y_k = \infty \ \&\ \overline{y} \in M_o\}$$

where

$$\overline{y}_i = \begin{cases} y_i & \text{if } i \neq k \\ a & \text{if } i = k. \end{cases}$$

Now to show that $\sigma_\mu(E)$ is either 0 or 1 we argue as follows :

For any $p \in Seq$, let $|\, p\,|$ denote the length of p. Now fix a $p \in Seq$ and let its length be n. We have the following relation:

$$\sigma_\mu(Ep) = \int_{i \in A^c} \sigma_\mu(Epi)d\mu(i) + \int_{i \in A} \sigma_\mu(Epi)d\mu(i)$$

For $i \in A^c$, $Epi = Ep$ by 1^o and 2^o. So, we get from the above equality,

$$\sigma_\mu(Ep)\mu(A) = \int_{i \in A} \sigma_\mu(Epi)d\mu(i)$$

Since μ is two-valued on A, this implies that $\mu\{i \in A : |\, \sigma_\mu(Ep) - \sigma_\mu(Epi)\,| > \frac{\epsilon}{2^{n+1}}\} = 0$ where $\epsilon > 0$ is any arbitrary number fixed beforehand. (Recall that $n = |\, p\,|$). As noted above, $Ep = Epi$ for all $i \in A^c$. Thus,

$$\mu\{i : |\, \sigma_\mu(Ep) - \sigma_\mu(Epi)\,| > \frac{\epsilon}{2^{n+1}}\} = 0.$$

Let I_p denote the set in braces and K_p the set of all those histories whose first coordinate is in I_p so that $\sigma_\mu(K_p) = 0$. This is all done for a fixed $p \in Seq$. Now having done this for each fixed $p \in Seq$ define

$$F_k = \cup_{|p|=k}(pK_p) \qquad \text{and} \qquad F = \cup_{k=0}^{\infty} F_k.$$

Note that $Ep = E$ when $|\, p\,| = 0$.

Also observe that $\sigma_\mu(F_k p_k(x)) = 0$ for all k and x. Now by 2(vi) we have $\sigma_\mu(F) = 0$. This can be restated as,

3^o. $\sigma_\mu\{x : \forall n \geq 0 : |\, \sigma_\mu(Ep_n(x)) - \sigma_\mu(Ep_{n+1}(x))\,| \leq \frac{\epsilon}{2^{n+1}}\} = 1$

To complete the proof of the Theorem fix $x \in F^c$ such that $\sigma_\mu(Ep_n(x)) \to 1_E(x)$. This is possible by Levy $0-1$ law (see 2(x)). Combining this with 3^o we get $|\, \sigma_\mu(E) - 1_E(x)\,| \leq \epsilon$. If, to start with, $0 < \sigma_\mu(E) < 1$, then an appropriate choice of ϵ would give rise to a contradiction. □

5. Remarks

1. Lemma 2 of Section 3 is perhaps true for general ϕ

2. There is perhaps a trite way to prove Theorem 3.1 without going through the detour as we did.

3. Ramakrishnan (1980) proved that if S is a G_δ set in H which is a countable intersection of symmetric open sets then $\sigma_\gamma(S)$ is either zero or one for any γ. It is quite likely that for a large class of sets the $0-1$ law holds whatever be γ.

4. The sets of interest in the context of Random Walks are G_δ sets of the form $S = \{h : \sum_1^n h_i \in A \ i.o.\}$ for some $A \subset \mathbf{N}$ (see Ramakrishnan, 1980, 1984,Gangopadhyay and Rao, 1998). It is interesting to note that such a set S is a countable intersection of symmetric open sets iff A is of the form $\{i : i \geq k\}$ for some k. But in that case S is already open.

5. Let $A = \{i : \gamma(i) = 0\}$. If γ restrited to subsets of A takes more than two values then the Hewitt-Savage $0-1$ law does not hold. However the Kolmogorov $0-1$ law holds. In this case there are symmetric sets which are not equivalent to any tail set under σ_γ. Of course if γ restricted to subsets of A is atmost two-valued then the Hewitt-Savage $0-1$ law holds. So trivially, any symmetric set is equivalent to a tail set under σ_γ.

6. If A_n consists of all histories with at least one coordinate smaller than n then A_n is a symmetric set and $A_n \uparrow H$. If γ is diffuse (that is every singleton gets γ mass zero) then clearly $\sigma_\gamma(A_n) = 0$ showing that σ_γ is not countably additive on the symmetric σ field. Contrast this with the well known fact that σ_γ is countably additive on the tail σ field (Theorem 2 of Purves and Sudderth, 1983). With a little more argument it can be shown that σ_γ is countably additive on the symmetric σ field iff $\gamma\{i : \gamma(i) = 0\} = 0$.

7. As noted earlier, $\mathbf{N}$ is taken for convenience but the theorems hold good for any countable set.

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