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# A BOSON FOCK SPACE REALIZATION OF ARCSINE BROWNIAN MOTION*

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*SUMMARY.* Within the framework of conventional quantum stochastic calculus in a boson Fock space we obtain an explicit realization of the noncommutative arcsine Brownian motion which was constructed by Muraki (1997a) in a monotone Fock space and by Lu in an interacting free Fock space. This, at once, simplifies the theory of stochastic integration with respect to the arcsine Brownian motion.

## 1. **Introduction**

In quantum (or noncommutative) probability theory several Brownian motions are known. As examples we have bosonic, fermionic, free and $q$-brownian motions (see Parthasarathy, 1992; Meyer, 1993; Schurmann, 1993; Speicher, 1990; Bozeiko and Speicher, 1991 etc.) To this list, recently, Muraki (1996, 1997a) and Lu have added a new example in which the distribution of the sum of the creation and annihilation operators at any time $t$ in the vacuum state is a scaled symmetric arcsine law. This new Brownian motion also satisfies the property of independent increments in the sense of Kummerer (see Speicher (1990)). Here, we continue the process of unification of these diverse Brownian motions initiated by Hudson and Parthasarathy (1986) and Parthasarathy and Sinha (1991). We realise explicitly the arcsine Brownian motion in terms of the usual bosonic (or Gaussian) Brownian motion by the method of stochastic integration. It turns out that the arcsine brownian motion is also the bosonic one with a random time change in the quantum probabilistic sense. As a consequence stochastic integration and quantum Ito's formula in monotone Fock space in the sense of Muraki turn out to be simple corollaries.

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## 2. The Arcsine Brownian Motion

Let $A^\dagger, \Lambda, A$ be respectively the creation, conservation and annihilation processes of boson Fock space quantum stochastic calculus with one degree of freedom in $\mathcal{H} = \Gamma(L^2(\mathbb{R}_+))$ . Denote by $\mathcal{E}$ the dense linear manifold in $\mathcal{H}$ generated by the set $\{e(f), f \in L^2(\mathbb{R}_+)\}$ of all exponential vectors. Let $\Omega_{t]}$ be the Fock vacuum vector in $\mathcal{H}_{t]} = \Gamma(L^2([0,t]))$ and let $P_0(t)$ be the projection on the subspace $\Omega_{t]} \otimes \mathcal{H}_{[t}$ where $\mathcal{H}_{[t} = \Gamma(L^2[t,\infty))$, with the usual identification of $\mathcal{H}$ with $\mathcal{H}_{t]} \otimes \mathcal{H}_{[t}$. Then there exists a spectral measure $\tau$ on the Borel $\sigma-$ algebra of $\mathbb{R}_+$ such that $\tau([t,\infty)) = P_0(t)$. In particular, $\tau$ is a quantum stop time in the sense of Hudson (1979), Parthasarathy and Sinha (1987). The quantum stochastic integrals

$$L_\varphi(t) = \int_0^t \bar{\varphi}(s) P_0(s) dA(s), \; L_\varphi^\dagger(t) = \int_0^t \varphi(s) P_0(s) dA^\dagger(s) \qquad \ldots (2.1)$$

are well-defined as operators on the domain $\mathcal{E}$ for every $\varphi \in L^2_{\text{loc}}(\mathbb{R}_+)$ and they are adjoint to each other on $\mathcal{E}$ for every t. Denote by the same symbols $L_\varphi(t), \; L_\varphi^\dagger(t)$ their respective closures.

Theorem 2.1. *For every $\varphi, \psi \in L^2_{loc}(\mathbb{R}_+)$ the operators $L_\varphi(t)$ and $L_\psi^\dagger(t)$ are defined on the whole space $\mathcal{H}$ as bounded operators satisfying the following:*

(i) $L_\varphi(t) L_\psi^\dagger(t) = \int_0^t \bar{\varphi}(s)\psi(s) P_0(s) ds$;

(ii) $\|L_\varphi(t)\|^2 = \|L_\varphi^\dagger(t)\|^2 = \int_0^t |\varphi(s)|^2 ds$;

(iii) $L_\varphi^\dagger(t)$ *is the adjoint of* $L_\varphi(t)$.

Proof. From (2.1) we have

$$dL_\varphi = \bar{\varphi} P_0 dA, \; dL_\psi^\dagger = \psi P_0 dA^\dagger.$$

By quantum Ito's formula (i.e. the second fundamental lemma in Parthasarathy (1992), page 191 ) we have for $f, g \in L^2(\mathbb{R}_+)$ $\langle L_\varphi^\dagger(t) e(f), L_\psi^\dagger(t) e(g)\rangle = \int_0^t \{\langle P_0(s) e(f), L_\psi^\dagger(s) e(g)\rangle (\bar{\varphi} g)(s)$ $\langle L_\varphi^\dagger(s) e(f), P_0(s) e(g)\rangle (\bar{f}\psi)(s) \; + \langle e(f), P_0(s) e(g)\rangle (\bar{\varphi}\psi)(s)\} ds.$

By definition $L_\varphi^\dagger(s) e(f),$ and$L_\psi^\dagger(s) e(g)$ belong to the orthogonal complement of $\Omega_{s]} \otimes \mathcal{H}_{[s}$ and therefore

$$\langle L_\varphi^\dagger(t) e(f), L_\psi^\dagger(t) e(g)\rangle = \int_0^t \langle e(f), P_0(s) e(g)\rangle (\bar{\varphi}\psi)(s) ds.$$

Thus

$$\langle L_\varphi^\dagger(t)u,\ L_\psi^\dagger(t)v\rangle = \langle u, \int_0^t (\bar{\varphi}\psi)(s)P_0(s)ds v\rangle \qquad \ldots(2.2)$$

for all $u, v \in \mathcal{E}$ and, in particular

$$\|L_\varphi^\dagger(t)u\|^2 = \int_0^t |\varphi(s)|^2 \, \|P_0(s)u\|^2 ds \leq \left(\int_0^t |\varphi(s)|^2 ds\right) \|u\|^2. \qquad \ldots(2.3)$$

This shows that the closure of $L_\varphi^\dagger(t)$ is a bounded operator on $\mathcal{H}$. When $u = e(0)$ is the vacuum vector, equality is attained in (2.3) and therefore $\|L_\varphi^\dagger(t)\|^2 = \int_0^t |\varphi(s)|^2 ds$. Since $L_\varphi(t)$ defined by (2.1) is adjoint to $L_\varphi^\dagger(t)$ in $\mathcal{E}$ it follows that the closure of $L_\varphi(t)$ is the adjoint of the closed operator $L_\varphi^\dagger(t)$. Now (2.2) implies the relation (i) of the theorem. □

Remark. If $A_\varphi(t) = \int_0^t \bar{\varphi}(s)dA(s)$, $A_\varphi^\dagger(t) = \int_0^t \varphi(s)dA^\dagger(s)$ then $L_\varphi(t) = A_\varphi(\tau \wedge t)$, $L_\varphi^\dagger(t) = A_\varphi^\dagger(\tau \wedge t)$ on $\mathcal{E}$, where $\tau \wedge t$ is the minimum of the two stop times $\tau$ and $t$. Thus $L_\varphi, L_\varphi^\dagger$ can be viewed as $A_\varphi, A_\varphi^\dagger$ stopped at the stop times $\tau \wedge t$ in the sense of Parthasarathy and Sinha (1987).

Relation (i) of Theorem 2.1 should be compared with the relation $L_\varphi(t)L_\psi^\dagger(t) = \int_0^t (\bar{\varphi}\psi)(s)ds$ for free Brownian motion in the sense of Speicher (1990).

Proposition 2.2. *Let*

$$X_\varphi(t) = L_\varphi(t) + L_\varphi^\dagger(t),\ \varphi \in L^2_{loc}(I\!R).$$

*Then, for any* $\varphi_1, \varphi_2, \ldots, \varphi_n \in L^2_{loc}(I\!R_+)$, *we have*

$$dX_{\varphi_1}X_{\varphi_2}\cdots X_{\varphi_n} = \sum_i X_{\varphi_1}X_{\varphi_2}\cdots X_{\varphi_{i-1}}P_0X_{\varphi_{i+1}}\cdots X_{\varphi_n}(\bar{\varphi}_i dA + \varphi_i dA^\dagger)$$

$$+ \left\{\sum_{i<j} \bar{\varphi}_i\varphi_j X_{\varphi_1}\cdots X_{\varphi_{i-1}}P_0X_{\varphi_{i+1}}\cdots X_{\varphi_{j-1}}P_0X_{\varphi_{j+1}}\cdots X_{\varphi_n}\right\} dt \qquad \ldots(2.4)$$

*where, in the case* $j = i+1$, *the summand within* $\{\}$ *is to be interpreted as* $\bar{\varphi}_i\varphi_{i+1}X_{\varphi_1}\cdots X_{\varphi_{i-1}}P_0X_{\varphi_{i+2}}\cdots X_{\varphi_n}$. Proof. By (2.1) we have $dX_\varphi = P_0(\varphi dA^\dagger + \bar{\varphi}dA)$. By quantum Ito's formula

$$dX_{\varphi_1}X_{\varphi_2} = X_{\varphi_1}P_0(\varphi_2 dA^\dagger + \bar{\varphi}_2 dA) + P_0X_{\varphi_2}(\varphi_1 dA^\dagger + \bar{\varphi}_1 dA) + \bar{\varphi}_1\varphi_2 P_0 dt.$$

Thus (2.4) holds for $n = 2$. Now (2.4) follows by induction and quantum Ito's formula. □

COROLLARY 2.3. *Define the vacuum expectation moments*

$$F(\varphi_1, \ldots, \varphi_n; t) = \langle e(0), X_{\varphi_1}(t) \cdots X_{\varphi_n}(t) e(0)\rangle,\ \varphi_i \in L^2_{loc}(\mathbb{R}_+).$$

*Then*

$$F(\varphi_1, \ldots, \varphi_n; t) = \int_0^t \sum_{i<j} (\bar{\varphi}_i \varphi_j)(s) F(\varphi_1, \ldots, \varphi_{i-1}; s) F(\varphi_{i+1}, \ldots, \varphi_{j-1}; s) \times\ F(\varphi_{j+1}, \ldots, \varphi_n; s) ds$$

*where inside the summation sign the functions $F$ are to be interpreted according to the convention*

$$\begin{aligned} F(\varphi_1, \ldots, \varphi_{i-1}, s) &= 1 \quad if \quad i = 1, \\ F(\varphi_{i+1}, \ldots, \varphi_{j-1}, s) &= 1 \quad if \quad j = i+1, \\ F(\varphi_{j+1}, \ldots, \varphi_n, s) &= 1 \quad if \quad j = n, \end{aligned}$$

PROOF. By equation (2.4) of Proposition 2.2 we have

$$F(\varphi_1, \ldots, \varphi_n; t) = \int_0^t \{\sum_{i<j} (\bar{\varphi}_i \varphi_j)(s) \langle e(0), X_{\varphi_1} \cdots X_{\varphi_{i-1}} P_0 X_{\varphi_{i+1}} \cdots X_{\varphi_{j-1}} P_0 X_{\varphi_{j+1}} \cdots X_{\varphi_n}(s) e(0)\rangle\} ds.$$

Since $P_0(s)e(0) = e(0)$ and

$$P_0(s)(X_{\psi_1} \cdots X_{\psi_k})(s) P_0(s) = F(\psi_1, \ldots, \psi_k; s) P_0(s)$$

the required result follows. □

COROLLARY 2.4. *Let $X_\varphi$ be as in Proposition 2.2. Then*

$$\langle e(0),\ X_\varphi(t)^n e(0)\rangle = \begin{cases} 0 & if \quad n \text{ is odd}, \\ (-2\int_0^t |\varphi(s)|^2 ds)^k \binom{-\frac{1}{2}}{k} & if \quad n = 2k \end{cases}$$

PROOF. By definition

$$\langle e(0), X_\varphi(t)^n e(0)\rangle = F(\varphi, \varphi, \ldots, \varphi; t)$$

where $\varphi$ is repeated n-fold. Write

$$f_0(t) \equiv 1,\ f_n(t) = \langle e(0), X_\varphi(t)^n e(0)\rangle,\ n \geq 1,\ t \geq 0.$$

Then by Corollary 2.3

$$f_n(t) = \int_0^t |\varphi(s)|^2 \left( \sum_{1\le i<j\le n} f_{i-1} f_{j-i-1} f_{n-j} \right)(s)ds.$$

Define

$$G(t,x) = \sum_{n=0}^{\infty} f_n(t)x^n$$

Then we have

$$G(t,x) = x^2 \int_0^t |\varphi(s)|^2 G(s,x)^3 ds.$$

Expressing this as a differential equation in the variable $t$ we have

$$-\frac{G'}{G^3} = x^2|\varphi|^2, G(0,x) = 1.$$

Thus

$$G(t,x) = \frac{1}{\sqrt{1 - 2x^2 \int_0^t |\varphi(s)|^2 ds}}$$

Identifying the coefficients of $x^n$ on both sides and using the binomial expansion on the right hand side we get the required result. □

Corollary 2.5. *The probability distribution of the observable* $(2\int_0^t |\varphi(s)|^2 ds)^{-\frac{1}{2}}$ $X_\varphi(t)$ *in the vacuum state* $e(0)$ *is the standard symmetric arcsine law with density function* $\pi^{-1}(1-x^2)^{-\frac{1}{2}}$ *in the interval* $(-1,1)$.

Proof. Immediate from Corollary 2.4 and the moment sequence of the arcsine law. □

Proposition 2.6. *For any* $\varphi, \psi \in L^2(I\!R_+)$ *define*

$$X(\varphi,\psi) = \int_0^t P_0(s)\{\varphi(s)dA^\dagger(s) + \bar{\psi}(s)dA(s)\}.$$

*If (supp* $\varphi) \cup$ *(supp*$\psi) \subset [a,b], 0 < a < b < \infty$ *then* $X(\varphi,\psi) = P_0(a)X(\varphi,\psi)$.

Proof. For any $a \le t_1 < t_2$ we have

$$P_0(t_1)\left\{\varphi(t_1)(A^\dagger(t_2) - A^\dagger(t_1)) + \bar{\psi}(t_1)(A(t_2) - A(t_1))\right\}$$

$$= P_0(a)P_0(t_1)\{\varphi(t_1)(A^\dagger(t_2) - A^\dagger(t_1)) + \bar{\psi}(t_1)(A(t_2) - A(t_1))\}$$

By the definition of stochastic integrals, $X(\varphi, \psi)$ is a limit of sums of terms of the kind on the left hand side of the equation above. This completes the proof. □

Proposition 2.7. *Let $X_i = X(\varphi_i, \psi_i)$, $1 \leq i \leq m+n$ where $\varphi_i, \psi_i \in L^2(\mathbb{R}_+)$ and*

$$\bigcup_{i=1}^{m} (\mathit{supp}\ \varphi_i) \cup (\mathit{supp}\ \psi_i) \subset [0, a],$$

$$\bigcup_{i=m+1}^{m+n} (\mathit{supp}\ \varphi_i) \cup (\mathit{supp}\ \psi_i) \subset [a, b].$$

*Then*

$$\langle e(0), X_1 \cdots X_{m+n} e(0) \rangle = \langle e(0), X_1 \cdots X_m e(0) \rangle \langle e(0), X_{m+1} X_{m+2} \cdots X_{m+n} e(0) \rangle.$$

Proof. By Proposition 2.6 we have

$$X_1 \cdots X_{m+n} = X_1 \cdots X_m P_0(a) X_{m+1} \cdots X_{m+n}.$$

Furthermore

$$P_0(a) X_1 \cdots X_m P_0(a) = \langle e(0), X_1 \cdots X_m e(0) \rangle P_0(a).$$

Thus $\langle e(0), X_1 \cdots X_{m+n} e(0) \rangle = \langle e(0), P_0(a) X_1 \cdots X_m P_0(a) X_{m+1} \cdots X_{m+n} P_0(a) e(0) \rangle$ $= \langle e(0), X_1 \cdots X_m e(0) \rangle \langle e(0), X_{m+1} \cdots X_{m+n} e(0) \rangle$. □

Corollary 2.8. *For any interval $I = [a, b) \subset \mathbb{R}_+$ denote by $\mathcal{A}_I$ the von Neumann algebra generated by all operators of the form $L(\varphi) = \int_0^\infty \overline{\varphi(s)} P_0(s) dA(s)$, $\varphi \in L^2(\mathbb{R}_+)$, supp $\varphi \subset I$. If $X_i \in \mathcal{A}_{I_i}$, $i = 1, 2, \ldots, n$, $I_i = [a_i, b_i) \subset \mathbb{R}_+$, $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots < b_{n-1} \leq a_n < b_n < \infty$ then*

$$\langle e(0), X_1 X_2 \cdots X_n e(0) \rangle = \prod_{i=1}^{n} \langle e(0), X_i e(0) \rangle.$$

Proof. Immediate from Proposition 2.7. □

Remark. If $L(t) = \int_0^t P_0(s) dA(s) = A(\tau \wedge t)$, then $L(s) L^\dagger(t) = \int_0^{s \wedge t} P_0(u) du$ where $s \wedge t = \min(s, t)$ and $\{L(t), L^\dagger(t),\ 0 \leq t < \infty\}$ is a process with independent increments in the state $e(0)$ in the sense of Kummerer. The following Ito's formula holds:

$$(dL(t))^2 = (dL^\dagger(t))^2 = (dL^\dagger(t))(dL(t)) = 0; \; dL(t)(t)dL^\dagger(t) = dt.$$

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