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PARTIAL HAUSDORFF SEQUENCES AND SYMMETRIC PROBABILITIES ON FINITE PRODUCTS OF $\{0,1\}$

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SUMMARY. Let $\mathbb{H}_n$ be the set of all partial Hausdorff sequences of order n, i.e., sequences $c_n(0), c_n(1), \ldots c_n(n), c_n(0) = 1$, with $(-1)^m \triangle^m c_n(k) \geq 0$ whenever $m + k \leq n$. Further, let $\prod_n$ be the set of all symmetric probabilities on $\{0,1\}^n$. We study the interplay between the sets $\mathbb{H}_n$ and $\prod_n$ to formulate and answer interesting questions about both. Assigning to $\mathbb{H}_n$ the uniform probability measure we show that, as $n \to \infty$, the fixed section $(c_n(1), c_n(2), \ldots, c_n(k))$, properly centered and normalized, is asymptotically normally distributed. That is, $\sqrt{n}(c_n(1) - c_0(1), c_n(2) - c_0(2), \ldots, c_n(k) - c_0(k))$, converges weakly to MVN $(0, \Sigma)$, where $c_0(i)$ correspond to the moments of the uniform law λ on $[0,1]$; the asymptotic covariances also depend on the moments of λ.

1. Introduction

We recall Hausdorff's solution to the moment problem on the unit interval. A sequence $c(n),\ n = 0, 1, 2, \ldots, c(0) = 1$, is called a completely monotone sequence if

$$(-1)^m \triangle^m c(k) \geq 0, \quad k, m = 0, 1, 2, \ldots, \qquad \ldots (1.1)$$

where $\triangle c(k) := c(k+1) - c(k)$ and $\triangle^m$ stands for m iterates of $\triangle$.

Theorem 1.1. (Hausdorff, 1923). *A sequence* $c(n), \quad n = 0, 1, 2, \ldots, c(0) = 1$*, is the moment sequence of some probability measure on [0,1] if and only if it is completely monotone.*

We call a sequence $c(0), c(1), \ldots, c(n), c(0) = 1$, a partial Hausdorff sequence of order n if (1.1) holds for all k and m such that $k + m \leq n$. Here $c(k),\ k = 1, 2, \ldots, n$ may not correspond to the moments of a probability measure on [0,1]. However, conditions (1.1) with $m = 0$ imply that $c(k) \geq 0$ for all $k \leq n$.

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Moreover, if a sequence $c(k),\ k = 0, 1, 2, \ldots$ is such that, for all n, $c(0), c(1), \ldots c(n)$ is a partial Hausdorff sequence, then it is a moment sequence.

We say that a probability on $\Omega_n = \{0, 1\}^n$ is symmetric if it is invariant under all permutations of coordinates of Ω_n. A symmetric probability on Ω_n is determined by constants $p_n(i),\ i = 1, 2, \ldots, n$ where $p_n(i)$ is the probability assigned to the set of all n-length sequences having exactly i 1's. Of course, a symmetric probability is not necessarily a mixture of i.i.d. probabilities.

This paper is organised as follows. Section 2 is devoted to a study of the set of partial Hausdorff sequences of order n on the one hand and the set of symmetric probabilities on $\{0, 1\}^n$ on the other. It turns out that these two sets, though seemingly unrelated, are affine equivalent and as such they are best studied in tandem. We exhibit an explicit affine correspondence between these sets and use it to obtain interesting results about both. In Section 3 we prove a normal limit theorem for partial Hausdorff sequences; this is inspired by the work of Chang, Kemperman and Studden (1993) who proved a similar theorem for moment sequences. Chang, Kemperman and Studden employ the canonical moments in their study while in our case the canonical coordinates $p_n(i),\ i = 1, 2, \ldots, n$ mentioned above play the central role.

2. Partial Hausdorff Sequences and Symmetric Probabilities

We introduce the notion of a partial Hausdorff sequence of order n.

Definition. A sequence $c_n(0), c_n(1), \ldots, c_n(n),\ c_n(0) = 1$, is called a *partial Hausdorff sequence* of order n if

$$(-1)^m \triangle^m c_n(k) \geq 0,\ k = 0, 1, \ldots, n;\ m = 0, 1, \ldots, n - k. \qquad \ldots (2.1)$$

The set

$$\mathbb{H}_n := \{(c_n(1), c_n(2), \ldots, c_n(n)) : (-1)^m \triangle^m c_n(k) \geq 0 \text{ if } m + k \leq n\} \quad \ldots (2.2)$$

with the understanding that $c_n(0) \equiv 1$, denotes the set of all partial Hausdorff sequences of order n.

We define

$$q_m(k) := (-1)^{m-k} \triangle^{m-k} c_n(k),\ k = 0, 1, \ldots, n;\ k \leq m \leq n, \qquad \ldots (2.3)$$

and observe that, by (2.1), they are all non-negative.

We define, for $m \geq k + 1$,

$$\bigtriangledown q_m(k) := q_m(k) + q_m(k + 1). \qquad \ldots (2.4)$$

By (2.3) and (2.4), it follows that

$$\bigtriangledown q_m(k) = q_{m-1}(k) \qquad \ldots (2.5)$$

and consequently, for $m \le n$,

$$q_m(k) = \nabla^{n-m} q_n(k) = \sum_{j=0}^{n-m} \binom{n-m}{j} q_n(k+j). \qquad \ldots(2.6)$$

By (2.3),

$$q_n(k) = (-1)^{n-k} \triangle^{n-k} c_n(k) = \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} c_n(k+j). \qquad \ldots(2.7)$$

By (2.3) and (2.4),

$$c_n(k) = q_k(k) = \nabla^{n-k} q_n(k) = \sum_{j=0}^{n-k} \binom{n-k}{j} q_n(k+j); \qquad \ldots(2.8)$$

in particular

$$c_n(0) = \nabla^n q_n(0) = \sum_{k=0}^{n} \binom{n}{k} q_n(k) = 1. \qquad \ldots(2.9)$$

We observe that, by (2.6), the non-negativity of $q_n(k),\ 0 \le k \le n$, implies that conditions (2.1) hold and consequently, we may redefine $\mathbb{H}_n$ as follows :

$$\mathbb{H}_n = \{(c_n(1), c_n(2), \ldots, c_n(n)) : q_n(k) \ge 0,\ 0 \le k \le n\}, \qquad \ldots(2.10)$$

where $q_n(k)$'s are given by (2.7).

Given an element of $\mathbb{H}_n$, we define a symmetric probability Q_n on $\Omega_n = \{0,1\}^n$ which, for each $0 \le k \le n$, assigns mass $q_n(k)$ to each $(\omega_1, \omega_2, \ldots, \omega_n) \in \Omega_n$ which has exactly k coordinates equal to 1. Conversely, given $q_n(k),\ 0 \le k \le n$, equations (2.8) give a partial Hausdorff sequence of order n. Thus there is a one-one correspondence between $\mathbb{H}_n$ and

$$\prod\nolimits_n := \{(q_n(1), q_n(2), \ldots q_n(n)) : q_n(k) \ge 0,\ \sum_{1}^{n} \binom{n}{k} q_n(k) \le 1\}, \qquad \ldots(2.11)$$

the set of all symmetric probabilities on $\{0,1\}^n$. By (2.9), of course, $q_n(0) = 1 - \sum_{1}^{n} \binom{n}{k} q_n(k)$. Equations (2.7) and (2.8) define maps

$$\phi_n : \mathbb{H}_n \longrightarrow \prod\nolimits_n$$

and

$$\psi_n : \prod\nolimits_n \longrightarrow \mathbb{H}_n \qquad \ldots(2.12)$$

respectively. Clearly these maps are one-one and onto and establish affine congruence of convex sets $\mathbb{H}_n$ and $\prod_n$. Further, the map ψ_n is the inverse of the map ϕ_n.

We define the projection map

$$\pi_n : \mathbb{H}_{n+1} \longrightarrow \mathbb{H}_n$$

by

$$(c(1), c(2), \ldots, c(n+1)) \mapsto (c(1), c(2), \ldots, c(n)). \qquad \ldots (2.13)$$

Likewise, we define

$$\tilde{\pi}_n : \prod_{n+1} \longrightarrow \prod_n$$

by

$$q^* \mapsto q,$$

where q is the n-dimensional marginal of q^* in $\prod_{n+1}$. We observe that both these projection maps are affine. We will now discuss the use of the maps ψ_n, ϕ_n, π_n and $\tilde{\pi}_n$ to answer questions about $\mathbb{H}_n$ and $\prod_n$.

(a) Extreme points of $\mathbb{H}_n$ and $\prod_n$. Clearly the extreme points of $\prod_n$ correspond to the probabilities $Q_n^k,\ k = 0, 1, \ldots, n$, where Q_n^k is the uniform distribution on the set of those elements of Ω_n which have exactly k coordinates equal to 1, i.e.,

$$\partial \prod_n = \{q_n^0, q_n^1, \ldots q_n^n\},$$

where $q_n^0 = (0, 0, \ldots, 0)$ and, for $j, k = 1, 2, \ldots, n$,

$$q_n^k(j) = \begin{cases} \frac{1}{\binom{n}{k}} & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases} \qquad \ldots (2.15)$$

The extreme points of $\mathbb{H}_n$, which are otherwise not so apparent, can be easily obtained by using the map ψ_n. The congruence ψ_n maps $\partial \prod_n$ onto $\partial \mathbb{H}_n$. Simple calculations show that

$$\partial \mathbb{H}_n = \{c_n^0, c_n^1, \ldots, c_n^n\},$$

where

$$c_n^k = (\frac{k}{n}, \frac{k(k-1)}{n(n-1)}, \ldots, \frac{k(k-1)\ldots 1}{n(n-1)\ldots(n-k+1)}, 0, \ldots, 0), \qquad \ldots (2.16)$$

$k = 0, 1, 2, \ldots, n$.

(b) Extendability of partial Hausdorff sequences. We define

$$\begin{aligned}\mathbb{H}_n^{n+1} : \quad = \quad & \{(c(1), c(2), \ldots c(n)) : \exists \ \ c(n+1) \text{ s.t.} \\ & (c(1), c(2), \ldots, c(n+1)) \in \mathbb{H}_{n+1}\}.\end{aligned} \quad \ldots (2.17)$$

Clearly, a partial Hausdorff sequence of order n can be extended to one of order $n+1$ if and only if it is in the range of the affine map π_n and consequently,

$$\mathbb{H}_n^{n+1} = \pi_n(\mathbb{H}_{n+1}) = \text{ Convex Hull } \{\pi_n(\partial \mathbb{H}_{n+1})\}. \quad \ldots (2.18)$$

Simple calculations show that

$$\partial \mathbb{H}_n^{n+1} = \{c_n^0, c_n^1, \ldots, c_n^{n+1}\},$$

where

$$c_n^k = (\frac{k}{n+1}, \frac{k(k-1)}{(n+1)n}, \ldots, \frac{k(k-1)\ldots 1}{(n+1).n\ldots(n-k)}, 0, \ldots, 0), \quad \ldots (2.19)$$

$k = 0, 1, 2, \ldots, n+1$.

In a similar fashion it is easy to figure out the extreme points of $\mathbb{H}_n^{n+k}$, the set of partial Hausdorff sequences of order n which can be extended by k steps.

(c) Extendability of symmetric probabilities. We define

$$\prod_n^{n+1} := \{q \in \prod_n : \exists q^* \in \prod_{n+1} \text{ s.t. its } n\text{-dim. marginal is } q\}. \quad \ldots (2.20)$$

Clearly, a symmetric probability on Ω_n is extendable to one on Ω_{n+1} if and only if it is in the range of $\tilde{\pi}_n$. Looking at the diagram

$$\begin{array}{ccccccc} \cdots & \longleftarrow & \mathbb{H}_n & \stackrel{\pi_n}{\longleftarrow} & \mathbb{H}_{n+1} & \longleftarrow & \cdots \\ & & \downarrow \phi_n & & \uparrow \psi_{n+1} & & \\ \cdots & \longleftarrow & \Pi_n & \stackrel{\tilde{\pi}_n}{\longleftarrow} & \Pi_{n+1} & \longleftarrow & \cdots \end{array}$$

it is readily seen that

$$\tilde{\pi}_n = \phi_n \circ \pi_n \circ \psi_{n+1}. \quad \ldots (2.21)$$

Easy calculations show that

$$\partial \prod_n^{n+1} = \{q_n^0, q_n^{0,1}, \ldots, q_n^{n-1,n}, q_n^n\}, \quad \ldots (2.22)$$

where q_n^0 and q_n^n are as defined in(2.15) and $q^{k,k+1}$, $k = 0, 1, \ldots, n-1$, corresponds to uniform distribution on the set of those elements of Ω_n which have either k or $k+1$ coordinates equal to 1.

Likewise, one can figure out the extreme points of $\prod_n^{n+k}$, the set of symmetric probabilities on Ω_n which are extendable to symmetric probabilities on Ω_{n+k}.

(d) We define

$$\mathbb{H}_2^\infty := \{(c_1, c_2) : \exists c_3, c_4, \ldots, \text{ s.t. } 1, c_1, c_2, \ldots \text{ is completely monotone }\}, \qquad \ldots (2.23)$$

Clearly,

$$\begin{aligned} \mathbb{H}_2^\infty &= \textstyle\bigcap_{k=0}^\infty \mathbb{H}_2^{2+k} \\ &= \textstyle\bigcap_{n=2}^\infty \text{ Convex Hull } \{(\frac{i}{n}, \frac{i(i-1)}{n(n-1)}) : \ i = 0, 1, 2, \ldots n\} \end{aligned}$$

and $(c_1, c_2) \in \mathbb{H}_2^\infty$ if and only if, for each $n \geq 2$, $\exists \lambda_{ni},\ i = 0, 1, \ldots, n\ \lambda_{ni} \geq 0, \sum_0^n \lambda_{ni} = 1$ such that

$$c_1 = \sum_0^n \lambda_{ni} \frac{i}{n} \ \text{ and } c_2 = \sum_{i=0}^n \lambda_{ni} \frac{i(i-1)}{n(n-1)}.$$

So

$$\frac{n-1}{n} c_2 = \Sigma \lambda_{ni} \frac{i^2}{n^2} - \Sigma \lambda_{ni} \frac{i}{n^2} \geq (\Sigma \lambda_{ni} \frac{i}{n})^2 - \Sigma \lambda_{ni} \frac{i}{n^2},$$

i.e.,

$$\tfrac{n-1}{n} c_2 + \tfrac{1}{n} c_1 \geq c_1^2. \text{ As } n \to \infty, \text{ we get } c_2 \geq c_1^2.$$

Thus

$$\mathbb{H}_2^\infty = \{(c_1, c_2) : c_1 \geq c_2 \geq c_1^2,\ , 0 \leq c_1 \leq 1\} \qquad \ldots (2.24)$$

This gives necessary and sufficient conditions on c_1 and c_2 to be the first two moments of some probability measure on $[0, 1]$. We do not know of a similar characterisation of $\mathbb{H}_3^\infty$.

While some of the results obtained in this section may perhaps be more readily accessible by other methods, we feel that the tools developed are indispensable for obtaining a normal limit theorem for partial Hausdorff sequences; see Section 3.

3. A Normal Limit Theorem for Partial Hausdorff Sequences

Our main result in this section is a normal limit theorem for partial Hausdorff sequences. This is inspired by a similar theorem proved by Chang, Kemperman and Studden (1993) for the moment space

$$\mathbb{M}_n := \{c_1, c_2, \ldots, c_n)|\lambda \in \Lambda\}, \qquad \ldots (3.1)$$

where

$$c_k = c_k(\lambda) = \int_0^1 x^k \lambda(dx),\ k = 1, 2, \ldots, n, \qquad \ldots (3.2)$$

and Λ is the space of all probability measures on $[0, 1]$.

They show, also see Karlin and Studden (1966), that

$$V_n = \text{ Volume } (\mathbb{M}_n) = \prod_{k=1}^{n} \frac{\Gamma(k)^2}{\Gamma(2k)} = exp[-n^2(\log 2 + o(1))] \qquad \ldots (3.3)$$

and, among other things, prove the following theorem.

THEOREM 3.1 (Chang, Kemperman and Studden, 1993). *As $n \to \infty$, the distribution of $\sqrt{n}(c_1 - c_1^0, c_2 - c_2^0, \ldots, c_k - c_k^0)$ converges to a multivariate normal distribution $MVN(0, ((\sigma_{ij})))$, where*

$$c_i^0 = \int_0^1 \frac{x_i\ dx}{\pi\sqrt{x(1-x)}} \ \ \textit{and} \ \ \sigma_{ij} = c_{i+j}^0 - c_i^0 c_j^0. \qquad \ldots (3.4)$$

In the proof of the above theorem the authors employ the canonical moments introduced by Skibinsky (1967). In our case we find it convenient to introduce a different set of canonical coordinates.

Let

$$\mathcal{S}_n = \{(p_n(1), p_n(2), \ldots p_n(n)) : p_n(k) \geq 0,\ \sum_1^n p_n(k) \leq 1\} \qquad \ldots (3.5)$$

be the standard simplex in $\mathbb{R}^n$. We put

$$p_n(0) = 1 - p_n(1) - p_n(2) - \ldots - p_n(n) \qquad \ldots (3.6)$$

and set up a one-one correspondence between $\mathcal{S}_n$ and $\prod_n$, as given by (2.11), by putting

$$p_n(k) = \binom{n}{k} q_n(k),\ k = 1, 2, \ldots, n; \qquad \ldots (3.7)$$

of course, by (2.9) and (3.6),

$$p_n(0) = q_n(0). \qquad \ldots (3.8)$$

This gives a one-one correspondence between $\mathbb{H}_n$ and $\mathcal{S}_n$. Explicitly, by (2.7), (2.8) and (3.7), we have

$$\begin{aligned} p_n(k) &= \binom{n}{k} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} c_n(k+j) \\ &= \binom{n}{k} \sum_{m=k}^{n} (-1)^{m-k} \binom{n-k}{n-m} c_n(m) \end{aligned}$$

and

$$c_n(k) = \sum_{j=0}^{n-k} \frac{\binom{n-k}{j} p_n(k+j)}{\binom{n}{k+j}} = \frac{1}{\binom{n}{k}} \sum_{m=k}^{n} \binom{m}{k} p_n(m), \qquad \ldots(3.9)$$

$k = 1, 2, \ldots, n$.

We will employ $p_n(k),\ k = 1, 2, \ldots, n$ as the canonical coordinates of the space $\mathbb{H}_n$. We observe that the matrices of transformations from $\mathcal{S}_n$ to $\mathbb{H}_n$ and vice-versa are upper triangular and , by (3.9),

$$\frac{\partial(c_n(1), c_n(2), \ldots, c_n(n))}{\partial(p_n(1), p_n(2), \ldots, p_n(n))} = [\prod_{k=1}^{n} \binom{n}{k}]^{-1}. \qquad \ldots(3.10)$$

We let

$$V_n^* = \text{ Volume } (\mathbb{H}_n). \qquad \ldots(3.11)$$

Proposition 3.2. *As* $n \to \infty$,

$$\frac{1}{n^2} \log V_n^* \longrightarrow -\frac{1}{2}. \qquad \ldots(3.12)$$

Proof.

$$\begin{aligned} V_n^* &= \int_{\mathbb{H}_n} dc_n(1) dc_n(2) \ldots dc_n(n) \\ &= [\textstyle\prod_{k=1}^{n} \binom{n}{k}]^{-1} \int_{\mathcal{S}_n} dp_n(1) dp_n(2) \ldots dp_n(n) \\ &= \frac{(\prod_{k=1}^{n} k!)^2}{(n!)^{n+2}} \\ &= \exp[-n^2(\tfrac{1}{2} + o(1))]. \end{aligned}$$

□

From (3.3) and (3.12) we get

$$\alpha_n = \frac{\text{Volume } (\mathbb{M}_n)}{\text{Volume } (\mathbb{H}_n)} = \frac{V_n}{V_n^*} = \exp[-n^2(\beta + o(1))], \qquad \ldots(3.13)$$

where

$$\beta = \log 2 - \frac{1}{2} > 0.$$

This shows that, volume-wise, $\mathbb{M}_n$ is a very small portion of $\mathbb{H}_n$. In the language of Section 2, α_n can be interpreted as the proportion of those partial

Hausdorff sequences of order n which can be extended to completely monotone sequences.

To get a better understanding of the shape and structure of the space $\mathbb{H}_n$ we would like to look at a typical point of it. For this purpose we put uniform probability measure on $\mathbb{H}_n$, i.e., the n-dimensional Lebesgue measure on $\mathbb{H}_n$ normalised by the volume V_n^* of $\mathbb{H}_n$.

PROPOSITION 3.3. *The uniform probability measure on the space* $\mathbb{H}_n$ *is equivalent to having the uniform probability measure on the space* $\mathcal{S}_n$ *of canonical coordinates.*

PROOF. This is an immediate consequence of (3.10) and the change of variables formula for an integral on $\mathbb{H}_n$ to $\mathcal{S}_n$ and vice versa. □

We will require the following combinatorial identity.

PROPOSITION 3.4. *For* $1 \le i \le j \le n$,

$$\sum_{m=j}^{n} \binom{m}{i}\binom{m}{j} = \sum_{m=j}^{i+j} \binom{n+1}{m+1}\binom{m}{m-i \quad m-j \quad i+j-m}. \quad \ldots (3.14).$$

PROOF. The equality follows by identifying the quantity on the LHS of (3.14) with the coefficient of $x^i y^j$ in

$$\sum_{m=j}^{n}(1+x)^m(1+y)^m = \sum_{k=0}^{j-1}\left\{\binom{n+1}{k+1} - \binom{j}{k+1}\right\}\rho^k + \sum_{k=j}^{n}\binom{n+1}{k+1}\rho^k,$$

where $\rho = (x+y+xy)$ and observing that the coefficient of $x^i y^j$ in $(x+y+xy)^k$ equals $\binom{k}{k-i \ \ k-j \ \ i+j-k}$ if $j \le k \le i+j$ and zero otherwise. □

Our main result is the following.

THEOREM 3.5. *For each* $k = 1, 2, \ldots$, *as* $n \to \infty$, *the law of* $\sqrt{n}[c_n(1), -c_0(1), c_n(2) - c_0(2), \ldots, c_n(k) - c_0(k)]$ *relative to the uniform distribution on* $\mathbb{H}_n$ *converges to a multivariate normal distribution* $MVN\,[0, \Sigma]$, *where*

$$c_0(i) = \int_0^1 x^i dx = \frac{1}{i+1}, \ \Sigma = ((\sigma_{ij})) \text{ with } \sigma_{ij} = c_0(i+j) = \frac{1}{i+j+1}. \quad \ldots (3.15)$$

PROOF. The uniform probability on the simplex $\mathcal{S}_n$ is just the Dirichlet $(1, 1, \ldots, 1)$ distribution on it. Let $Z_0, Z_1, \ldots$ be a sequence of i.i.d. standard exponential random variables defined on, say, the probability space $(\Omega, \mathcal{F}, P)$. Then the law, under P, of

$$\left(\frac{Z_1}{Z_0+Z_1+\ldots+Z_n}, \frac{Z_2}{Z_0+Z_1+\ldots+Z_n}, \frac{Z_n}{Z_0+Z_1+\ldots+Z_n}\right)$$

is Dirichlet $(1,1,\ldots,1)$ Hence, by (3.9) and Proposition 3.3, the law of $(c_n(1), c_n(2), \ldots c_n(n))$, under uniform probability on $\mathbb{H}_n$, is same as the law, under P, of

$$Y_n(j) = [\sum_{m=j}^{n} \binom{m}{j} Z_m]/[\binom{n}{j} (Z_0 + Z_1 + \ldots + Z_n)],\ j = 1,2,\ldots,n. \quad \ldots (3.16)$$

Now choose and fix an integer k and consider $n \geq k$. We observe that

$$E(Y_n(j)) = [\sum_{m=j}^{n} \binom{m}{j}]/[(n+1)\binom{n}{j}] = \frac{1}{j+1} = c_0(j),\ j = 1,2,\ldots,n. \quad \ldots (3.17)$$

and

$$\frac{Z_0 + Z_1 + \ldots + Z_n}{n} \xrightarrow{P} 1. \quad \ldots (3.18)$$

Hence, by (3.16), (3.17) and (3.18), to prove the stated weak convergence of $\sqrt{n}(c_n(1) - c_0(1), c_n(2), -c_0(2), \ldots, c_n(k) - c_0(k))$ it suffices to prove that $(T_n(1), T_n(2), \ldots, T_n(k))$ converges weakly to MVN $[0, \Sigma]$, where

$$T_n(j) := \frac{1}{\sqrt{n}}[\sum_{m=j}^{n} \binom{m}{j} (Z_m - 1)]/\binom{n}{j},\ j = 1,2,\ldots,k. \quad \ldots (3.19)$$

For $1 \leq i \leq j \leq n$,

$$\begin{aligned} Cov(T_n(i), T_n(j)) &= \tfrac{1}{n}\cdot\frac{1}{\binom{n}{i}}\cdot\frac{1}{\binom{n}{j}}\cdot\textstyle\sum_{m=j}^{n} \binom{m}{i}\binom{m}{j} \\ &= \tfrac{1}{n}\cdot\frac{1}{\binom{n}{i}}\cdot\frac{1}{\binom{n}{j}}\cdot\textstyle\sum_{m=j}^{i+j} \binom{n+1}{m+1} \\ &\quad \cdot\binom{m}{m-i \quad m-j \quad i+j-m} \text{ by (3.14)} \\ &\sim \frac{1}{i+j+1} = c_0(i+j). \end{aligned} \quad \ldots (3.20)$$

By the Cramér-Wold theorem it suffices to prove that $\sum_{i=1}^{k} \alpha_i T_n(i)$ converges weakly to $N[0, \Sigma\Sigma\alpha_i\alpha_j c_0(i+j)]$ for all $\alpha_1, \alpha_2, \ldots, \alpha_k$.

We have

$$\begin{aligned} \textstyle\sum_{i=1}^{k} \alpha_i T_n(i) &= \tfrac{1}{\sqrt{n}} \textstyle\sum_{i=1}^{k} \alpha_i [\binom{n}{i}]^{-1} \sum_{m=1}^{n} \binom{m}{i} (Z_m - 1) \\ &= \tfrac{1}{\sqrt{n}} \textstyle\sum_{m=1}^{n} b_{n,m}(Z_m - 1) \end{aligned}$$

where

$$b_{n,m} = \sum_{i=1}^{m\wedge k} \frac{\alpha_i \binom{m}{i}}{\binom{n}{i}}, \ 1 \le m \le n. \qquad \ldots (3.21)$$

We write

$$S_n := \sum_{m=1}^{n} X_m \quad \text{with } X_m = b_{n,m}(Z_m - 1)$$

and observe that S_n is a sum of independent random variables centered at expectations. Further, we have the following :

(i) $s_n^2 = Var(S_n) = n \ \ Var(\sum_1^k \alpha_i T_n(i)) \sim n\Sigma\Sigma\alpha_i\alpha_j c_0(i+j)$ by (3.20)

(ii) $\frac{1}{s_n^3}\sum_{m=1}^n E|X_m|^3 = \frac{1}{s_n^3}\sum_{m=1}^n |b_{n,m}|^3 E|Z_m - 1|^3 \to 0$, as $n \to \infty$, since $|b_{n,m}| \le \sum_1^k |\alpha_i|$ by (3.21) and $s_n^3 = O(n^{3/2})$ by (i).

Hence, by Loéve (1963) p. 275, $\frac{S_n}{s_n}$ converges weakly to $N(0,1)$, or equivalently, $\sum_1^k \alpha_i T_n(i) = \frac{S_n}{\sqrt{n}}$ converges weakly to $N[0, \Sigma\Sigma\alpha_i\alpha_j c_0(i+j)]$.

This completes the proof. □

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