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INFERENCE ABOUT THE TRANSITION-POINT IN NBUE-NWUE OR NWUE-NBUE MODELS

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SUMMARY. A life distribution F is called NBUE-NWUE if for some $t_0 \in (0, \infty)$, its mean residual life function $e(t) = E_F(X - t|X \geq t)$ satisfies $e(t) < e(0)$ for $0 < t < t_0$ and $e(t) > e(0)$ for $t > t_0$. If the inequalities for $e(t)$ are reversed on these time intervals, it is called NWUE-NBUE. Using a characterization of such distributions in terms of the scaled total-time-on-test transform (STTT), we first give tests of exponentiality versus NBUE-NWUE or NWUE-NBUE with t_0 unknown. This extends the work of Klefsjö (1989), who devised tests assuming that $p_0 = F(t_0)$ is known. Then, assuming that F is either NBUE-NWUE or NWUE-NBUE, we give point estimates and asymptotic confidence intervals for t_0 and p_0. The point estiamtes are asymptotically normal. We rely heavily on the theory of the empirical STTT process discussed in Csörgo, Csörgo and Horváth (1986).

1. Introduction

Let $\mathcal{F}$ denote the set of absolutely continuous strictly increasing c.d.f.'s on $\mathbb{R}$ with $F(0) = 0$ and $\int_0^\infty x^2 dF(x) < \infty$. For $F \in \mathcal{F}$ define the mean residual life function $e(t) = E_F(X - t|X \geq t) = [\bar{F}(t)]^{-1} \int_t^\infty \bar{F}(x)dx, t \geq 0$ where $\bar{F}(t) = 1 - F(t)$. F is said to be "new better than used in expectation" (NBUE) if $e(t) < e(0)$ for $t > 0$ and "new worse than used in expectation" (NWUE) if $e(t) > e(0)$ for $t > 0$. These classifications of life distributions are useful in reliability theory. See Barlow and Proschan (1981).

Let $\mathcal{E}$ denote the family of exponential distributions (i.e. $F \in \mathcal{E}$ implies $F(x) = 1 - e^{-x/\lambda}, x \geq 0$ for some $\lambda > 0$). Recently, Klefsjö (1989) has proposed tests of $H_0 : F \in \mathcal{E}$ versus either of $H_{BW} : F \in C_{BW}$ or $H_{WB} : F \in C_{WB}$, where $C_{BW} = \{F \in \mathcal{F}$: there exists a $t_0 > 0$ such that $e(t) < e(0)$ for $0 < t < t_0, e(t) > e(0)$ for $t > t_0\}$ and $C_{WB} = \{F \in \mathcal{F}$: there exists $t_0 > 0$ such that $e(t) > e(0)$ for $0 < t < t_0, e(t) < e(0)$ for $t > t_0\}$. Distributions in $C_{BW}(C_{WB})$ are called

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NBUE-NWUE (NWUE-NBUE). t_0 is called the transition point, $p_0 = F(t_0)$ the transition quantile. To obtain his tests, Klefsjö assumed that p_0 is known. In most applications this is an unrealistic assumption.

In this paper, we first develop tests of H_0 versus H_{BW} and H_{WB} which do not require any assumptions about t_0 or p_0. Next, given that $F \in C_{BW} \cap \mathcal{F}^*$ or $F \in C_{WB} \cap \mathcal{F}^*$ ($\mathcal{F}^*$ as defined below), we obtain point estimates and asymptotic confidence intervals for t_0 and p_0. The point estimates are shown to be asymptotically normal. Throughout we assume that we have a random sample $X_1, \dots, X_n$ from $F \in \mathcal{F}$ (or $\mathcal{F}^*$, where F is unknown.

The rest of the paper is organized as follows. Section 2 contains the main results. Section 3 contains a real-data application of these methods. Section 4 contains proofs of the theorems. The proofs of all lemmas are omitted here for brevity, but may be found in the technical report by Hawkins and Kochar (1992) (henceforth HK92).

To push through our asymptotic results we rely heavily on results of Cšorgo, Csörgo and Horvath (1986) (henceforth CCH). These results require the following condition of F (here $f(x) = F'(x)$) :

$$J = \sup_{0<u<1} \frac{q(u)(1-u)}{f(F^{-1}(u))} < \infty \qquad \dots (1.1)$$

for *some* function q satisfying $q(t) > 0, 0 < t < 1, q$ symmetric about $t = \frac{1}{2}, q(t)$ nondecreasing on $[0, \frac{1}{2})$ and

$$\int_0^{1/2} t^{-1} exp\{-\epsilon q^2(t)/t\} dt < \infty \text{ for all } \epsilon > 0. \qquad \dots (1.2)$$

Functions q satisfying (1.2) are called Chibisov-O'Reilly weight functions; see CCH p. 22.

Note that any $F \in \mathcal{E}$ trivially satisfies (1.1) with $q(u)1$ for $0 \leq u \leq 1$. However, any lognormal cdf F_{LN} satisfies $F_{LN} \in C_{BW}$ (see Klefsjö (1989), p. 566), but does not satisfy (1.1); see Lemma 0 in HK92. For later reference, define $\mathcal{F}^* = \{F \in \mathcal{F} : F \text{ satisfies (1.1) }\}$.

2. Main results

We introduce the so-called scaled total time on test (STTT) transform (in centered form)

$$\bar{\phi}_F(u) = \frac{1}{\mu} \int_0^{F^{-1}(u)} \bar{F}(t)dt - u, \ 0 \leq u \leq 1, \qquad \dots (2.1)$$

where $\mu = e(0) = \int_0^\infty \bar{F}(t)dt$. Bergman (1979) and Klefsjö (1982) have shown that

$$F \in \mathcal{E} \text{ iff } \bar{\phi}_F(u) = 0,\ 0 \le u \le 1,$$

$$F \text{ is NBUE iff } \bar{\phi}_F(u) > 0,\ 0 < u \le 1, \qquad \ldots(2.2)$$

$$F \text{ is NWUE iff } \bar{\phi}_F(u) < 0,\ 0 < u \le 1.$$

It follows in a similar way that

$$F \in C_{BW} (F \in C_{WB}) \text{ iff} \begin{cases} \bar{\phi}_F(u) > (<) 0,\ 0 < u < p_0 \\ \bar{\phi}_F(u) < (>) 0,\ p_0 < u \le 1,\ p_0 = F(t_0). \end{cases} \qquad \ldots(2.3)$$

Following Klefsjö (1989), we introduce the functional

$$\psi_F(p) = \int_0^p \bar{\phi}_F(u)du - \int_p^1 \bar{\phi}_F(u)du,\ 0 \le p \le 1. \qquad \ldots(2.4)$$

All inferences in this paper are based on ψ_F.

2.1. *Hypothesis tests.* For $F \in \mathcal{E}, \psi_F(p) = 0$ for all $0 \le p \le 1$. However, for $F \in C_{BW}$ it follows from (2.3) and the fact that $\frac{d}{dp}\psi_F(p) = 2\bar{\phi}_F(p)$ that $\psi_F(p)$ is increasing for $0 \le p < p_0$ and decreasing for $p_0 < p \le 1$, with $\psi_F(p_0) = \sup\{\psi_F(p) : 0 \le p \le 1\} > 0$. Similarly, for $F \in C_{WB}, \psi_F(p)$ is, respectively, decreasing and increasing on these intervals, with $\psi_F(p_0) = \inf\{\psi_F(p) : 0 \le p \le 1\} < 0$. These observations suggest the following test statistics:

$$\text{for } H_0 \text{ vs. } H_{BW}: \quad T_n^{BW} = n^{\frac{1}{2}} \sup\{\psi_{F_n}(p) : 0 \le p \le 1\},$$

$$\text{for } H_0 \text{ vs. } H_{WB}: \quad T_n^{WB} = n^{\frac{1}{2}} \inf\{\psi_{F_n}(p) : 0 \le p \le 1\},$$

where $F_n(x) = n^{-1}\sum_{i=1}^{n} I(X_i \le x)$ is the empirical cdf. Now it follows from result 4, p. 65 of CCH and the Skorokhod continuity of the integral functional in (2.4) that

$$\sup\{|\psi_{F_n}(p) - \psi_F(p)| : 0 \le p \le 1\} = O_p(n^{-\frac{1}{2}}),\ F \in \mathcal{F}^*. \qquad \ldots(2.5)$$

Thus, T_n^{BW} and T_n^{WB} will lie close to zero under H_0, but $T_n^{BW} (T_n^{WB})$ will be large positive (negative) under $H_{BW} (H_{WB})$.

The exact distributions of T_n^{BW} and T_n^{WB} under H_0 are intractable, so their limit distributions are obtained in the following result. In this direction, let $\tilde{Z} = \{\tilde{Z}(u) : 0 \le u \le 1\}$ denote a mean-zero Gaussian process with covariance $\{\tilde{Z}(v)\tilde{Z}(u)\} = \frac{1}{3}(u^3 - v^3) - \frac{1}{2}(u^2 + v^2) + 2uv^2 - u^2v^2 + \frac{1}{12}$ for $0 \le v \le u \le 1$..

THEOREM 1. *Under* H_0, *as* $n \to \infty, T_n^{BW} \xrightarrow{\mathcal{L}} Z_s =: sup\{\tilde{Z}(u) : 0 \le u \le 1\}$ *and* $T_n^{WB} \xrightarrow{\mathcal{L}} Z_I =: inf\{\tilde{Z}(u) : 0 \le u \le 1\}$.

TABLE 1. APPROXIMATE CRITICAL VALUES

β	.90	.95	.99
$Z_{s;\beta}$	0.50	0.57	0.74

Table 1 contains Monte-Carlo-estimated 100β quantiles $Z_{s;\beta}$ of the distribution of Z_s, obtained in Hawkins and Kochar (1991). The test for H_{BW} rejects H_0 at level α if $T_n^{BW} > Z_{s;1-\alpha}$. The test for H_{WB} rejects H_0 at level α if $T_n^{WB} < Z_{I;\alpha}$. (Since $Z_I \stackrel{d}{=} -Z_s$, we have $Z_{I;\alpha} = -Z_{s;1-\alpha}$).

A brief Monte Carlo study comparing the power of the T_n^{BW}-test with the test of Klefsjö (1989) for F lognormal is given in HK92, and shows basically that the price of not knowing t_0 or p_0 is a slight loss in power.

2.1.1. *Computing the test statistics.* One easily checks that $\psi_{F_n}(p)$, $0 \le p \le 1$ is almost surely continuous, and defining $A_{nk} = ((k-1)/n, k/n), 1 \le k \le n$, that

$$\bar{X}_n \psi_{F_n}(p) = 2I_n^*(p) - I_n^*(1) + \bar{X}_n(\frac{1}{2} - p^2),\ p \in A_{nk}, \qquad \dots(2.6)$$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean, $X_{(j)}$ denotes the j-th order statistic, $X_{(0)} =: 0$ and

$$\begin{aligned} I_n^*(p) &= \int_{u=0}^{p}\int_{t=0}^{F_n^{-1}(u)} \bar{F}_n(t)dtdu \\ &= n^{-2}\left\{\sum_{j=1}^{[np]}\sum_{l=1}^{j}(n-l+1)\left[X_{(l)} - X_{(l-1)}\right]\right. \qquad \dots(2.7) \\ &\quad \left. +(np-[np])\sum_{l=1}^{m_n(p)}(n-l+1)\left[X_{(l)} - X_{(l-1)}\right]\right\} \end{aligned}$$

(Here $[s]$ denotes integer part of s, and $m_n(p) = \min([np]+1, n)$.) It follows from (2.6) that

$$\frac{d}{dp}\psi_{F_n}(p) = 2\{U_{nk} - p\},\ p \in A_{nk}, \qquad \dots(2.8)$$

where for $1 \le k \le n$,

$$U_{nk} = (n\bar{X}_n)^{-1}\left\{(n-k+1)X_{(k)} + \sum_{j=1}^{k-1} X_{(j)}\right\} \qquad \dots(2.9)$$

From (2.8) and (2.9) it follows that for $p \in A_{nk}$,

$$\frac{d}{dp}\psi_{F_n}(p) \text{ is } \begin{cases} > 0, & \text{if } p < U_{nk} \\ = 0, & \text{if } p = U_{nk} \\ < 0, & \text{if } p > U_{nk} \end{cases} \qquad \ldots (2.10)$$

Since $\frac{d^2}{dp^2}\psi_{F_n}(p) = -2 < 0$ for $p \in A_{nk}$, it follows from (2.8) - (2.10) by ordinary calculus that any minimizer of $\psi_{F_n}(p)$, say $\hat{p}_n^m$, almost surely occurs in the set $G_n =: \{\frac{k}{n} : 0 \le k \le n\}$, and that any maximizer, say $\hat{p}_n^M$, a.s. falls into $\mathcal{L}_n^* = G_n \cup \{U_{nk} : U_{nk} \in A_{nk}\}$. In fact, a closer look shows that the set of possibilities for $\hat{p}_n^M$ is even smaller.

LEMMA 1 *For any absolutely continuous* $F, \hat{p}_n^M \in \mathcal{L}_n =: \{1\} \cup \{U_{nk} : U_{nk} \in A_{nk}\}$ *a.s.* .

A FORTRAN program for computing T_n^{BW}, T_n^{WB} and all other statistics in this paper is available from the first author.

Finally, one notes from (2.6) and (2.7) that $\psi_{F_n}(p)$ is scale-invariant, and hence is distribution-free over $\mathcal{E}$.

2.2. *Point estimation of* t_0 *and* p_0. Here we assume that $F \in C_{BW}^* := C_{BW} \cap \mathcal{F}^*$ or $F \in C_{WB}^* =: C_{WB} \cap \mathcal{F}^*$, and that we know which is the case.

For $F \in C_{BW}^*$, in view of (2.5) and the fact that $\psi_F(p_0) = \sup\{\psi_F(p) : 0 \le p \le 1\}$, it is natural to estimate p_0 by any value, say $\hat{p}_{0n}^{BW}$, which maximizes $\psi_{F_n}(p)$ over $0 \le p \le 1$. It may be checked that the values of $\psi_{F_n}(p)$ for $p \in \mathcal{F}_n$ are a.s. distinct, so that this maximum will be uniquely attained a.s. Hence, for $F \in C_{BW}^*$ we define the point estimate $\hat{p}_{0n}^{BW}$ of p_0 by

$$\psi_{F_n}(\hat{p}_{0n}^{BW}) = \sup\{\psi_{F_n}(p) : 0 \le p \le 1\}.$$

Since $t_0 = F^{-1}(p_0)$, it is natural to define a point estimate of t_0 by $\hat{t}_{0n}^{BW} =: F_n^{-1}(\hat{p}_{0n}^{BW})$.

By similar considerations for $F \in C_{WB}^*$ it is natural to define the point estimates $\hat{p}_{0n}^{WB}$ and $\hat{t}_{0n}^{WB}$ of p_0 and t_0 by $\psi_{F_n}(\hat{p}_{0n}^{WB}) = \inf\{\psi_{F_n}(p) : 0 \le p \le 1\}$ and $\hat{t}_{0n}^{WB} =: F_n^{-1}(\hat{p}_{0n}^{WB})$.

The following result shows that all of these estimators are asymptotically normal. Let $Q(t) = F^{-1}(t), h(t) = 1/f(Q(t)), 0 < t < 1$. Let $\hat{p}_{0n}$ denote either $\hat{p}_{0n}^{BW}$ or $\hat{p}_{0n}^{WB}$, and let t_{0n} denote either $\hat{p}_{0n}^{BW}$ or $\hat{t}_{0n}^{WB}$, as $F \in C_{BW}^*$ or $F \in C_{WB}^*$.

THEOREM 2. *As* $n \to \infty$,

(i) $n^{\frac{1}{2}}\{\hat{p}_{0n} - p_0\} \xrightarrow{\mathcal{L}} N(0, \gamma^2(F, p_0))$, *where*

$$\begin{aligned}\gamma^2(F,p_0) &= \{(1-p_0)h(p_0)-\mu\}^{-2} \\ &\times\left\{(1-2p_0)\left[(1-p_0)Q^2(p_0)+\int_0^{Q(p_0)} x^2dF(x)\right]\right. \\ &+(1-p_0)^3[2Q(p_0)h(p_0)+p_0h^2(p_0)] \\ &\left.-2\mu p_0(1-p_0)[Q(p_0)+(1-p_0)h(p_0)]+p_0^2\int_0^\infty x^2dF(x)\right\}.\end{aligned}$$

(ii) $n^{\frac{1}{2}}\{\hat{t}_{0n} - t_0\} \xrightarrow{\mathcal{L}} N(0, \tau^2(F, t_0))$, where

$$\tau^2(F,t_0) = (1,1)\mathbf{J}_\Gamma^T \Sigma \mathbf{J}_\Gamma \begin{pmatrix}1\\1\end{pmatrix}$$

$\mathbf{J}_\Gamma$ is the Jacobian of the function Γ in (4.16) and $\Sigma = (\sigma_{ij} : i,j = 1,2,3)$ with

$$\begin{aligned}\sigma_{11} &= F(t_0)[1-F(t_0)], \sigma_{12} = [1-F(t_0)]\{t_0-\mu F(t_0)\}, \\ \sigma_{13} &= t_0[1-F(t_0)], \sigma_{22} = [1-F(t_0)]t_0^2 - \mu^2F^2(t_0) + \int_0^{F(t_0)} Q^2(y)dy, \\ \sigma_{23} &= t_0[1-F(t_0)][t_0+\mu] + \int_0^{F(t_0)} Q^2(y)dy - \mu^2F(t_0), \sigma_{33} = Var(F).\end{aligned}$$

2.2.1. *Monte Carlo results.* To verify the asymptotic results in Theorem 2, a small Monte Carlo study was undertaken. A parametric family of distributions in C^*_{BW} was constructed based on the scaled total time on test transform (STTT : $\phi_F(u) =: \mu^{-1}\int_0^{Q(u)}\{1-F(t)\}dt$ in general)

$$\phi_0(u) = \left(\frac{s_0-1}{p_0}\right)u^3 - \frac{(p_0+1)(s_0-1)}{p_0}u^2 + s_0u,\ 0 \le u \le 1,$$

where $0 < p_0 < 1$ is the transition quantile, $s_0 = \phi_0'(0) \in (1, d_0)$, $d_0 = \delta_0/(\delta_0 - 1)$ and $\delta_0 = (p_0+1)^2/3p_0$. One may check that ϕ_0 corresponds to the distribution F_0 on $[0,\infty)$ with quantile function

$$F_0^{-1}(u) = a_1u + a_2u^2 + a_3ln(1-u), 0 \le u < 1,$$

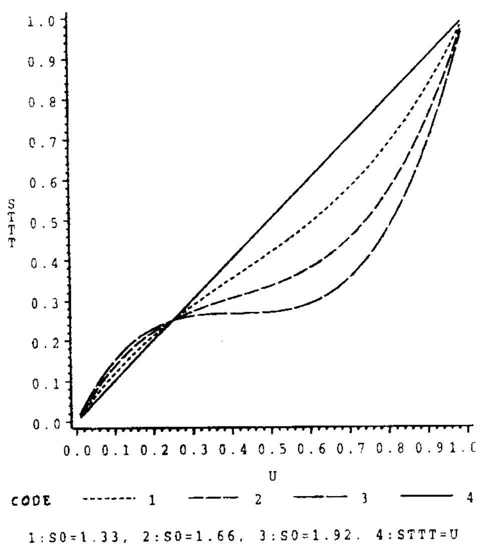
where $a_1 = (2p_0-1)(s_0-1)/p_0, a_2 = -\frac{3}{2}(s_0-1)/p_0$ and $a_3 = a_1 - s_0$. The vector (p_0, s_0) acts as the parameter indexing the family. Each distribution in this family has mean one. Verifying that F_0 has finite variance and a positive density is straightforward. Varifying condition (1.1) is easy using CCH's sufficient condition on p. 63 and the relation $\int_0^1(1-u)^2/f^2(Q(u))du = \mu^2\int_0^1[\phi'(u)]^2du$, which holds for any distribution F with density f, quantile function Q, mean μ and STTT ϕ.

TABLE 2. MONTE CARLO RESULTS FOR POINT ESTIMATORS ($n = 100$)

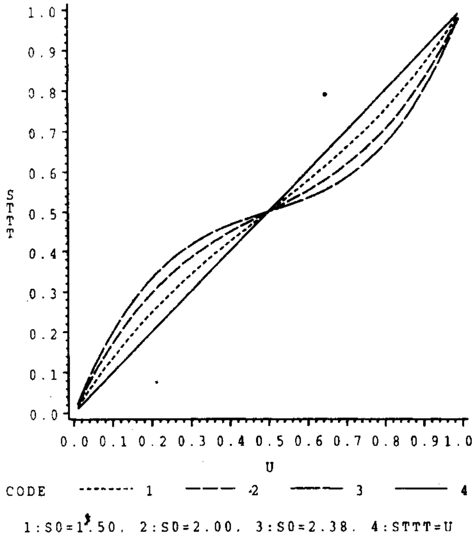
					$\hat{p}^{BW}_{n0}$ mean		$\hat{p}^{BW}_{n0}$ var		$\hat{t}^{BW}_{n0}$ mean		$\hat{t}^{BW}_{n0}$ var	
p_0	s_0	$\psi''_{(p_0)}$	$Var(F)$	PNM	Asym.	Emp.	Asym.	Emp.	Asym.	Emp.	Asym.	Emp.
.25	1.33	-0.5	1.72	.009	.250	.253	.026	.027	.284	.299	.038	.055
	1.66	-1.0	2.77	.000	.250	.243	.007	.011	.280	.261	.005	.011
	1.92	-1.4	3.83	.000	.250	.244	.005	.008	.277	.254	.001	.005
.50	1.50	-0.5	1.20	.109	.500	.489	.037	.048	.665	.735	.108	.240
	2.00	-1.0	1.47	.027	.500	.512	.011	.017	.636	.673	.015	.054
	2.38	-1.4	1.71	.010	.500	.514	.007	.009	.615	.637	.003	.019
.75	2.00	-0.5	0.91	.293	.750	.730	.032	.018	1.22	1.29	.323	.275
	3.00	-1.0	0.87	.190	.750	.756	.010	.008	1.06	1.16	.044	.099
	3.76	-1.4	0.86	.137	.750	.759	.007	.007	0.94	1.03	.010	.062

Table 2 displays the results for $n = 100$ and a selection of (p_0, s_0) choices, based on 1000 replications. Graphs of the corresponding STTT's appear in Figure 1. As it is intuitive that estimation should be easier when the criterion functional $\psi_F(p)$ has a more distinct peak at $p = p_0$, we have chosen s_0 to maintain particular values of $\psi''_{F_0}(p_0) = 2(s_0 - 1)(p_0 - 1)$. Also included is the variance of $F_0(Var(F))$.

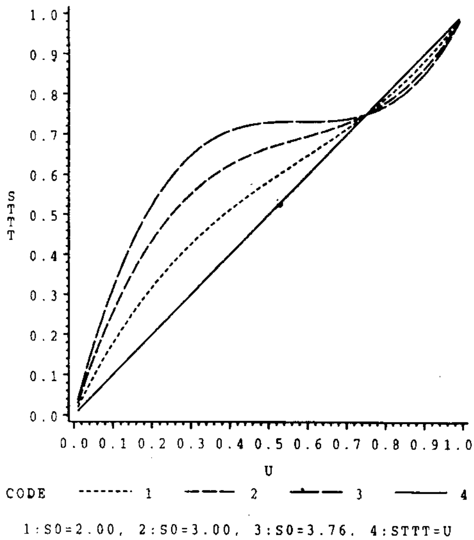
STTT'S WITH P0=.25



STTT'S WITH PO=.50



STTT'S WITH PO=.75



When the peak of $\psi_F(p)$ is not steep ($\psi''_{F_0}(p_0)$ is small), the empirical function $\psi_{F_n}(p)$ tends with positive probability to be nondecreasing on $[0,1]$, with no maximizer in $(0,1)$. This results in $\hat{p}_{0n}^{BW} = 1$. As the peak gets steeper (ψ''_{F_0} gets more negative), this probability decreases. We include an empirical estimate of this probability of no interior maximizer (PNM) in Table 2. When $\hat{p}_{0n}^{BW} = 1$ occurs, in our opinion the estimation has failed and we have evidence that either : (1) p_0 is near 1 or (2) $\psi_F(p)$ has a poorly-defined peak. An examination of $\psi_{F_n}(p)$ can help shed light on this question in practice.

The rate of convergence in Theorem 2 also (apparently) depends on the steepness of $\psi_F(p)$ at p_0. When $\psi_F''(p_0)$ is large negative, the empirical means and variances are in closer agreement with their asymptotic counterparts than when $\psi_F''(p_0)$ is nearer to zero. The agreement of the empirical with the asymptotic variance is particularly sensitive to this situation, due to the occurrence of $\hat{p}_{0n}^{BW} = 1$ noted above. For this reason, the empirical mean and variance estimates in Table 2 are from those replications where $\hat{p}_{0n}^{BW} < 1$; i.e. in those cases where the estimation is considered "successful".

REMARK. Estimation of p_0 and t_0 was attempted for the lognormal *cdf* prior to (and actually motivated) our discovery that $F_{LN} \notin \mathcal{F}^*$. The results were striking : the empirical function $\psi_{F_n}(p)$ was increasing on $0 \le p \le 1$ in almost every Monte Carlo experiment we ran, regardless of the lognormal parameters, resulting in $\hat{p}_{0n}^{BW} = 1$ in almost all cases. In other words, Theorem 2 failed completely. These results support the conjecture that condition (1.1) is not only sufficient for the CCH-results we used to prove Theorem 2 (see section 4), but may be almost necessary.

2.3. *Interval estimation.* Again we assume that $F \in C_{BW}^*$ or $F \in C_{WB}^*$, and that we know which is the case. The discussion is in terms of the generic $\hat{p}_{n0}$ and $\hat{t}_{n0}$, since all the results are the same for $F \in C_{BW}^*$ or $F \in C_{WB}^*$.

The idea is to construct consistent estimators of $\gamma^2(F, p_0)$ and $\tau^2(F, t_0)$, from which large-sample confidence intervals follow by Theorem 2 in the usual way. These consistent estimators, $\hat{\gamma}_n^2$ and $\hat{\tau}_n^2$, say, are formed by substituting $\hat{p}_{0n}$ for $p_0, \hat{t}_{0n}$ for $t_0, \bar{X}_n$ for μ and F_n for F in the formulae in Theorem 2. The only troublesome part is estimating $h(p_0) = 1/f(F^{-1}(p_0)) = 1/f(t_0)$, which apparently requires estimation of the density f.

To avoide density estimation, we note that

$$\begin{aligned} \psi_F'(p) &= 2\bar{\phi}_F(p),\ \psi_F'(p_0) = 0, \\ \psi_F''(p) &= 2\bar{\phi}_F'(p) = 2\{\mu^{-1}(1-p)h(p) - 1\}, \end{aligned} \qquad \ldots(2.11)$$

and hence that for p near p_0,

$$\psi_F(p) - \psi_F(p_0) = \frac{1}{2}\psi_F''(p_0)(p-p_0)^2 + o((p-p_0)^2). \qquad \ldots(2.12)$$

Expressions (2.11) and (2.12) suggest estimating $h(p_0)$ as follows. First, estimate $\beta =: \psi_F''(p_0)/2$ using the empirical analog of (2.12):

$$\psi_{F_n}(\frac{i}{n}) - \psi_{F_n}(\hat{p}_{0n}) \simeq \hat{\beta}_n(\frac{i}{n} - \hat{p}_{0n})^2,\ |\frac{i}{n} - \hat{p}_{0n}| \le \Delta_n. \qquad \ldots(2.13)$$

Then estimate $h(p_0)$ (ala (2.11)) by

$$\hat{h}_n = \{1 + \hat{\beta}_n\}\bar{X}_n/(1 - \hat{p}_{0n}). \qquad \ldots(2.14)$$

This is a convenient approach since : (1) the quantities on the left side of (2.13) are already computed as by-products of the computation of $\hat{p}_{0n}$; and (2) (2.13) has the form of a linear regression model. The sequence Δ_n in (2.13) must be chosen carefully to make $\hat{\beta}_n$, and hence $\hat{h}_n$, consistent. Since we only need $\hat{\beta}_n$ to be consistent for β, for simplicity we take the ordinary least squares estimate

$$\hat{\beta}_n = \sum_{i \in A_n} (\frac{i}{n} - \hat{p}_{0n})^2 \{\psi_{F_n}(\frac{i}{n}) - \psi_{F_n}(\hat{p}_{0n})\} / \sum_{i \in A_n} (\frac{i}{n} - \hat{p}_{0n})^4$$

where $A_n = \{i : |\frac{i}{n} - \hat{p}_{0n}| \leq \Delta_n\}$. The right rate for Δ_n is given by

LEMMA 2. *If $F \in C^*_{BW}$ (or $F \in C^*_{WB}$) and $\Delta_n = O(n^{\frac{1}{4}+\delta})$ for some $\delta > 0$, then $\hat{\beta}_n \xrightarrow{P} \beta$ as $n \to \infty$.*

In practice, we find that taking $\Delta_n = 5n^{\frac{1}{4}}$ works reasonably well for finite n.

The following result gives the desired confidence intervals. Let I_n and J_n denote the intervals $\hat{p}_{0n} \pm n^{-\frac{1}{2}} z_{\alpha/2}\hat{\gamma}_n$ and $\hat{t}_{0n} \pm n^{-\frac{1}{2}} z_{\alpha/2}\hat{\tau}_n$, where z_β denotes the 100β quantile of $N(0, 1)$.

THEOREM 3. *If $F \in C^*_{BW}$ (or $F \in C^*_{WB}$) and $\Delta_n = O(n^{\frac{1}{4}+\delta})$ for some $\delta > 0$, then as $n \to \infty$,*

(i) $P\{p_0 \in I_n\} \to 1 - \alpha$.
(ii) $P\{t_0 \in J_n\} \to 1 - \alpha$.

2.3.1. *Monte Carlo results.* To check the performance of our confidence intervals, the intervals (with $\Delta_n = 5n^{\frac{1}{4}}$) were computed along with the point estimates in the experiments reported in Section 2.2. The results are given in Table 3. Since the standard error estimates $\hat{\gamma}_n$ and $\hat{\tau}_n$ cannot be computed when $\hat{p}^{BW}_{0n} = 1$, the results in Table 3 are, as in Table 2, only for the replications in which $\hat{p}^{BW}_{0n} < 1$. Also, the "plug-in-type" variance estimates $\hat{\gamma}^2_n$ and $\hat{\tau}^2_n$ can occasionally be negative, so we simply took the absolute value, which of course, preserves consistency.

TABLE 3. MONTE CARLO RESULTS FOR ASYMPTOTIC 95% CONFIDENCE INTERVALS ($n = 100$)

p_0	s_0	$\psi''_{(p_0)}$	$Var(F)$	PNM	p_0 cov. prob.	p_0 median length	t_0 cov. prob.	t_0 median length
.25	1.33	-0.5	1.72	.009	.828	.588	.828	.721
	1.66	-1.0	2.77	.000	.862	.324	.854	.279
	1.92	-1.4	3.83	.000	.873	.255	.867	.136
.50	1.50	-0.5	1.20	.109	.900	.955	.916	1.885
	2.00	-1.0	1.47	.027	.949	.600	.979	.948
	2.38	-1.4	1.71	.010	.953	.471	.991	.609
.75	2.00	-0.5	0.91	.293	.941	.625	.975	1.983
	3.00	-1.0	0.87	.190	.923	.361	.974	.780
	3.76	-1.4	0.86	.137	.900	.291	.889	.373

The result in Table 3 show that the performance of the confidence intervals is sensitive to the peakedness of $\psi_F(p)$, with coverage probabilities and median lengths generally improving as this peakedness increases (i.e. $\psi''_F(p_0)$ becomes more negative).

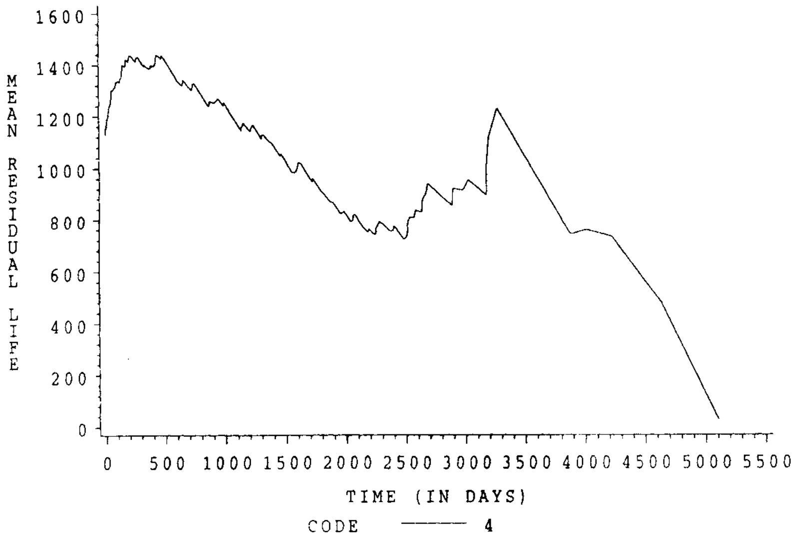
3. Examples

EXAMPLE 1. Figure 2 displays the empirical mean residual life function (i.e. $e(t)$ with F replaced by F_n) of post-heart-transplant survival data for 174 patients treated by the Stanford Heart Transplant Center during the years 1976-1985. For the purpose of illustrating our methods, the estimates in Figure 2 and the ensuing analysis ignore the presence of a small amount of censoring in these data, primarily for the years 1983-1985. I.e. censored cases were deleted.

Figure 2 suggests (apart from the slight bump at time around 3500 days) that the distribution F of post-transplant survival time falls into the C_{WB} family. Further, the value of T_n^{WB} is -1.06, which by Table 1 has a p-value less than .01, so the exponential model is strongly rejected. The point estimates of p_0 and t_0 (based on the raw data) are $\hat{p}_{0n}^{WB} = 0.603$ and $\hat{t}_{0n}^{WB} = 1150.0$, with respective 95% confidence intervals [0.39, 0.81] and [516.3, 1783.7]. In this situation, $t = 0$ represents the transplant time and the interval $(0, t_0]$ represents the duration of benefit of the transplant - i.e. the time post-surgery for which the population mean residual life exceeds its value at transplant time. By our estimates from these data, this duration of benefit is about $\hat{t}_{0n}^{WB} = 1150$ days (about 3.2 years). The estimate $\hat{p}_{0n}^{WB} = .603$ means that about 60% of the population dies during this duration of benefit.

EXAMPLE 2. One of the referees asked us to provide an example of a distribution in C_{WB} which is not in the IDMRL (increasing-decreasing mean residual life) family. The latter family, which as has been extensively studied (see e.g. Guess, Hollander and Proschan (1986) and Hawkins, Kochar and Loader

(1992)), is characterized by $e(t)$ being strictly increasing (respectively decreasing) for $t < t^*$ (respectively $t > t^*$), for some change point t^*. The following example of such a distribution was constructed by Professor Ramesh Korwar.



Let

$$\bar{F}(t) = \begin{cases} e^{-t}, 0 \le t \le 1 \\ e^{-1}, 1 \le t \le 2 \\ e^{-(2t-3)}, 2 \le t \le 3 \\ e^{-t}, t \ge 3. \end{cases}$$

Then the corresponding mean residual life function is

$$e(t) = \begin{cases} 1 + (\frac{1}{2}e^{-1} + \frac{1}{2}e^{-3})e^t, 0 \le t \le 1 \\ \frac{5}{2} - t + \frac{1}{2}e^{-2}, 1 \le t \le 2 \\ \frac{1}{2} + \frac{1}{2}e^{2(t-3)}, 2 \le t \le 3 \\ 1, t \ge 3. \end{cases}$$

One easily checks that $F \in C_{WB}$ with $t_0 = \frac{3}{2} + \frac{1}{2}(e^{-2} - e^{-1} - e^{-3})$, but F is not IDMRL sice $e(t)$ is increasing for $t \in (0,1)$, decreasing on (1,2) and increasing on (2,3).

4. Proofs of theorems

All lemmas stated here are proved in HK92.

PROOF OF THEOREM 1. Since $\psi_F(p) = 0$ for $0 \le p \le 1$ if $F \in \mathcal{E}$, we have by (2.4) that

$$\begin{aligned} n^{\frac{1}{2}}\psi_{F_n}(p) &= n^{\frac{1}{2}}\{\psi_{F_n}(p) - \psi_F(p)\} \\ &= \int_0^p s_n(u)du - \int_p^1 s_n(u)du \\ &= :(T_{s_n})(p), \end{aligned} \qquad \ldots(4.1)$$

where $s_n(u) =: n^{\frac{1}{2}}\{\bar{\phi}_{F_n}(u) - \bar{\phi}_F(u)\}, 0 \le u \le 1$ is the scaled total time on test empirical process (see CCH, p. 10) and $T : D([0,1]) \to D([0,1])$ by $(Th)(p) = \int_0^p h(u)du - \int_p^1 h(u)du$. Further, CCH p. 65 item (5) gives that for $F \in \mathcal{E}(\xrightarrow{\omega}$ denotes weak convergence in $D([0,1])$),

$$s_n \xrightarrow{\omega} W^o \qquad \ldots(4.2)$$

where W^o is a Brownian bridge process on $[0,1]$. The result follows by the Skorokhod continuity of T and that of the "sup" functional, using the argument following (4.7) in the proof of Theorem 1 of Hawkins and Kochar (1991). *Q.E.D.*

PROOF OF THEOREM 2(I). Assume $F \in C^*_{BW}$. The proof for $F \in C^*_{WB}$ is almost identical. We write $\hat{p}_{n0}$ for $\hat{p}^{BW}_{n0}$, $\hat{t}_{n0}$ for $\hat{t}^{BW}_{n0}$, etc. .

(i) First, by Taylor's theorem for some p^*_n between $\hat{p}_{0n}$ and p_0,

$$n^{\frac{1}{2}}\{\bar{\phi}_F(\hat{p}_{0n}) - \bar{\phi}_F(p_0)\} = \bar{\phi}'_F(p^*_n).n^{\frac{1}{2}}\{\hat{p}_{0n} - p_0\}, \qquad \ldots(4.3)$$

where

$$\bar{\phi}'_F(p) = \mu^{-1}(1-p)/f(F^{-1}(p)) - 1 = \frac{1}{2}\psi''_F(p) \qquad \ldots(4.4)$$

satisfies $\bar{\phi}'_F(p_0) = \frac{1}{2}\psi''_F(p_0) < 0$ since p_0 is a maximizer of $\psi_F(p)$ by (definition of C_{BW}). We next require

LEMMA 3. *For $F \in C^*_{BW} \cup C^*_{WB}$, as $n \to \infty$,*

$$n^{\frac{1}{2}}\{\bar{\phi}'_F(\hat{p}_{0n}) - \bar{\phi}_F(p_0)\} = -n^{\frac{1}{2}}\{\bar{\phi}_{F_n}(p_0) - \bar{\phi}_F(p_0)\} + o_p(1).$$

Using Lemma 3, (4.3) and the fact (which follows from (2.5) by the argument used in the proof of Theorem 2 in Hawkins and Kochar (1991)) that $\hat{p}_{0n} \xrightarrow{P} p_0$, it holds that

$$\begin{aligned} n^{\frac{1}{2}}\{\hat{p}_{0n} - p_0\} &= -\{\bar{\phi}'_F(p_0)\}^{-1}.n^{\frac{1}{2}}\{\bar{\phi}_{F_n}(p_0) - \bar{\phi}_F(p_0)\} + o_p(1) \\ &= -\{\bar{\phi}'_F(p_0)\}^{-1}s_n(p_0) + o_p(1), \end{aligned} \qquad \ldots(4.5)$$

where $s_n(\cdot)$ was defined following (4.1). The result follows by tedious but easy calculations from (4.5) and the fact (see item (4) on p. 65 of CCH) that for $F \in \mathcal{F}^*$,

$$s_n \xrightarrow{\omega} Z(F) \quad \text{as } n \to \infty, \qquad \ldots (4.6)$$

where $Z(F)$ is a zero-mean Gaussian process with covariance (for $0 \le s \le t \le 1$)

$$\begin{aligned} \sigma_2(s,t) &= \mu^{-2}\sigma_1(s,t) + \mu^{-4}\phi_F^*(s)\phi_F^*(t)\sigma_1(1,1) \\ &\quad -\mu^{-3}\phi_F^*(t)\sigma_1(s,1) - \mu^{-3}\phi_F^*(s)\sigma_1(t,1). \end{aligned} \qquad \ldots (4.7)$$

Here $\phi_F^*(t) = \int_0^{Q(t)}\{1 - F(x)\}dx$ is the total time on test transform of F, and for $0 \le s \le t$,

$$\begin{aligned} \sigma_1(s,t) &= \phi_F^*(t)\{Q(s) - \phi_F^*(s)\} - Q(s)\phi_F^*(s) + (1-s)[Q(s)]^2 \\ &\quad + \int_0^s [Q(y)]^2 dy + (1-t)^2 h(t)\{Q(s) - \phi_F^*(s)\} \\ &\quad + (1-s)^2 h(s)\{Q(s) - \phi_F^*(s)\} + s(1-s)h(s)\{\phi_F^*(t) - \phi_F^*(s)\} \\ &\quad + (1-s)(1-t)^2 s h(s) h(t). \end{aligned} \qquad \ldots (4.8)$$

Q.E.D. (Theorem 2 (i)).

Proof of Theorem 2 (ii). Let $Q_n(t) = F_n^{-1}(t), 0 \le t \le 1$ denote the empirical quantile function. Then by Theorem D, p. 101 of Serfling (1981), we have

$$\begin{aligned} \hat{t}_{0n} &= Q_n(\hat{p}_{0n}) \\ &= Q(\hat{p}_{0n}) + h(\hat{p}_{0n})\{\hat{p}_{0n} - F_n(Q(\hat{p}_{0n}))\} + o_p(n^{-\frac{1}{2}}). \end{aligned} \qquad \ldots (4.9)$$

Thus, since $t_0 = Q(p_0)$, we have

$$\begin{aligned} n^{\frac{1}{2}}\{\hat{t}_{0n} - t_0\} + o_p(1) &= n^{\frac{1}{2}}\{Q_n(\hat{p}_{0n}) - Q(p_0)\} \\ &= n^{\frac{1}{2}}\{Q(\hat{p}_{0n}) - Q(p_0)\}(=: D_{1n}) \end{aligned} \qquad \ldots (4.10)$$

$$+ n^{\frac{1}{2}}\{\hat{p}_{0n} - F_n(Q(\hat{p}_{0n}))\}h(\hat{p}_{0n})(=: D_{2n}). \qquad \ldots (4.11)$$

We see that $n^{\frac{1}{2}}\{\hat{t}_{0n} - t_0\}$ is the sum of D_{1n} and D_{2n}, so we will need to obtain the limiting joint distribution of (D_{1n}, D_{2n}).

In this direction, define the empirical process $E_n(x) = n^{\frac{1}{2}}\{F_n(x) - F(x)\}$, $0 \le x < \infty$. Then

$$\begin{aligned} D_{2n} &= n^{\frac{1}{2}}\{F(Q(\hat{p}_{0n})) - F_n(Q(\hat{p}_{0n})\}h(\hat{p}_{0n}) \\ &= -h(p_0)E_n(Q(\hat{p}_{0n})) + o_p(1), \end{aligned} \qquad \ldots (4.12)$$

since $\hat{p}_{0n} \xrightarrow{P} p_0$ and E_n weakly converges. Further, by the differentiability of Q, the delta method and Theorem 2(i), there is a finite constant $K > 0$ such that in probability,

$$|E_n(Q(\hat{p}_{0n})) - E_n(Q(p_0))| \leq \sup_{0 \leq s \leq K/\sqrt{n}} |E_n(Q(p_0) + s) - E_n(Q(p_0))|, \quad \ldots (4.13)$$

which is $p_p(1)$ by the tightness of the sequence $\{E_n : n \geq 1\}$. Combining (4.13) into (4.12) gives that

$$D_{2n} = -h(p_0)E_n(Q(p_0)) + o_p(1). \quad \ldots (4.14)$$

On the other hand, by Taylor's theorem for α_n^* between $\hat{p}_{0n}$ and p_0, and for all sufficiently large n,

$$\begin{aligned} D_{1n} &= h(\alpha_n^*)n^{\frac{1}{2}}\{\hat{p}_{0n} - p_0\} \\ &= \{h(\alpha_n^*)/\bar{\phi}_F'(p_n^*)\}.n^{\frac{1}{2}}\{\bar{\phi}_F(\hat{p}_{0n}) - \bar{\phi}_F(p_0)\} \\ &= -\{h(\alpha_n^*)/\bar{\phi}_F'(p_n^*)\}.n^{\frac{1}{2}}\{\bar{\phi}_{F_n}(p_0) - \bar{\phi}_F(p_0)\} + o_p(1) \\ &= -\{h(\alpha_n^*)/\bar{\phi}_F'(p_n^*)\}.s_n(p_0) + o_p(1), \end{aligned} \quad \ldots (4.15)$$

where we have used (4.3) for the second equality and Lemma 3 for the third one. We want to write D_{1n} in terms of E_n to put it in the same terms as D_{2n} in (4.14). For this we need the following lemma.

LEMMA 4. *For* $F \in \mathcal{F}^*$, $s_n(p_0) = \mu^{-1}\{-(1 - p_0)h(p_0)E_n(Q(p_0))$

$$- \int_0^{p_0} E_n(Q(u))h(u)du + p_0 \int_0^1 E_n(Q(u))h(u)du\Big\} + o_p(1).$$

Now define the random vector

$$\mathbf{U}_n = \left[E_n(Q(p_0)), \int_0^{p_0} E_n(Q(u))h(u)du, \int_0^1 E_n(Q(u))h(u)du\right]^T.$$

Then from (4.13)-(4.15) and lemma 4 it follows that

$$n^{\frac{1}{2}}\{\hat{t}_{0n} - t_0\} + o_p(1) = (1\ 1)\begin{bmatrix} D_{1n} \\ D_{2n} \end{bmatrix} = (1,\ 1)\Gamma(\mathbf{U}_n).$$

where $\Gamma : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\Gamma(t_1, t_2, t_3) = \begin{bmatrix} \left\{\frac{-h(p_0)}{\bar{\phi}_F'(p_0)\mu}\right\}\{-(1 - p_0)h(p_0)t_1 - t_2 + p_0 t_3\} \\ -h(p_0)t_1 \end{bmatrix}. \quad \ldots (4.16)$$

Further,

$$\mathbf{U}_n \xrightarrow{\mathcal{L}} \mathbf{U} =: \left[W^o(p_0), \int_0^{p_0} W^o(u)h(u)du, \int_0^1 W^o(u)h(u)du \right],$$

where W^o denotes a Brownian bridge process on $[0,1]$. Of course, $\mathbf{U}$ is multivariate normal with zero mean, and the covariance matrix may be verified to be Σ as stated in the theorem. *Q.E.D.*

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