Three-dimensional noncommutative bosonization

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Abstract

We consider the extension of the $(2+1)$-dimensional bosonization process in noncommutative (NC) spacetime. We show that the large mass limit of the effective action obtained by integrating out the fermionic fields in NC spacetime leads to the NC Chern–Simons action. The present result is valid to all orders in the noncommutative parameter θ. We also discuss how the NC Yang–Mills action is induced in the next to leading order.

1. Introduction

Field theories in odd $(2+1)$-dimensional spacetime [1,2] provide interesting features that are linked to the presence of the Chern–Simons three form. It is known that even if one does not include such a term from the beginning, it gets induced as a result of quantum (one loop) effects [2,3]. This, in essence, is the phenomenon of bosonization in $(2+1)$ dimensions [4,5] since the Chern–Simons term appears in the effective theory as one computes the fermion determinant in large mass limit. This last point differentiates between bosonization in $(1+1)$ dimensions [6] where the fermion determinant is exactly solvable and its $(2+1)$-dimensional counterpart, that can only be determined locally as a power series in inverse fermion mass.

In this Letter we discuss the issue of bosonization where the underlying fermion model lives in a noncommutative (NC) spacetime, with the coordinates obeying

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}. \tag{1}$$

The noncommutativity parameter $\theta_{\mu\nu}$ is a constant antisymmetric object. In recent years NC quantum field theories have captured the interest of theoretical physics community ever since their presence was established [7] in certain low energy limits of open string theory in background field. In particular NC theories in $(2+1)$ dimensions bear a special interest since if one is only restricted to spatial NC with $\theta_{0i} = 0$ (as is generally the case), the minimal spatial dimensionality has to be two. In fact the prototype of NC theories is the Landau problem which deals with planar motion of charged particles in a strong magnetic field.

A few years back a number of articles appeared [8–10] that attempted to extend the well-known duality, either in the Lagrangian [11] or Hamiltonian [12] formulations, between Maxwell–Chern–Simons theory and Self-Dual theory to their respective NC generalizations. The issue was not conclusively settled mainly for two reasons: firstly, computation of the NC fermion determinant was not considered and the duality was studied between the NC extended Maxwell–Chern–Simons theory and Self-Dual theory. Secondly, for an explicit comparison one had to exploit the Seiberg–Witten map [7] that connects NC fields to their normal counterpart in the usual gauge theory. This makes the previous analysis [8–10] perturbative in nature and only $O(\theta)$ effects were taken into account since at higher orders the Seiberg–Witten map itself is not unique.

In this perspective the present work becomes significant since we have been able to rectify both the above mentioned drawbacks in a single stroke. We have computed the $(2+1)$-dimensional NC fermion determinant that is valid to all orders in the NC parameter θ. Obviously our NC extended result is also valid in the long wavelength (large fermion mass) limit. We have followed the idea proposed in [13] where NC fermion effective actions in $(3+1)$ dimensions were considered. It was demonstrated in [13] that it is in fact enough to consider the existence of an exact Seiberg–Witten map, valid to all orders in θ, in a formal way and the explicit form of the map is not required. As far as calculating the fermion effective action is considered, the above procedure captures the NC effects in a non-perturbative way. This makes the present analysis, i.e., NC bosonization, exact to all orders in θ.

To be more specific, we compute to all orders in θ, the one loop effective action in the large mass limit, that is obtained by integrating out the fermionic matter sector in the NC gauge theory. Due to the peculiarities of $(2+1)$ dimensions, the large mass limit of the effective action turns out to be finite. In the leading order, $\mathcal{O}(m^0)$, this is precisely the NC Chern–Simons action. We also discuss, how in the next to leading $\mathcal{O}(m^{-1})$ order, the NC Yang–Mills term can be obtained.

Our analysis shows that, in spite of the complexities of NC spacetime, the one loop effective action (in the large mass limit) yields exactly the NC extension of the commutative space result. It might be recalled that the latter was also obtained in the large mass limit [4,5].

2. The effective action

In NC spacetime, the action for the fermionic fields, in the fundamental representation, in the presence of an external non-Abelian gauge potential, is given by

$$S=\int d^3x\,\big[\bar{\Psi}\star i\gamma^\mu D_\mu\Psi - m\bar{\Psi}\star\Psi\big], \tag{2}$$

where $(D_\mu)_{IJ}=\delta_{IJ}\partial_\mu - i(A_\mu)_{IJ}\star$ and the star product is defined as,

$$(A\star B)(x)=\lim_{y\to x} e^{\frac{i}{2}\theta^{\alpha\beta}\partial^x_\alpha\partial^y_\beta}A(x)B(y). \tag{3}$$

The one-loop effective action is defined as

$$i\Gamma[A]=\ln\frac{\det[\partial\!\!\!/ + im - iA\!\!\!/\star]}{\det[\partial\!\!\!/ + im]} = -\sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{Tr}\big[(\partial\!\!\!/ + im)^{-1}iA\!\!\!/\star\big]^n. \tag{4}$$

The operator $(\partial\!\!\!/ + im)^{-1}$ gives the propagator for the fermions

$$\langle y|(\partial\!\!\!/ + im)^{-1}|x\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{i(p\!\!\!/ + m)}{p^2-m^2+i\epsilon}e^{-ip(y-x)}, \tag{5}$$

which is identical to the commutative space expression. Hence,

$$i\Gamma[A]\equiv\sum_{n=1}^{\infty}\int d^3x_1\cdots\int d^3x_n\,\mathrm{Tr}\big[A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\big]\Gamma^{\mu_1\cdots\mu_n}[x_1,\ldots,x_n], \tag{6}$$

where,

$$\Gamma^{\mu_1\cdots\mu_n}[x_1,\ldots,x_n]=\frac{(-1)^{n+1}}{n}\int\prod_{i=1}^{n}\frac{d^3p_i}{(2\pi)^3}(2\pi)^3\delta\Big(\sum_i p_i\Big)e^{i\sum_{i=1}^n p_ix_i}e^{-i/2\sum_{i<j}^n p_i\times p_j}$$
$$\times\int\frac{d^3q}{(2\pi)^3}\frac{\mathrm{tr}[(q\!\!\!/ + p\!\!\!/_1+m)\gamma^{\mu_1}(q\!\!\!/ + m)\gamma^{\mu_2}(q\!\!\!/ - p\!\!\!/_2+m)\gamma^{\mu_3}\cdots(q\!\!\!/ - \sum_{i=2}^{n-1}p\!\!\!/_i+m)\gamma^{\mu_n}]}{[(q+p_1)^2-m^2][q^2-m^2][(q-p_2)^2-m^2]\cdots[(q-\sum_{i=2}^{n-1}p_i)^2-m^2]}. \tag{7}$$

Note that the effect of the star product in (4) is manifested in (7) through the exponential phase factor with,

$$p_i\times p_j=\theta^{\alpha\beta}p_{(i)\alpha}p_{(j)\beta}. \tag{8}$$

The consistency of this formalism in NC spacetime and the fact that there is no UV/IR mixing can be justified by exploiting the Seiberg–Witten map that connects the NC field variables to their commutative counterpart and vice versa [13]. As we have emphasized before, only a formal definition of the exact Seiberg–Witten map is needed in this process.

The above two Eqs. (6), (7) form the basis of our calculations. Furthermore, for the sake of notational simplicity, we have not written down the $i\epsilon$ terms in the propagators and it is to be understood that they exist.

We now explicitly compute the effective action in the long wavelength (i.e., large m) limit. This is done to the leading $\mathcal{O}(m^0)$ and next to leading $\mathcal{O}(m^{-1})$ orders. In the leading order there are only two contributions that arise from the two-A and three-A terms in (6). The other pieces drop out due to the peculiarities of three-dimensional spacetime and the trace properties of gamma matrices. In the next to leading order $\mathcal{O}(m^{-1})$, apart from the contributions coming from the two-A and three-A terms, there is also a piece from the four-A term. Eq. (6) is now explicitly evaluated order by order in the background fields. The term with one gauge field vanishes because of the tracelessness of the group matrices. Next, the effect of the two-A term is considered.

3. Contribution from two-A terms

First, we compute the odd parity contribution. The relevant expression is obtained from (6),

$$i\Gamma[AA] = \int d^3x_1\, d^3x_2\, \mathrm{Tr}\big[A_{\mu_1}(x_1)A_{\mu_2}(x_2)\big]\Gamma^{\mu_1\mu_2}[x_1,x_2], \tag{9}$$

with

$$\Gamma^{\mu_1\mu_2}[x_1,x_2] = -\frac{1}{2}\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}(2\pi)^3\delta(p_1+p_2)e^{i(p_1\cdot x_1+p_2\cdot x_2)}e^{-i(p_1\times p_2)/2}I^{\mu_1\mu_2}, \tag{10}$$

where,

$$I^{\mu_1\mu_2} = \int \frac{d^3q}{(2\pi)^3}\frac{\mathrm{tr}[(\not{q}+\not{p}_1+m)\gamma^{\mu_1}(\not{q}+m)\gamma^{\mu_2}]}{[(q+p_1)^2-m^2][q^2-m^2]}. \tag{11}$$

In what follows we will concentrate on the q integral and its explicit evaluation. The trace can be performed using the identities provided in Appendix A. We thus get

$$I^{\mu_1\mu_2} = 2\int \frac{d^3q}{(2\pi)^3}\frac{[2q^{\mu_1}q^{\mu_2}+p_1^{\mu_1}q^{\mu_2}+q^{\mu_1}p_1^{\mu_2}+g^{\mu_1\mu_2}(m^2-q\cdot p_1-q^2)+mp_{1\alpha}\epsilon^{\alpha\mu_1\mu_2}]}{[(q+p_1)^2-m^2][q^2-m^2]}. \tag{12}$$

Here we take the leading term in the large mass limit. In this limit the $g^{\mu_1\mu_2}$ term and the last term contribute. Using the integral Eq. (B.1) in Appendix B, the odd parity contribution is given by,

$$I^{\mu_1\mu_2} = \frac{m}{|m|}\frac{ip_{1\alpha}\epsilon^{\mu_1\mu_2\alpha}}{4\pi}. \tag{13}$$

Substituting Eqs. (13) and (10) in Eq. (9) we get

$$i\Gamma[AA] = -\frac{m}{|m|}\frac{\epsilon^{\mu_1\mu_2\mu_3}}{8\pi}\int d^3x_1\, d^3x_2\, \mathrm{Tr}\big[A_{\mu_1}(x_1)A_{\mu_2}(x_2)\big]\partial^{x_1}_{\mu_3}\delta(x_1-x_2). \tag{14}$$

In deriving the above result we have performed both the p integrations in (10) that has yielded the delta function. Performing an integration by parts and neglecting the surface terms we get the final form,

$$i\Gamma[AA] = +\frac{m}{|m|}\frac{\epsilon^{\mu\nu\lambda}}{8\pi}\int d^3x\, \mathrm{Tr}\big[A_\mu(x)\partial_\nu A_\lambda(x)\big]. \tag{15}$$

It should be mentioned that the effect of the phase factor in (7) trivializes so that no star product occurs in (15). This is expected because of the special property

$$\int d^3x\,(A\star B)(x) = \int d^3x\,(AB)(x). \tag{16}$$

Observe that (15) is precisely the two-A part of the NC Chern–Simons action, which is of $\mathcal{O}(m^0)$.

We now consider the normal parity part in (12). This is essentially the next to leading order $\mathcal{O}(m^{-1})$ contribution.

Using the integrals given in Appendix B, one obtains,

$$I^{\mu_1\mu_2} = -\frac{i}{16\pi}\frac{1}{|m|}\big[p_1^{\mu_1}p_1^{\mu_2}-m^2g^{\mu_1\mu_2}\big].$$

Putting the above result in Eq. (10), we obtain,

$$\Gamma^{\mu_1\mu_2}[x_1,x_2] = +\frac{i}{32\pi}\frac{1}{|m|}\int \frac{d^3p_1}{(2\pi)^3}e^{ip_1\cdot(x_1-x_2)}\big[p_1^{\mu_1}p_1^{\mu_2}-m^2g^{\mu_1\mu_2}\big], \tag{17}$$

which in turn is utilized in Eq. (9) to yield,

$$i\Gamma[AA] = \frac{i}{32\pi}\frac{1}{|m|}\int d^3x_1\, d^3x_2\, \mathrm{Tr}\big[A_{\mu_1}(x_1)A_{\mu_2}(x_2)\big]\int \frac{d^3p_1}{(2\pi)^3}e^{ip_1\cdot(x_1-x_2)}\big[p_1^{\mu_1}p_1^{\mu_2}-m^2g^{\mu_1\mu_2}\big]. \tag{18}$$

Writing the momenta as derivatives, performing the p_1 integral to yield a delta function, and finally doing an integration by parts, the effective action simplifies to,

$$i\Gamma[AA] = -\frac{i}{32\pi}\frac{1}{|m|}\int d^3x_1\,d^3x_2\,\mathrm{Tr}\big[\big(\partial_{x_1}^{\mu_1}A_{\mu_1}(x_1)\big)\big(\partial_{x_2}^{\mu_1}A_{\mu_2}(x_2)\big) - g^{\mu_1\mu_2}\big(\partial_{x_1}^{\alpha}A_{\mu_1}(x_1)\big)\big(\partial_{\alpha}^{x_2}A_{\mu_2}(x_2)\big)\big]\delta(x_1-x_2) \tag{19}$$

$$= -\frac{i}{32\pi}\frac{1}{|m|}\int d^3x\,\mathrm{Tr}\big[\big(\partial^{\mu}A_{\mu}\big)\big(\partial^{\nu}A_{\nu}\big) - g^{\mu\nu}\big(\partial^{\alpha}A_{\mu}\big)\big(\partial^{\alpha}A_{\nu}\big)\big]$$

$$= \frac{i}{64\pi}\frac{1}{|m|}\int d^3x\,\mathrm{Tr}\big[2\big(\partial^{\mu}A_{\nu}\big)\big(\partial_{\mu}A^{\nu}\big) - 2\big(\partial^{\mu}A_{\nu}\big)\big(\partial^{\nu}A_{\mu}\big)\big]. \tag{20}$$

The above is the two-A term of the NC Yang–Mills action.

4. Contribution from three-A terms

The contribution of the three-A term in the effective action is obtained from (6),

$$i\Gamma[AAA] = \int d^3x_1\,d^3x_2\,d^3x_3\,\mathrm{Tr}\big[A_{\mu_1}(x_1)A_{\mu_2}(x_2)A_{\mu_3}(x_3)\big]\Gamma^{\mu_1\mu_2\mu_3}[x_1,x_2,x_3], \tag{21}$$

where

$$\Gamma^{\mu_1\mu_2\mu_3}[x_1,x_2,x_3] = \frac{1}{3}\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_3}{(2\pi)^3}(2\pi)^3\delta(p_1+p_2+p_3)$$
$$\times\, e^{i(p_1\cdot x_1+p_2\cdot x_2+p_3\cdot x_3)}e^{-i(p_1\times p_2+p_1\times p_3+p_2\times p_3)/2}I^{\mu_1\mu_2\mu_3}. \tag{22}$$

In the present case

$$I^{\mu_1\mu_2\mu_3} = \int\frac{d^3q}{(2\pi)^3}\frac{\mathrm{tr}[(\not q+\not p_1+m)\gamma^{\mu_1}(\not q+m)\gamma^{\mu_2}(\not q-\not p_2+m)\gamma^{\mu_3}]}{[(q+p_1)^2-m^2][q^2-m^2][(q-p_2)^2-m^2]}. \tag{23}$$

As before, we first compute the odd parity terms. These are basically the leading order $\mathcal{O}(m^0)$ pieces. The integrand is simplified as,

$$2im\big[\epsilon^{\alpha\mu_1\mu_3}\big(q^{\mu_2}q_\alpha + q^{\mu_2}p_{1\alpha} - p_2^{\mu_2}p_{1\alpha}\big) + \epsilon^{\alpha\mu_2\mu_3}\big(q^{\mu_1}p_{2\alpha} + p_1^{\mu_1}p_{2\alpha} - q^{\mu_1}q_\alpha\big)$$
$$- \epsilon^{\alpha\mu_1\mu_2}\big(p_2^{\mu_3}p_{1\alpha} + q^{\mu_3}q_\alpha\big) + m^2\epsilon^{\mu_1\mu_2\mu_3}\big]. \tag{24}$$

The terms we are interested in are the ones independent of the p's and the last term of the above expression. Using (24) and the integrals provided in Eqs. (B.2) and (B.3), we obtain from (23),

$$I^{\mu\nu\lambda} = +\frac{m}{|m|}\frac{\epsilon^{\mu\nu\lambda}}{4\pi}. \tag{25}$$

Substituting Eqs. (25) and (22) in Eq. (21), and using the definition (3) of the star product on a chain of fields,

$$\int dx\,\big[\mathcal{O}_1(x)\cdots\star\mathcal{O}_n(x)\big] = \int\left[\prod_{i=1}^{n}dx_i\right]\left[\prod_{j=1}^{n}\frac{dk_j}{(2\pi)^D}\right]\exp\big(ik_i^{\mu}\cdot x_\mu^i\big)\exp\left[-\frac{i}{2}\sum_{i<j=1}^{n}k_\mu^i k_\nu^j\Theta^{\mu\nu}\right]$$
$$\times\,(2\pi)^D\delta(k_1+k_2+\cdots+k_n). \tag{26}$$

we obtain the final result,

$$i\Gamma[AAA] = +\frac{m}{|m|}\frac{2}{3}\frac{\epsilon^{\mu\nu\lambda}}{8\pi}\int d^3x\,\mathrm{Tr}\big[A_\mu(x)\star A_\nu(x)\star A_\lambda(x)\big], \tag{27}$$

where the star product now appears explicitly. Combining Eqs. (15) and (27) we get the complete NC Chern–Simons action.

In a similar way it is possible to compute the normal parity contribution associated with the three-A term. In the order $\mathcal{O}(m^{-1})$ expansion this yields the three-A piece in the NC Yang–Mills action. Furthermore, there is an analogous $\mathcal{O}(m^{-1})$ contribution from the four-A term in (6). Taking the two-A piece explicitly computed in (20), one obtains the complete structure of the NC Yang–Mills action.

5. Discussions

In the present work we have studied the noncommutative extension of the bosonization phenomenon in $(2+1)$ dimensions. We have derived the noncommutative (NC) Chern–Simons action by integrating out the fermionic matter in the one loop effective action in the long wavelength limit. This result is valid to all orders in the NC parameter θ. Moreover the limit $\theta \to 0$ is smooth so that the NC expressions reduce to their commutative versions and we recover the bosonization in commutative spacetime.

Finally we return to the ambiguity [8–10] surrounding the issue of duality in noncommutative Maxwell–Chern–Simons theory and Self-Dual theory and point out that the analysis of [8] holds as far as the duality of the noncommutative versions of the above two bosonic theories are concerned. However, if one wants to tie up the bosonization of the noncommutative fermion theory as well, (as is true in the commutative spacetime), the present work shows that one needs to consider the noncommutative Chern–Simons theory [9,10].

Appendix A. Some trace identities

We consider a Hermitian representation of the 2×2 gamma matrices satisfying $\gamma^\mu\gamma^\nu = g^{\mu\nu} + i\epsilon^{\mu\nu\sigma}\gamma^\sigma$. This implies the following trace identities in $(2+1)$ dimensions:

$$
\begin{aligned}
&\mathrm{tr}(\gamma^\mu\gamma^\nu) = 2g^{\mu\nu},\\
&\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma) = 2i\epsilon^{\mu\nu\sigma},\\
&\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda) = 2(g^{\mu\nu}g^{\sigma\lambda} + g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\lambda}),\\
&\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda\gamma^\alpha) = 2i(g^{\mu\nu}\epsilon^{\sigma\lambda\alpha} + g^{\lambda\alpha}\epsilon^{\mu\nu\sigma} + g^{\sigma\lambda}\epsilon^{\mu\nu\alpha} - g^{\sigma\alpha}\epsilon^{\mu\nu\lambda}),\\
&\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda\gamma^\alpha\gamma^\beta) = 2(g^{\mu\nu}g^{\sigma\lambda}g^{\alpha\beta} + g^{\mu\nu}g^{\sigma\beta}g^{\lambda\alpha} - g^{\mu\nu}g^{\sigma\alpha}g^{\lambda\beta} - g^{\alpha\beta}g^{\mu\sigma}g^{\nu\lambda} + g^{\alpha\beta}g^{\mu\lambda}g^{\nu\sigma} - g^{\lambda\alpha}g^{\mu\sigma}g^{\nu\beta}\\
&\qquad + g^{\lambda\alpha}g^{\mu\beta}g^{\nu\sigma} + g^{\lambda\beta}g^{\mu\sigma}g^{\nu\alpha} - g^{\lambda\beta}g^{\mu\alpha}g^{\nu\sigma} - g^{\mu\lambda}g^{\nu\alpha}g^{\sigma\beta} - g^{\mu\alpha}g^{\nu\beta}g^{\sigma\lambda} - g^{\mu\beta}g^{\nu\lambda}g^{\sigma\alpha}\\
&\qquad + g^{\mu\lambda}g^{\nu\beta}g^{\sigma\alpha} + g^{\mu\beta}g^{\nu\alpha}g^{\sigma\lambda} + g^{\mu\alpha}g^{\nu\lambda}g^{\sigma\beta}),\\
&\mathrm{tr}(\gamma^\rho\gamma^\tau\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\lambda\gamma^\alpha) = 2i(g^{\rho\tau}g^{\mu\nu}\epsilon^{\sigma\lambda\alpha} + g^{\rho\tau}g^{\lambda\alpha}\epsilon^{\mu\nu\sigma} + g^{\rho\tau}g^{\sigma\lambda}\epsilon^{\mu\nu\alpha} - g^{\rho\tau}g^{\sigma\alpha}\epsilon^{\mu\nu\lambda} + g^{\mu\nu}g^{\sigma\lambda}\epsilon^{\rho\tau\alpha} + g^{\mu\nu}g^{\lambda\alpha}\epsilon^{\rho\tau\sigma}\\
&\qquad - g^{\mu\nu}g^{\sigma\alpha}\epsilon^{\rho\tau\lambda} - g^{\mu\sigma}g^{\nu\lambda}\epsilon^{\rho\tau\alpha} + g^{\mu\lambda}g^{\nu\sigma}\epsilon^{\rho\tau\alpha} - g^{\lambda\alpha}g^{\mu\sigma}\epsilon^{\rho\tau\nu} + g^{\lambda\alpha}g^{\nu\sigma}\epsilon^{\rho\tau\mu} + g^{\mu\sigma}g^{\nu\alpha}\epsilon^{\rho\tau\lambda}\\
&\qquad - g^{\mu\alpha}g^{\nu\sigma}\epsilon^{\rho\tau\lambda} - g^{\mu\lambda}g^{\nu\alpha}\epsilon^{\rho\tau\sigma} - g^{\mu\alpha}g^{\sigma\lambda}\epsilon^{\rho\tau\nu} - g^{\nu\lambda}g^{\sigma\alpha}\epsilon^{\rho\tau\mu}\\
&\qquad + g^{\mu\lambda}g^{\sigma\alpha}\epsilon^{\rho\tau\nu} + g^{\nu\alpha}g^{\sigma\lambda}\epsilon^{\rho\tau\mu} + g^{\mu\alpha}g^{\nu\lambda}\epsilon^{\rho\tau\sigma}).
\end{aligned} \tag{A.1}
$$

Appendix B. Some important integrals

We also need the following forms of the integrals [14]. Note that the results presented below hold in the large mass limit, i.e., $m \to \infty$

$$\int \frac{d^3q}{(2\pi)^3} \frac{1}{[(q+p_1)^2 - m^2 + i\epsilon][q^2 - m^2 + i\epsilon]} = \frac{i}{8\pi m}, \tag{B.1}$$

$$\int \frac{d^3q}{(2\pi)^3} \frac{1}{[(q+p_1)^2 - m^2 + i\epsilon][q^2 - m^2 + i\epsilon][(q-p_2)^2 - m^2 + i\epsilon]} = -\frac{1}{4}\left(\frac{i}{8\pi m^3}\right), \tag{B.2}$$

$$\int \frac{d^3q}{(2\pi)^3} \frac{q^2}{[(q+p_1)^2 - m^2 + i\epsilon][q^2 - m^2 + i\epsilon][(q-p_2)^2 - m^2 + i\epsilon]} = \frac{3}{4}\left(\frac{i}{8\pi m}\right). \tag{B.3}$$

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