Causality, joint measurement and Tsirelson's bound

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Abstract

Tsirelson showed that $2\sqrt{2}$ is the maximum value that CHSH expression can take for quantum correlations [B.S. Tsirelson, Lett. Math. Phys. 4 (1980) 93]. This bound simply follows from the algebra of observables. Recently by exploiting the physical structure of quantum mechanics like unitarity and linearity, Buhrman and Massar [H. Buhrman, S. Massar, Phys. Rev. A 72 (2005) 052103] have established that violation of Tsirelson's bound in quantum mechanics will imply signalling. We prove the same with the help of realistic joint measurement in quantum mechanics and a Bell's inequality which has been derived under the assumption of existence of joint measurement and no signalling condition.

1. Introduction

There exists quantum mechanical states shared between two parties which exhibit nonlocal character. This nonlocality is quantified by using 'Bell's expression'. This is an expression which is bounded by a certain value for 'Local Hidden Variable (LHV) models'; but can exceed this value in case of quantum correlations. Consider for example a setting of two parties, Alice and Bob; sharing a quantum state ρ and each has a choice of two local measurements. Alice can measure the observables A and A' whereas Bob's observables are B and B'. The measured values of all the observables can be 1 or -1. One relevant Bell's expression in this case is the Clauser–Horn–Shimony–Holt (CHSH) expression [1]. For local hidden variable models, this expression is bounded by 2 but in case of entangled quantum systems, this bound can be violated. For example, on the singlet state of two qubits there exist observables (A, A', B, B') for which value of the above expression is $2\sqrt{2}$.

In fact as shown later by Tsirelson [2] that $2\sqrt{2}$ is the maximum quantum value of the CHSH expression. Tsirelson's bound is a simple mathematical consequence of the axioms of quantum theory, but it would be interesting to know that whether there is some deeper reason why a violation greater than $2\sqrt{2}$ is unphysical. It is known in this connection that a violation greater than $\sqrt{\frac{32}{3}} \simeq 3.27$ would imply that any communication complexity problem can be solved using a constant amount of communication [3]. But this does not answer the question that what odd would have happened for a violation just greater than $2\sqrt{2}$.

Recently by exploiting the physical structure of quantum mechanics like unitary dynamics and linearity; Buhrman and Massar [4] have shown that exceeding Tsirelson's bound by quantum mechanics will imply signalling in quantum mechanics. Here we provide a simple proof of the same by exploiting nice results of existence of joint measurement for spin along two different directions in quantum mechanics [5–9] and a Bell's inequality derived under assumptions different than that of the local-realism.

2. Joint measurement, no signalling and Bell's inequality

Usually, Bell's inequality is derived under the notion of local-realism. So, its violation by a theory will imply that the theory is incompatible with the notion of local-realism. For example, quantum mechanics violates it and so we conclude that quantum theory is not local-realistic. Tsirelson observed that for quantum mechanical states and observables, Bell's expression can go maximum up to $2\sqrt{2}$. Here arises an interesting question that what unphysical would have happened if quantum mechanics had violated Tsirelson's bound. One cannot answer this question on the basis of a Bell's inequality derived under the notion of local-realism. Bell's inequality in this context can only tell that quantum mechanics is not local-realistic, it cannot tell more than this.

Recently Andersson et al. [10] have derived Bell's inequality by assuming the existence of joint measurement (not necessarily revealing the pre-existing value) and no signalling condition. This is not a trivial assumption. In case of classical system it is always possible to measure two different observables jointly, but it is not always the case with quantum systems, where there exist noncommuting observables. At the moment, we do not need to think about how to achieve this joint measurement, rather we simply assume that this can be achieved.

In the framework of a general non-signalling probabilistic theory, we consider a physical system consisting of two subsystems shared between Alice and Bob. The two observers (Alice and Bob) have access to one subsystem each. Assume that Bob can measure two observables B or B' on his subsystem and Alice can measure A and A' on hers. The measured values of all the observables can be 1 or -1. We further assume that Alice can measure the observables A and A' jointly. Let us now consider a situation where the system is always prepared in the same state and Alice measures A and A' jointly (we shall use the subscript J to denote the joint measurement) and Bob measures the observable B.

The probability that Alice will obtain the result $A_J = A'_J$ can be written as

$$p(A_J = A'_J; B) = p(A_J = A'_J = B) + p(A_J = A'_J = -B). \tag{1}$$

As these probabilities are non-negative, hence:

$$\begin{aligned} &p(A_J = A'_J = B) + p(A_J = A'_J = -B) \\ &\quad \geqslant \left|p(A_J = A'_J = B) - p(A_J = A'_J = -B)\right|. \end{aligned} \tag{2}$$

Now the term in the right hand side can be written as

$$\begin{aligned} &\left|p(A_J = A'_J = B) - p(A_J = A'_J = -B)\right| \\ &\quad = \frac{1}{2}\left|E(A_J, B) + E(A'_J, B)\right|, \end{aligned} \tag{3}$$

where the correlation function $E(A, B)$ is defined as

$$E(A, B) = p(A = B) - p(A = -B) = \overline{AB}.$$

The above three equations finally give us

$$p(A_J = A'_J; B) \geqslant \frac{1}{2}\left|E(A_j, B) + E(A'_J, B)\right|. \tag{4}$$

Similarly, if we assume that Bob measures for the observable B', we will obtain

$$p(A_J = -A'_J; B') \geqslant \frac{1}{2}\left|E(A_j, B') - E(A'_J, B')\right|. \tag{5}$$

Adding inequalities (4) and (5) we get:

$$\begin{aligned} &p(A_J = A'_J; B) + p(A_J = -A'_J; B') \\ &\quad \geqslant \frac{1}{2}\left[\left|E(A_J, B) + E(A'_J, B)\right| \right. \\ &\qquad \left. + \left|E(A_j, B') - E(A'_J, B')\right|\right]. \end{aligned} \tag{6}$$

Because of the no signalling constraint the probability of Alice getting $A_J = -A'_J$ must be independent of the fact that Bob measured spin along B or B', i.e.

$$p(A_J = -A'_J; B) = p(A_J = -A'_J; B'). \tag{7}$$

Putting for $p(A_J = -A'_J; B')$ from Eq. (7) into inequality (6), we get:

$$\begin{aligned} &p(A_J = A'_J; B) + p(A_J = -A'_J; B) \\ &\quad \geqslant \frac{1}{2}\left[\left|E(A_J, B) + E(A'_J, B)\right| \right. \\ &\qquad \left. + \left|E(A_j, B') - E(A'_J, B')\right|\right]. \end{aligned} \tag{8}$$

Now, noting that, $p(A_J = A'_J; B) + p(A_J = -A'_J; B) = 1$; inequality (8), ultimately reduces to:

$$\left|E(A_J, B) + E(A'_J, B)\right| + \left|E(A_J, B') - E(A'_J, B')\right| \leqslant 2. \tag{9}$$

One should note that the above inequality is an usual Bell's inequality but derived under the assumptions that there exists joint measurement and there can be no superluminal signalling. So, violation of this inequality in a physical theory will imply that some or all of the assumptions used in the derivation of it are inconsistent with that particular theory. For example if joint measurement really exists in a physical theory then violation of this inequality will imply signalling in that physical theory.

It is well known that there are quantum mechanical states which violate this inequality. Now, in this particular context of Bell's inequality, if no-signalling is considered to be a principle, then violation will imply that there can be no joint measurement in quantum mechanics. On the other hand to address the question of signalling in quantum mechanics with the help of Bell's inequality, one will have to consider a situation in quantum mechanics where joint measurement exists. The next two sections deal with this situation.

3. Quantum measurements

Usual quantum measurements are projective measurements which project the initial state of a system to one of the eigenstates of the observables being measured. For example in a measurement for spin along direction $\hat{\alpha}$ the projectors onto the eigenstates are:

$$E(\pm\hat{\alpha}) = \frac{1}{2}[I \pm \hat{\alpha} \,.\, \vec{\sigma}]. \tag{10}$$

But further progress had shown that the most general quantum measurements are positive operator valued measures (POVM). These generalized measurements allow us to describe any measurement that can be performed within the limits of quantum mechanics.

In this more general framework of quantum theory, the states of a quantum system are represented by positive trace class operators. Most general observable is represented by a collection of positive operators $\{E_i\}$ where $0 \leqslant E_i \leqslant I$ for all i and $\sum E_i = I$, I being an unit operator on the Hilbert space. In a measurement for this observable for the state ρ (say), the probability of occurrence of the ith result is given by $\mathrm{Tr}[\rho E_i]$. In the case of spin-1/2 particles, P. Busch [7,8] had first introduced collection of positive operators with the above said properties in a particular form which can be interpreted as unsharp spin observables. This particular unsharp observables are represented in the following form:

$$E_\lambda(\hat{\alpha}) = \frac{1}{2}[I + \lambda\hat{\alpha} \,.\, \vec{\sigma}], \tag{11}$$

where $0 < \lambda \leqslant 1$ and $\hat{\alpha}$ is a unit vector. Here $\vec{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ denotes the usual Pauli spin operator. The spectral decomposition of $E_\lambda(\hat{\alpha})$ is given by

$$E_\lambda(\hat{\alpha}) = \left(\frac{1+\lambda}{2}\right)\frac{1}{2}[I + \hat{\alpha} \,.\, \vec{\sigma}] + \left(\frac{1-\lambda}{2}\right)\frac{1}{2}[I - \hat{\alpha} \,.\, \vec{\sigma}]. \tag{12}$$

Here $\frac{1}{2}[I + \hat{\alpha} \,.\, \vec{\sigma}]$ and $\frac{1}{2}[I - \hat{\alpha} \,.\, \vec{\sigma}]$ are the one dimensional orthogonal spin-projection operators on H^2. From this representation it is clear that the POVM $\{E_\lambda(\hat{\alpha}), E_\lambda(-\hat{\alpha})\}$ is a smeared version of the projective measurement $\{\frac{1}{2}[I + \hat{\alpha} \,.\, \vec{\sigma}], \frac{1}{2}[I - \hat{\alpha} \,.\, \vec{\sigma}]\}$. This is the formal sense in which the former represents unsharp spin measurement in the direction $\hat{\alpha}$. Noteworthy here is that for $\lambda = 1$, it represents the usual sharp (projective) spin measurement along $\hat{\alpha}$. The eigenvalues r and u of $E_\lambda(\hat{\alpha})$ where

$$r = \frac{1}{2}(1+\lambda) > \frac{1}{2}$$

and

$$u = \frac{1}{2}(1-\lambda) < \frac{1}{2}$$

are interpreted respectively as reality degree and the degree of unsharpness of the spin property along $\hat{\alpha}$.

Keeping the above interpretation for unsharp measurement in mind it is easy to show that expectation value of an unsharply measured spin observable with respect to an initial state ρ is proportional to the expectation value of the corresponding spin observable when measured sharply over the same state ρ, the coefficient of proportionality being equal to the unsharp parameter (for example λ in this case), as

$$\begin{aligned}\mathrm{Tr}\big[\rho(\hat{\alpha} \,.\, \vec{\sigma})_U\big] &= (+1)\,\mathrm{Tr}\big[\rho E_\lambda(\hat{\alpha})\big] + (-1)\,\mathrm{Tr}\big[\rho E_\lambda(-\hat{\alpha})\big] \\ &= \lambda\,\mathrm{Tr}\big[\rho\hat{\alpha} \,.\, \vec{\sigma}\big].\end{aligned} \tag{13}$$

4. Existence of joint measurement in quantum mechanics

Projective measurements are too restrictive. In the framework of projective measurements, there are observables which cannot be measured jointly. This distinguishing feature of quantum mechanics is popularly known as complementarity. Examples of complementary observables are position and momentum observables, spin observables in different directions, etc. But in the more general framework, it has been shown that certain complementary observables (in standard measurement) can be measured jointly if they are represented by a particular form of POVM (having an interpretation in terms of unsharpness) instead of being represented by projection operators [5,6].

Joint measurement of spin observables in different directions has been extensively studied by P. Busch [7]. He, by exploiting the necessary and sufficient condition for co-existence of two *effects* as given by Kraus [5], showed that a pair of unsharp spin properties $E_{\lambda_1}(\hat{\alpha}_1)$ and $E_{\lambda_2}(\hat{\alpha}_2)$ are co-existent (i.e. can be jointly measured) if and only if:

$$\big|(\lambda_1\hat{\alpha}_1 + \lambda_2\hat{\alpha}_2)\big| + \big|(\lambda_1\hat{\alpha}_1 - \lambda_2\hat{\alpha}_2)\big| \leqslant 2. \tag{14}$$

For $\lambda_1 = \lambda_2 = \lambda$, i.e. for equal unsharpness for both the spin properties, the condition reduces to:

$$\lambda\big[|\hat{\alpha}_1 + \hat{\alpha}_2| + |\hat{\alpha}_1 - \hat{\alpha}_2|\big] \leqslant 2. \tag{15}$$

The term in brackets has maximum value $2\sqrt{2}$. Hence the co-existence condition is satisfied for all pairs of directions $\hat{\alpha}_1$ and $\hat{\alpha}_2$ if and only if $\lambda \leqslant \frac{1}{\sqrt{2}}$.

5. Violation of Tsirelson bound in quantum mechanics implies violation of causality

Now we consider a situation where the system consists of two, two level quantum systems in a state ρ (say). Out of the two observers Alice and Bob, Alice, on her subsystem, measures for the unsharp spin observables A_U or A'_U (whose joint measurement is possible in quantum mechanics) where

$$A_U = \frac{1}{2}[I + \lambda\hat{a} \,.\, \vec{\sigma}]$$

and

$$A'_U = \frac{1}{2}[I + \lambda\hat{a}' \,.\, \vec{\sigma}].$$

We will denote the sharp counterparts of these observables by A and A' respectively.

Bob on his subsystem measures either

$$B = \frac{1}{2}[I + \hat{b} \,.\, \vec{\sigma}]$$

or

$$B' = \frac{1}{2}[I + \hat{b}' \,.\, \vec{\sigma}].$$

For these observables inequality (9) will read as

$$\big|E(A_U, B) + E\big(A'_U, B\big)\big| + \big|E(A_U, B') - E\big(A'_U, B'\big)\big| \leqslant 2, \tag{16}$$

here $E(A_U, B)$ stands for $\mathrm{Tr}(\rho A_U B)$, $E(A'_U, B)$ for $\mathrm{Tr}(\rho A'_U B)$ and so on.

Now from Eq. (13) as $\mathrm{Tr}(\rho A_U B) = \lambda\mathrm{Tr}(\rho AB)$, hence we can write $E(A_U, B) = \lambda E(A, B)$ where $E(A, B) = \mathrm{Tr}(\rho AB)$. Similarly $E(A'_U, B) = \lambda E(A', B)$ and so on. It is noteworthy here that $E(A, B)$, $E(A', B)$, etc., denote the usual quantum mechanical expectations.

With the help of above analysis, inequality (16) can be rewritten as

$$\lambda\left[\left|E(A,B)+E(A',B)\right|+\left|E(A,B')-E(A',B')\right|\right]\leqslant 2. \quad (17)$$

As we have seen in the previous discussion that value of λ can go maximum up to $\frac{1}{\sqrt{2}}$ in order to make joint measurement of spin along any two different directions possible within quantum mechanics. Hence, for no violation of the 'no signalling condition' the term in the parentheses of inequality (17) should be either less than or equal to $2\sqrt{2}$; i.e. there will be no superluminal signalling in quantum mechanics as long as

$$\left[\left|E(A,B)+E(A',B)\right|+\left|E(A,B')-E(A',B')\right|\right]\leqslant 2\sqrt{2}, \quad (18)$$

i.e. as long as quantum correlations satisfy Tsirelson's bound.

6. Discussion

In the present work, we have shown that violation of Tsirelson's bound in quantum mechanics will result in signalling. This we have shown with the help of (a) POVM formalism of quantum mechanics which limits to what extent one can simultaneously measure two noncommuting observables in quantum mechanics and (b) an inequality due to Anderson et al. [10] derived under the assumptions of existence of joint measurement and nonexistence of superluminal signalling in a physical theory.

Fortunately, the bound on correlation function under this newer (than the original local-realistic) assumptions and the bound on correlation function under the assumption of local-realism, come out to be same; i.e. both of these assumptions lead to the same inequality (the Bell's inequality).[1]

This new derivation of Bell's inequality can be exploited to search out whether a theory permits signalling or not, for it's violation in a theory will imply that either there can be no joint measurement in that theory or if it (joint measurement) exists, then the theory is signalling.

The generalized formalism of quantum mechanics allows joint measurement of certain unsharp observables (not necessarily revealing the pre-existing value) provided the degree of sharpness is sufficiently small. If such cases where joint measurement exists are considered in quantum mechanics then violation of Bell's inequality will imply signalling in quantum mechanics. As its consequence, we have found that violation of Tsirelson's bound by quantum mechanical correlation functions will result in signalling in quantum mechanics.

Generalized observable in quantum mechanics, i.e. POVM formalism of observable captures features of quantum mechanics in a more comprehensive way. In this context it would be worth mentioning that Bell could construct a Hidden Variable Theory for two dimensional quantum system by using standard observables but it has been shown recently that if one uses formalism of generalized observables (i.e. the POVM formalism), then even for two dimensional quantum system, Gleason's theorem as well as Kochen–Specker theorem hold true [12,13]. Furthermore, this formalism creates the possibility of certain joint measurements of complementary observables like position and momentum; spin along two different directions, etc. In particular, joint measurement of spin along different directions are possible if standard (sharp) measurements are replaced by their unsharp counterparts. In our case we have used this feature of POVM formalism and it, together with a new derivation of Bell's inequality has manifested its power by answering an important question that the CHSH expression should be bounded by $2\sqrt{2}$ for quantum systems to avoid superluminal signalling in quantum mechanics.

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[1] The original assumption of local-realism and this newer assumption of joint measurement together with no signaling may seem identical prima facie, but it is not the case as the later is silent over the issue of measurement revealing pre-existing values. But interestingly these two different types of assumptions lead to the same bound on correlation function for the case considered in the Letter. However, there may be cases where the condition for joint measurement to exist (together with the no signalling constraint) may not lead to the local-realistic bound on correlation functions because this assumption is not equivalent to the assumption of local-realism. Example of this sort is Gisin's Bell inequality for three coplaner measurement directions where the inequivalency of these two types of assumptions can be seen (vide [11]).