Local cloning of genuinely entangled states of three qubits

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We discuss the (im)possibility of the exact cloning of orthogonal but genuinely entangled three qubit states aided with entangled ancilla under local operation and classical communication. Whereas any two orthogonal Greenberger-Horne-Zeilinger (GHZ) states taken from the canonical GHZ basis can be cloned with the help of a known GHZ state, surprisingly we find that no two W states can be cloned by using any known three qubit (possibly entangled) state as a blank copy.

I. INTRODUCTION

Manipulation of multiparty entanglement by local operation and classical communication (LOCC) is an open area in quantum information. Distinguishability of orthogonal entangled states by LOCC and their cloning under LOCC with the help of appropriate entanglement are two important and closely connected areas in this field. Some interesting results have been obtained in the case of LOCC discrimination of orthogonal entangled states [1–7]. The concept of entanglement cloning under LOCC aided with entanglement, henceforth will be called "local cloning," was first introduced by Ghosh *et al.* [8]. Since then much work has been done in this direction [9–11]. These works are important not only due to the fact that they are helpful in understanding the nonlocality of a set [12] but also because of the fact that local cloning is very closely connected with many important information processing tasks, such as channel copying, entanglement distillation, error correction, and quantum key distribution [11]. However, most of these works deal mainly with maximally entangled states of two qubits. Recently, Choudhary *et al.* [13] have discussed the impossibility of local cloning of arbitrary entangled states shared between two parties. They, by entanglement considerations, have obtained the necessary amount of entanglement in the blank copy for exact local cloning of two orthogonal nonmaximally entangled bipartite states. However, a common scenario in quantum information processing is where a multipartite entangled state is distributed among a number of spatially separated parties. Each of these parties are able to perform only local operations on the subsystem they possess and can send only classical information to each other. So exploring the possibility of local cloning of multipartite entangled states is in fact of greater interest.

Although some results are known for local discrimination of a set of orthogonal multiparty entangled state, no result is known for local copying. In this work we concentrate on

three qubit pure states. W and Greenberger-Horne-Zeilinger (GHZ) states are the two extreme representatives of the inequivalent kinds of genuine three qubit entangled states [14]. Our result shows that whereas any two GHZ states from the canonical set of eight orthogonal GHZ states can be cloned locally with the help of a GHZ state as the blank copy, no two W states, taken from the complete set of orthogonal W states, can be cloned with the help of any three-qubit entangled state. We also find the condition under which a set of three orthogonal GHZ states cannot be cloned with the help of any three-qubit entangled state.

II. CLONING OF GHZ STATES

The full orthogonal canonical set of tripartite GHZ states can be written as (up to a global phase):

$$|\Psi_{p,i,j}\rangle_{ABC} = \frac{1}{\sqrt{2}}[|0ij\rangle + (-1)^p|1\bar{i}\bar{j}\rangle], \quad (1)$$

where $p,i,j=0,1$ and a bar over a bit value indicates its logical negation.

Consider any pair from Eq. (1). Let one state of this pair is shared among three parties; Alice, Bob, and Charlie. They share another known GHZ state as blank copy. As illustrated below, Controlled-NOT (CNOT) operations $(C|i\rangle|j\rangle=|i\rangle|(j+i)\mathrm{mod}\,2\rangle)$ by each of the parties will make cloning possible.

By appropriate basis transformation any pair of the above states (GHZ) can be written either as

$$(\mathrm{I})\ |\Psi_{000}\rangle_{ABC} = \frac{1}{\sqrt{2}}[|000\rangle + |111\rangle],$$

$$|\Psi_{010}\rangle_{ABC} = \frac{1}{\sqrt{2}}[|010\rangle + |101\rangle]$$

or

$$(\mathrm{II})\ |\Psi_{000}\rangle_{ABC} = \frac{1}{\sqrt{2}}[|000\rangle + |111\rangle],$$

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$$|\Psi_{100}\rangle_{ABC} = \frac{1}{\sqrt{2}}[|000\rangle - |111\rangle].$$

Without loss of generality, one can consider the blank state as $|\Psi_{000}\rangle$.

For pair (I); Alice, Bob, and Charlie each apply CNOT operations on their respective qubits by taking the unknown original state's qubit as source and blank state's qubit as target to achieve the required cloning. For pair (II), however, each of them (Alice, Bob, and Charlie) will apply the CNOT by taking the blank state's qubit as source and original state's qubit as target.

Existence of the three GHZ states that cannot be cloned by LOCC. We would like to mention a necessary condition for cloning of a three-qubit entangled state under LOCC with the help of a three-qubit state as blank copy which is required for our investigation. A necessary condition for cloning of a three-qubit state under the usual LOCC (where all the three qubits of the state is operated separately) would be that: the states should remain copiable when the two qubits are operated jointly at one place whereas the third undergo a separate local operation at a different place and there can be classical communication between these two places.

Consider three states from set (1). The first two qubits of these states are put together in lab A whereas the remaining third qubit in a different lab (B).

These three states are equivalent to the three Bell states in the abovementioned bipartite cut if and only if, all of them have same i and two among them have same j but different p. Now as even 1 free ebit is not sufficient to clone three Bell states [11] (an alternative proof is given in the Appendix) and as the maximum bipartite entanglement that a three-qubit state can have is 1 ebit, so we conclude that these GHZ states with any three-qubit ancilla state cannot be cloned by LOCC. Any set of three states which are not in the above form in any bipartite cut can always be cloned by LOCC using a known GHZ state as ancilla, where every party uses CNOT. Our main objective in this paper is to explore the possibility of cloning of W states under LOCC.

III. CLONING OF W STATES

A full set of tripartite $|W\rangle$ states is given as

$$|W_1\rangle_{123} = \sqrt{\frac{1}{3}}(|001\rangle + |100\rangle + |111\rangle),$$

$$|W_2\rangle_{123} = \sqrt{\frac{1}{3}}(|011\rangle + |101\rangle + |110\rangle),$$

$$|W_3\rangle_{123} = \sqrt{\frac{1}{3}}(|001\rangle - |100\rangle + |010\rangle),$$

$$|W_4\rangle_{123} = \sqrt{\frac{1}{3}}(|011\rangle - |101\rangle + |000\rangle),$$

$$|W_5\rangle_{123} = \sqrt{\frac{1}{3}}(|001\rangle - |010\rangle - |111\rangle),$$

$$|W_6\rangle_{123} = \sqrt{\frac{1}{3}}(|011\rangle - |000\rangle - |110\rangle),$$

$$|W_7\rangle_{123} = \sqrt{\frac{1}{3}}(|100\rangle - |111\rangle + |010\rangle),$$

$$|W_8\rangle_{123} = \sqrt{\frac{1}{3}}(|101\rangle - |110\rangle + |000\rangle).$$

We put our main result in the following theorem.

Theorem: No set of orthogonal W states can be cloned by LOCC with the help of any three qubit state as a blank copy.

Lemma. States belonging to the W class, unless it is a W state, has at least one bipartite cut for which the entanglement, the "bipartite entanglement" $E < -\frac{1}{3}\log_2 \frac{1}{3} - \frac{2}{3}\log_2 \frac{2}{3}$.

Proof. A generic W class state [14] shared among three parties is

$$|\Psi_W\rangle = \sqrt{a}|001\rangle + \sqrt{b}|010\rangle + \sqrt{c}|100\rangle + \sqrt{d}|000\rangle,$$

where $a,b,c > 0$, and $d \equiv 1-(a+b+c) \geq 0$.

If it is possible to let in all the three bipartite cuts, the entanglement of the above W-class state is greater than or equal to $\frac{1}{3}\log_2 \frac{1}{3} - \frac{2}{3}\log_2 \frac{2}{3}$. The entanglement in 1 vs 2-3 cut $E_{1:23}$ is

$$-\frac{1-\sqrt{(1-2c)^2+4cd}}{2}\log_2\frac{1-\sqrt{(1-2c)^2+4cd}}{2},$$

$$-\frac{1+\sqrt{(1-2c)^2+4cd}}{2}\log_2\frac{1+\sqrt{(1-2c)^2+4cd}}{2}.$$

Now $E_{1:23} \geq -\frac{1}{3}\log_2\frac{1}{3} - \frac{2}{3}\log_2\frac{2}{3} \Rightarrow$,

$$\frac{1}{3} \leq \frac{1\pm\sqrt{(1-2c)^2+4cd}}{2} \leq \frac{2}{3}; \text{ i.e. } \frac{1}{3} \leq c \leq \frac{2}{3}. \quad (2)$$

Similarly, for other cuts, the previous assumption will lead to

$$\frac{1}{3} \leq b \leq \frac{2}{3} \quad \text{and} \quad \frac{1}{3} \leq a \leq \frac{2}{3}. \quad (3)$$

Both the inequalities (2) and (3), cannot hold simultaneously, unless $d=0$ and $a=b=c$ (i.e., a W state). Q.E.D.

Proof of the theorem. One needs an entangled blank state to clone an entangled state or else, entanglement of the entire system will increase under LOCC which is impossible. So, let us try to clone the W states with the help of a known genuine tripartite entangled state as the blank copy. Recently, Dür *et al.* [14] have shown that any genuine tripartite entangled state can have entanglement either of the W kind or GHZ kind. So our blank copy is either of the W class or GHZ class.

(i) *Blank copy having GHZ kind of entanglement.* In this case we will show that even a known W state cannot be cloned by LOCC. The minimum number of product terms for a given state cannot be altered by LOCC [14]. But such a cloning would imply that the minimal number of product term for a given state is increased from 6 (minimum number of product terms in the input to the cloner) to 9 (correspond-

ing number in the output), by LOCC which is impossible.

(ii) *Blank copy having W kind of entanglement.* We first consider a *W*-class state which is not a *W* state as our blank copy. Here too a known *W* state cannot be cloned by LOCC. Keeping the lemma in mind, we consider a situation where those two qubits of the blank copy are kept together in lab A for which the entanglement in that bipartite cut of the blank copy (*W* kind of state in this case) is less than that of corresponding *W* state (the state proposed to be cloned). Corresponding qubits of the state to be cloned are also put in lab A. Another lab (B) contains the remaining third qubits of these states. As LOCC cannot increase entanglement hence the *W* state is not copiable under LOCC between these labs. But as mentioned earlier this is necessary for local cloning of any three-qubit state, hence we conclude that a *W* kind of state (unless it is a *W* state) is not helpful in LOCC cloning of *W* states.

(iii) *W state as a blank copy.* In this case we will prove the theorem by showing the impossibility of cloning any pair of *W* states from the abovementioned *W* basis by LOCC. There are 28 such pairs. Consider one pair $|W_m\rangle_{123}$ and $|W_n\rangle_{123}$. Let E_{ij}^{mn} denotes the subspace generated by the support of ρ_{ij}^m and ρ_{ij}^n, where $\rho_{ij}^m = \text{Tr}_k\{|W_m\rangle_{123}\langle W_m|\}$ (similar is ρ_{ij}^n). Here $i,j,k=1,2,3$ and $i \neq j \neq k$. A close inspection will reveal that these pairs fall broadly into three categories:

(A) Pairs for which $\dim(E_{ij}^{mn})=2$ for at least one value of k are (a) (W_1,W_2), (W_3,W_6) for $k=2$, (b) (W_1,W_4), (W_6,W_7) for $k=1$, (c) (W_2,W_7), (W_3,W_4) for $k=3$.

(B) Pairs for which $\dim(E_{ij}^{mn})=3$ for at least one value of k are (a) (W_1,W_6), (W_1,W_8), (W_5,W_6), (W_5,W_8) for $k=3$, (b) (W_2,W_3), (W_2,W_5), (W_3,W_8) for $k=1$, (c) (W_4,W_5), (W_4,W_7), (W_7,W_8) for $k=2$ are such pairs.

(C) Pairs which do not fall under above categories are (W_1,W_3), (W_1,W_5), (W_1,W_7), (W_2,W_4), (W_2,W_6), (W_2,W_8), (W_3,W_5), (W_3,W_7), (W_4,W_6), (W_4,W_8), (W_5,W_7), (W_6,W_8)

We consider the above three types of pairs separately.

A-type pairs. The *k*th qubits of the pair for which $\dim(E_{ij}^{mn})=2$ is put in lab B, whereas the *i*th and *j*th together in lab A. Under this arrangement, any given pair of this type, for a proper choice of basis, reduces to the form

$$|W_m\rangle_{123} = \sqrt{\frac{1}{3}}|0\rangle_A|0\rangle_B + \sqrt{\frac{2}{3}}|1\rangle_A|1\rangle_B,$$

$$|W_n\rangle_{123} = \sqrt{\frac{1}{3}}|0\rangle_A|1\rangle_B + \sqrt{\frac{2}{3}}|1\rangle_A|0\rangle_B.$$

(The subscripts *A* and *B* indicates the laboratories occupying the qubits.)

We assume now the existence of a cloner which, by LOCC between the labs, can clone a pair $|W_m\rangle_{123}$ and $|W_n\rangle_{123}$ when a known *W* state (suppose W_1) is supplied to it as blank copy. If we supply to the cloner an equal mixture of $|W_n\rangle_{123}$ and $|W_m\rangle_{123}$ together with the blank copy W_1, i.e., if the input state to this LOCC cloner is

$$\rho_{\text{in}} = \frac{1}{2}P[(|W_m\rangle_{123} \otimes (W_1)] + \frac{1}{2}P[(|W_n\rangle_{123} \otimes W_1],$$

the output of the cloner will be

$$\rho_{\text{out}} = \frac{1}{2}P[(|W_m\rangle_{123} \otimes (|W_m\rangle_{123}] + \frac{1}{2}P[(|W_n\rangle_{123}$$
$$\otimes (|W_n\rangle_{123}].$$

Here *P* stands for projector.

For proving impossibility of LOCC cloning of these states, we make use of the fact that negativity of a bipartite quantum state ρ, $N(\rho)$ cannot increase under LOCC [15]. $N(\rho)$ is given by [16]

$$N(\rho) \equiv \|\rho^{T_B}\| - 1, \tag{4}$$

where ρ^{T_B} is the partial transpose with respect to system B and $\|\cdots\|$ denotes the trace norm which is defined as

$$\|\rho^{T_B}\| = \text{tr}(\sqrt{\rho^{T_B^\dagger}\rho^{T_B}}). \tag{5}$$

Numerical calculations for negativities give $N(\rho^{\text{in}}) = 0.942809$, $N(\rho^{\text{out}})=0.993808$. As negativity cannot be increased under LOCC between the two labs hence we conclude that the states belonging to this pair cannot be cloned under the LOCC between the two labs and hence by the usual LOCC.

B-type pairs. Once again the *k*th qubits of the pair for which $\dim(E_{ij}^{mn})=3$ is put in lab B, whereas the *i*th and the *j*th together in lab A. For *ij* vs *k* cut and for a proper choice of basis [18], a given pair of this type can be written either as

(I) $(W_m)_{\text{I}} = \sqrt{\frac{1}{3}}|0\rangle_A|0\rangle_B + \sqrt{\frac{2}{3}}|1\rangle_A|1\rangle_B,$

$(W_n)_{\text{I}} = \sqrt{\frac{2}{3}}|1\rangle_A|0\rangle_B + \sqrt{\frac{1}{3}}|2\rangle_A|1\rangle_B$

or

(II) $(W_m)_{\text{II}} = \sqrt{\frac{1}{3}}|0\rangle_A|0\rangle_B + \sqrt{\frac{2}{3}}|1\rangle_A|1\rangle_B,$

$(W_n)_{\text{II}} = \sqrt{\frac{1}{3}}|0\rangle_A|1\rangle_B + \sqrt{\frac{2}{3}}|2\rangle_A|0\rangle_B.$

For showing impossibility of local cloning of these pairs, this time we suppose the existence of a cloner which, by LOCC between the labs, can clone a pair $(W_m)_{\text{I}}$ and $(W_n)_{\text{I}}$ when a known *W* state (suppose W_1) is supplied to it as blank copy, i.e., the state supplied to the cloner is

$$\rho_{\text{in}} = \frac{1}{2}P[(W_m)_{\text{I}} \otimes (W_1)] + \frac{1}{2}P[(W_n)_{\text{I}} \otimes W_1].$$

We then should have the output of the cloner as

$$\rho_{\text{out}} = \frac{1}{2}P[(W_m)_{\text{I}} \otimes (W_m)_{\text{I}}] + \frac{1}{2}P[(W_n)_{\text{I}} \otimes (W_n)_{\text{I}}]$$

(*P* stands for projector as usual). Calculations for negativities of the input and the output of the cloner give us $N(\rho^{\text{in}}) = 1.89097$; $N(\rho^{\text{out}}) = 2.14597$. As $N(\rho^{\text{in}}) < N(\rho^{\text{out}})$. So, these *W* states cannot be cloned by LOCC.

Negativity calculations for type (II) pairs gives $N(\rho^{\text{in}}) = 2.23802$; $N(\rho^{\text{out}}) = 2.49298$, where

$$\rho_{\text{in}} = \frac{1}{2}P[(W_m)_{\text{II}} \otimes (W_{\text{I}})] + \frac{1}{2}P[(W_n)_{\text{II}} \otimes W_{\text{I}}],$$

$$\rho_{\text{out}} = \frac{1}{2}P[(W_m)_{\text{II}} \otimes (W_m)_{\text{II}}] + \frac{1}{2}P[(W_n)_{\text{II}} \otimes (W_n)_{\text{II}}]$$

and *P* as usual stands for the projector. As $N(\rho^{\text{in}}) < N(\rho^{\text{out}})$, hence states belonging to this pair too cannot be cloned.

C-type pairs. Every pair of this set has an important feature that there is one value of *k* for which $\dim(E_{ij}^{mn}) = 4$ and $[\rho_{ij}^m, \rho_{ij}^n] \neq 0$. For showing impossibility of cloning, we put those two qubits together in lab A for which the reduced density matrices of the corresponding W pairs are noncommuting. The states of a given pair under this arrangement reduce to the following representative form:

$$|W_m\rangle = \sqrt{\frac{1}{3}}|0\rangle_A|0\rangle_B + \sqrt{\frac{2}{3}}|1\rangle_A|1\rangle_B$$

and

$$|W_n\rangle = \sqrt{\frac{2}{3}}|0'\rangle_A|0\rangle_B + \sqrt{\frac{1}{3}}|1'\rangle_A|1\rangle_B$$

for proper choice of basis [19], where $\langle 0|1\rangle_A = \langle 0'|1'\rangle_A = 0$, $\langle 0|1'\rangle_A = \langle 0'|1\rangle_A = 0$, $\langle 0|0'\rangle_A = -\langle 1|1'\rangle_A$ and $|\langle 0|0'\rangle_A| = \frac{1}{\sqrt{2}}$. Analysis similar to the previous one shows that negativities of the input (equal mixture of W_m and W_n together with a known *W* state) and output of the assumed cloner (equal mixture of W_m and W_n) are $N(\rho^{\text{in}}) = 2.23802$ and $N(\rho^{\text{out}}) = 2.55185$, respectively, again denying the existence of such a cloner. ∎

IV. DISCUSSION

One of the outstanding feature of quantum mechanics is that nonorthogonal states can not be cloned. But cloning of orthogonal entangled states using LOCC with appropriate supply of entanglement is another area which would further reveal nature of (multipartite) entanglement and as well as of LOCC. The result of this letter established one stark differ-

ence between two kind of symmetric three partite genuine entanglement, namely, *W*-type and GHZ-type entanglement even for a pair of entangled state (where LOCC distinguishability is blunt).

APPENDIX: PROOF OF IMPOSSIBILITY OF CLONING THREE BELL STATES PERFECTLY BY LOCC

The four Bell states are given as

$$|B_{mn}\rangle = \frac{1}{\sqrt{2}}\sum_{j=0}^{1} e^{2\pi ijn/2}|j\rangle|j \oplus m\rangle, n, m = 0, 1. \quad (A1)$$

First, because of nonincrease of entanglement under LOCC, one needs at least 1 free ebit in the blank state to copy any set of the above states.

It has been shown recently by Anselmi *et al.* [9] that if local cloning of a set of maximally entangled state is possible then it is possible only with local unitary operation (LU) (a small subclass of LOCC). So, proving the impossibility of local cloning of Bell states reduces to the proof of impossibility of cloning of these states by local unitary operations.

For proving such an impossibility, we would make use of the fact that the relative entropy of entanglement cannot be altered by any local unitary operation [17]. If cloning of three Bell states (e.g., $|B_{00}\rangle, |B_{01}\rangle, |B_{10}\rangle$) is possible with a 1 free ebit (say $|B_{00}\rangle$ as blank copy) then the following state $\rho_{\text{in}}(= \frac{1}{3}[|B_{00}\rangle\langle B_{00}| + |B_{01}\rangle\langle B_{01}| + |B_{10}\rangle\langle B_{10}|])$ along with the blank state $|B_{00}\rangle$ given as the input to the cloner will provide the ouput as

$$\rho_{\text{in}}\left(= \frac{1}{3}[|B_{00}\rangle\langle B_{00}| + |B_{01}\rangle\langle B_{01}| + |B_{10}\rangle\langle B_{10}|] \otimes |B_{00}\rangle\langle B_{00}|\right)$$

$$\rightarrow \rho_{\text{out}}\left(= \frac{1}{3}[|B_{00}^{\otimes 2}\rangle\langle B_{00}^{\otimes 2}| + |B_{01}^{\otimes 2}\rangle\langle B_{01}^{\otimes 2}| + |B_{10}^{\otimes 2}\rangle\langle B_{10}^{\otimes 2}|]\right).$$

Now as the relative entropy of a separable state is zero and as equal mixture of three Bell states is separable, so the relative entropy of the left hand side of the above equation is one $(0+1)$ ebit. But the relative entropy in the right-hand side is $2 - \log_2 3$ [2] which is strictly less than 1. But this is not possible. Hence we conclude that local cloning of three bell states is not possible even with one free ebit.

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[18] (i) Take for example the pair (W_1, W_6). These states in the "12 vs 3" cut reduce to form (I) for the substitution $|0\rangle_A$ for $-|10\rangle_{12}$, $|1\rangle_A$ for $\left(\frac{|00+11\rangle}{\sqrt{2}}\right)_{12}$, $|2\rangle_A$ for $|01\rangle_{12}$, $|0\rangle_B$ for $-|0\rangle_3$, and $|1\rangle_B$ for $|1\rangle_3$. (ii) Take another pair (W_1, W_8). This for the "12 vs 3" cut can be written as (II) under the substitution $|0\rangle_A$ for $|10\rangle_{12}$, $|1\rangle_A$ for $\left(\frac{|00+11\rangle}{\sqrt{2}}\right)_{12}$, $|2\rangle_A$ for $\left(\frac{|00-11\rangle}{\sqrt{2}}\right)_{12}$, $|0\rangle_B$ for $|0\rangle_3$, and $|1\rangle_B$ for $|1\rangle_3$.

[19] For example, the reduced density matrices of W_1 and W_3 are noncommuting when the third qubits are traced out. So we keep the first and the second qubits in lab A and the third in lab B for LOCC between these labs. Under the substitution $|0\rangle_A$ for $|10\rangle_{12}$, $|1\rangle_A$ for $\left(\frac{|00+11\rangle}{\sqrt{2}}\right)_{12}$, $|0'\rangle_A$ for $\left(\frac{|01-10\rangle}{\sqrt{2}}\right)_{12}$, $|1'\rangle_A$ for $|00\rangle_{12}$, $|0\rangle_B$ for $|0\rangle_3$, and $|1\rangle_B$ for $|1\rangle_3$, W_1 and W_3 reduce to the said form.