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Energy identities in water wave theory for free-surface boundary condition with higher-order derivatives

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Abstract

A modified form of Green's integral theorem is employed to derive the energy identity in any water wave diffraction problem in a single-layer fluid for free-surface boundary condition with higher-order derivatives. For a two-layer fluid with free-surface boundary condition involving higher-order derivatives, two forms of energy identities involving transmission and reflection coefficients for any wave diffraction problem are also derived here by the same method. Based on this modified Green's theorem, hydrodynamic relations such as the energy-conservation principle and *modified* Haskind–Hanaoka relation are derived for radiation and diffraction problems in a single as well as two-layer fluid.

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1. Introduction

In the linearized theory of water waves, the reflection and transmission coefficients R, T in any wave diffraction problem involving a finite number of bodies present in a single-layer fluid with a free-surface

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satisfy the energy identity $|R|^2 + |T|^2 = 1$ which can be derived by a simple use of Green's integral theorem in the fluid region. For a two-layer fluid with a free surface there exist two different modes at which time-harmonic progressive waves can propagate, and as such two energy identities involving reflection and transmission coefficients of two different modes corresponding to incident waves of again two different modes, can also be derived by using Green's integral theorem in both the fluid regions (cf. Linton and McIver, 1995). The various hydrodynamic relations such as the energy-conservation principle and Haskind–Hanaoka relation in any radiation problem involving a free surface can be derived by a simple use of Green's integral theorem in the fluid region (cf. Mei, 1982). For a two-layer fluid with a free surface, the energy-conservation principle and Haskind–Hanaoka relations can also be derived by using Green's integral theorem in both the fluid region (cf. Newman, 1976; Yeung and Nguyen, 2000; Ten and Kashiwagi, 2004; Kashiwagi et al., 2006). In these problems the free-surface condition involves first-order partial derivative. However, if the free-surface involves higher-order partial derivatives (cf. Manam et al., 2006), then the derivation of the energy identity cannot be achieved by a straightforward application of the standard form of Green's integral theorem. The same is true also for a two-layer fluid wherein the free surface of the upper fluid involves higher-order partial derivatives. Such higher-order derivatives in the free-surface condition arise for a large class of problems in the area of ocean structure interaction, e.g. ice sheet covering a vast area of the ocean surface in the cold regions (Antarctic region), the ice sheet being regarded as thin elastic plate, very large floating structure constructed for the purpose of using it as a large floating air port, etc. (cf. Kashiwagi, 1998; Gayen (Chowdhury) et al., 2005; Manam et al., 2006; Fox and Squire, 1994; Lawrie and Abrahams, 1999; Evans and Porter, 2003 and others). Although some part of the free surface may be covered by a floating elastic plate, the remaining part of the free surface is of ordinary gravity waves. Thus there may co-exist two different free-surface conditions. For such a case, the energy identity for a single-layer fluid of uniform finite depth has been derived by Balmforth and Craster (1999) (cf. the relation (7.10) in their paper).

In this paper a modified form of Green's integral theorem designed appropriately to take care of the free-surface condition with higher-order derivatives throughout the entire free-surface, is employed to derive the energy identity for any diffraction problem, the energy-conservation principle and the *modified* Haskind–Hanaoka relation for any radiation problem in a single-layer fluid. For a two-layer fluid again with higher-order derivatives in the free-surface condition, two forms of energy identities satisfied by the reflection and transmission coefficients of different wave modes exist and these are also derived by appropriate uses of the modified form of Green's integral theorem in the upper and lower fluid layers. Also are derived the energy-conservation principle and the *modified* Haskind–Hanaoka relations in such a two-layer fluid.

In fact, for a general boundary value problem involving a single-layer or a two-layer fluid with higher-order derivatives in the free-surface condition, a relation involving the wave amplitudes at either infinities on the right and left sides of any finite number of bodies present in the fluid is first obtained. Then, from this general result, the energy identity for a single-layer fluid or the energy identities for a two-layer fluid, for a general wave diffraction problem relating the reflection and transmission coefficients and the other hydrodynamic relations for radiation problem, are derived. In Section 2, the description of a general wave propagation problem for time-harmonic progressive waves in the presence of a finite number of bodies in a single as well as two-layer fluid with higher-order derivatives in the free-surface condition is described. In Section 3, a relation between the wave amplitudes produced at either infinities due to prescribed normal velocity at the body boundaries is obtained by employing the aforesaid modified form of Green's integral theorem in the fluid region for a single-layer fluid. The energy identity for a diffraction problem involving a finite number of bodies present in a single-layer fluid is then derived as a special case. Similarly, two different energy identities for a diffraction problem involving a finite number of bodies present in a two-layer fluid are also derived as special cases. It is emphasized that the energy identity for a single-layer fluid or the energy identities for a two-layer fluid are of great use in checking the correctness of the analytical as well as numerical results determining the reflection and transmission coefficients for any diffraction problem involving prescribed body boundaries. This necessitates the derivation of these identities for fluid involving higher-order derivatives in the free-surface condition.

2. Formulation of a general boundary value problem

2.1. Single-layer fluid

A general boundary value problem describing wave propagation in the presence of a finite number of bodies is considered here assuming linear theory. The usual assumptions of incompressible, homogeneous and inviscid fluid, irrotational and simple harmonic motion with angular frequency ω under gravity only, are made. The y-axis is chosen vertically upwards and the plane $y = 0$ is taken as the mean horizontal position of the upper surface of the fluid. Two-dimensional motion depending on x, y only is considered. The fluid occupies the region $y < 0$ if it is infinitely deep, or $-h < y < 0$ if it is of uniform finite depth h. If $\mathrm{Re}\{\phi(x, y)e^{-i\omega t}\}$ denotes the velocity potential describing the motion in the fluid, then ϕ satisfies

$$\nabla^2\phi = 0 \quad \text{in the fluid region,} \tag{2.1}$$

where ∇^2 denotes the two-dimensional Laplace operator. On the upper surface having the mean position $y = 0$, ϕ satisfies the free-surface condition with higher-order derivatives of the form

$$\left(D\frac{\partial^4}{\partial x^4} + (1 - \varepsilon K)\right)\frac{\partial\phi}{\partial y} - K\phi = 0 \quad \text{on } y = 0 \tag{2.2}$$

if the free-surface has an ice-cover modelled as a thin elastic plate, where $D = Eh_0^3/12(1 - \nu^2)\rho g$, $\varepsilon = \rho_0 h_0/\rho$, ρ_0 is the density of ice, ρ is density of water, h_0 is the small thickness of ice-cover, E, ν are the Young's modulus and Poission's ratio of the ice and $K = \omega^2/g$, g being the acceleration due to gravity. A generalization of (2.2) for more higher-order derivatives has been introduced by Manam et al. (2006) and has the form

$$\mathscr{L}\phi_y - K\phi = 0 \quad \text{on } y = 0, \tag{2.2$'$}$$

where $\mathscr{L}$ is a linear differential operator of the form

$$\mathscr{L} = \sum_{m=0}^{m_0} c_m \frac{\partial^{2m}}{\partial x^{2m}}. \tag{2.3}$$

In (2.3) c_m $(m=0, 1, \ldots, m_0)$ are known constants. Keeping in mind various physical problems involving fluid structure interaction, only the even order partial derivatives in x are considered in the differential operator $\mathscr{L}$. The bottom condition is given by

$$\nabla\phi \longrightarrow 0 \quad \text{as } y \to -\infty \tag{2.4a}$$

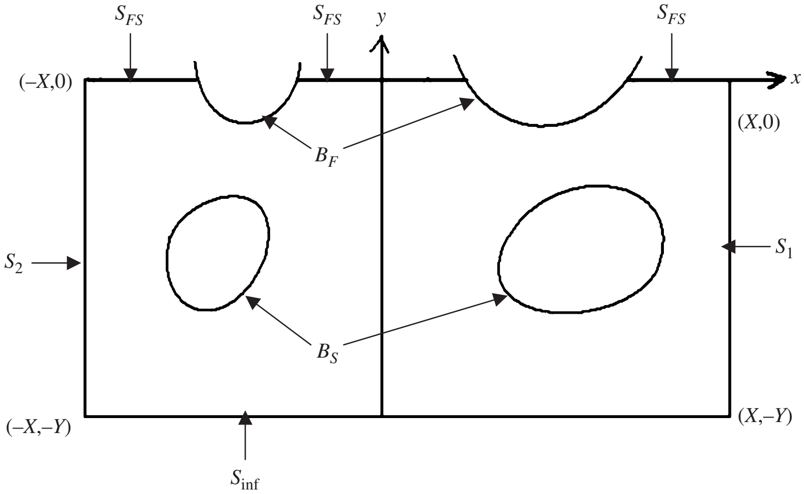

Fig. 1. Boundaries of a single-layer fluid.

for infinitely deep water, or by

$$\phi_y = 0 \quad \text{on } y = -h \tag{2.4b}$$

for water of uniform finite depth h. Finally, the body boundary conditions are given by

$$\phi_n \quad \text{is prescribed on } B = B_{\mathrm{F}} \bigcup B_{\mathrm{S}}, \tag{2.5}$$

where B_{F} denotes the wetted parts of the floating bodies while B_{S} denotes the submerged body boundaries and ϕ_n denotes the derivative normal to B (see Fig. 1).

The forms of the far-field on the two sides of B are given by

$$\phi(x, y) \longrightarrow \begin{cases} (A^{\pm}\mathrm{e}^{\pm \mathrm{i} p_0 x} + B^{\pm}\mathrm{e}^{\mp \mathrm{i} p_0 x})\mathrm{e}^{p_0 y} & \text{as } x \to \pm\infty \text{ for deep water}, & (2.6a) \\ (A^{\pm}\mathrm{e}^{\pm \mathrm{i} p_0 x} + B^{\pm}\mathrm{e}^{\mp \mathrm{i} p_0 x})g(y) & \text{as } x \to \pm\infty \text{ for finite depth water}, & (2.6b) \end{cases}$$

where

$$g(y) = \frac{\cosh p_0(y+h)}{\cosh p_0 h}, \tag{2.7}$$

and p_0 satisfies the transcendental equation

$$\sum_{m=0}^{m_0} (-1)^m c_m p^{2m} = K \quad \text{for deep water}, \tag{2.8a}$$

$$\left(\sum_{m=0}^{m_0} (-1)^m c_m p^{2m}\right) \tanh ph = K \quad \text{for finite depth water}. \tag{2.8b}$$

Under specific assumptions involving the constants c_m $(m=0, 1, \ldots, m_0)$, Eq. (2.8a) or (2.8b) is assumed to possess only one real positive root. This is also physically realistic since wave of only one wavenumber can propagate on the upper surface.

A convenient short notation for (2.6) is (cf. Linton and McIver, 1995),

$$\phi \sim (A^-, B^-; A^+, B^+), \tag{2.9}$$

where $A^\pm$, $B^\pm$ denote the amplitudes as $x \to \pm\infty$ of the outgoing and incoming waves set up at either infinities.

2.2. Two-layer fluid

In a two-layer fluid, both the upper and lower fluids are assumed to be homogeneous, incompressible and inviscid. Let ρ^{I} be the density of the upper fluid and ρ^{II} $(>\rho^{\mathrm{I}})$ be the same for the lower fluid. Let the lower fluid extend infinitely downwards while the upper one has a finite height h above the mean interface. Let y-axis points vertically upwards from the undisturbed interface $y=0$. Thus the upper layer occupies the region $0<y<h$ while the lower layer occupies the region $y<0$. Under the usual assumption of linear theory and irrotational two-dimensional motion, velocity potentials $\mathrm{Re}\{\phi^{\mathrm{I,II}}(x, y)\mathrm{e}^{-\mathrm{i}\omega t}\}$ describing the fluid motion in the upper and lower layers exist. For a general boundary value problem, $\phi^{\mathrm{I,II}}$ satisfy

$$\nabla^2\phi^{\mathrm{I}} = 0, \quad 0<y<h, \tag{2.10a}$$

$$\nabla^2\phi^{\mathrm{II}} = 0, \quad y<0. \tag{2.10b}$$

The linearized boundary conditions at the interface $y=0$ are

$$\phi_y^{\mathrm{I}} = \phi_y^{\mathrm{II}} \quad \text{on } y=0, \tag{2.11a}$$

$$\rho(\phi_y^{\mathrm{I}} - K\phi^{\mathrm{I}}) = \phi_y^{\mathrm{II}} - K\phi^{\mathrm{II}} \quad \text{on } y=0, \tag{2.11b}$$

where $\rho = \rho^{\mathrm{I}}/\rho^{\mathrm{II}}(<1)$, while the free-surface condition with higher-order derivatives at $y=h$ is

$$\mathscr{L}\phi_y^{\mathrm{I}} - K\phi^{\mathrm{I}} = 0 \quad \text{on } y=h, \tag{2.12}$$

where the differential operator $\mathscr{L}$ has the same form as given in (2.3). The bottom condition is given by

$$\nabla\phi^{\mathrm{II}} \longrightarrow 0 \quad \text{as } y \to -\infty. \tag{2.13}$$

The conditions on the body boundary are given by

$$\phi_n^{\mathrm{I}} \quad \text{is prescribed on } B_{\mathrm{I}} = B_{\mathrm{IF}} \bigcup B_{\mathrm{IS}}, \tag{2.14a}$$

where B_{IF} denotes the wetted parts of the floating bodies, while B_{IS} denotes the body boundaries submerged in the upper fluid (see Fig. 2) and

$$\phi_n^{\mathrm{II}} \quad \text{is prescribed on } B_{\mathrm{II}}, \tag{2.14b}$$

where B_{II} represents the body boundaries submerged in the lower fluid.

It is well known that there exists two distinct values of the wavenumbers for time-harmonic progressive waves of a particular frequency, one propagating at the upper surface and the other at the interface. Thus for any wave propagation problem, the far-field consists of outgoing and incoming waves at each of the

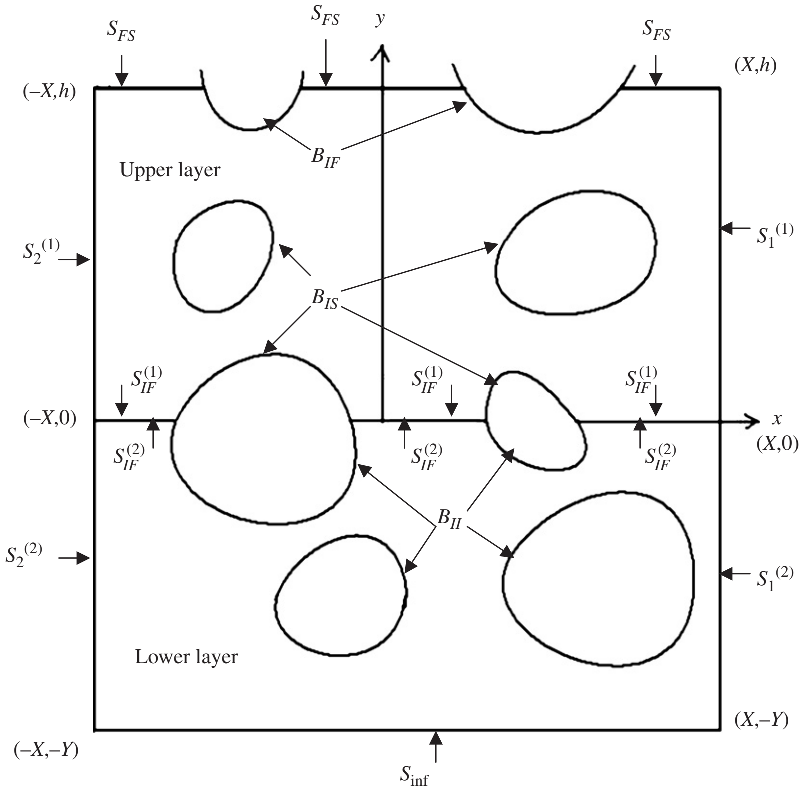

Fig. 2. Boundaries of a two-layer fluid.

two wavenumbers λ_1, λ_2, say, where λ_1, λ_2 are defined in (2.17). Thus it is given by

$$\phi^{\mathrm{I}} \to (A^{\pm}\mathrm{e}^{\pm\mathrm{i}\lambda_1 x} + C^{\pm}\mathrm{e}^{\mp\mathrm{i}\lambda_1 x})g_1(y) + (B^{\pm}\mathrm{e}^{\pm\mathrm{i}\lambda_2 x} + D^{\pm}\mathrm{e}^{\mp\mathrm{i}\lambda_2 x})g_2(y) \quad \text{as } x \to \pm\infty, \tag{2.15a}$$

$$\phi^{\mathrm{II}} \to (A^{\pm}\mathrm{e}^{\pm\mathrm{i}\lambda_1 x} + C^{\pm}\mathrm{e}^{\mp\mathrm{i}\lambda_1 x})\mathrm{e}^{\lambda_1 y} + (B^{\pm}\mathrm{e}^{\pm\mathrm{i}\lambda_2 x} + D^{\pm}\mathrm{e}^{\mp\mathrm{i}\lambda_2 x})\mathrm{e}^{\lambda_2 y} \quad \text{as } x \to \pm\infty, \tag{2.15b}$$

where $g_1(y)$, $g_2(y)$ are defined in (2.18).

In the notation of Linton and McIver (1995), a short-hand version of (2.15) is

$$\phi \sim (A^-, B^-, C^-, D^-; A^+, B^+, C^+, D^+). \tag{2.16}$$

In (2.15), the real positive number λ_1, $\lambda_2 (> \lambda_1)$ are the only two real positive roots of the transcendental equation

$$\left(\sum_{m=0}^{m_0}(-1)^m c_m \lambda^{2m+1}\sinh\lambda h - K\cosh\lambda h\right)(\lambda - \rho\lambda - K)$$
$$- \rho K\left(\sum_{m=0}^{m_0}(-1)^m c_m \lambda^{2m+1}\cosh\lambda h - K\sinh\lambda h\right) = 0, \tag{2.17}$$

and the functions $g_j(y)$ $(j=1,2)$ are given by

$$g_j(y)=\frac{\{(1-\rho)\lambda_j-K\}}{\rho K[\sum_{m=0}^{m_0}(-1)^m c_m\lambda_j^{2m+1}\cosh\lambda_j h-K\sinh\lambda_j h]}\times\left[\left\{\sum_{m=0}^{m_0}(-1)^m c_m\lambda_j^{2m+1}+K\right\}\mathrm{e}^{\lambda_j(y-h)}+\left\{\sum_{m=0}^{m_0}(-1)^m c_m\lambda_j^{2m+1}-K\right\}\mathrm{e}^{-\lambda_j(y-h)}\right]. \tag{2.18}$$

The constants c_m $(m=0,1,\ldots,m_0)$ are assumed to be such that Eq. (2.17) possesses only two real positive roots, which correspond to the two different wavenumbers (modes) at which progressive waves propagate at the upper surface and the interface of the two-layer fluid. It is emphasized that the physical constants $c_m\,(m=0,1,\ldots,m_0)$ are such that only two real positive roots of Eq. (2.17) exist due to physical reasons.

3. Derivation of energy identity (identities)

3.1. Modified form of Green's integral theorem

In the case of a single-layer fluid having free-surface condition with first-order derivative, a relation exists between the various hydrodynamics quantities that arise in a general boundary value problem involving a finite number of body boundaries, and can be determined by a judicious application of the standard Green's integral theorem for harmonic functions in the form

$$\int_S(\phi\psi_n-\psi\phi_n)\,\mathrm{d}s=0. \tag{3.1}$$

In (3.1), S denotes the boundary of the fluid region and ϕ_n, ψ_n denote partial derivatives along the normal to S. A similar relation can also be determined in the case of a two-layer fluid having free-surface condition with first-order derivative.

For the determination of a similar relation for a general boundary value problem in a single-layer fluid or a two-layer fluid with higher-order derivatives in the free-surface condition, Green's theorem (3.1) has to be modified taking into account this higher-order derivatives. For boundary condition of the form as given by (2.2′) for a single-layer fluid or by (2.12) for a two-layer fluid, the modified form of (3.1) is given by

$$\int_S(\phi L_n\psi-\psi L_n\phi)\,\mathrm{d}s=0, \tag{3.2}$$

where the operator L_n is of the form

$$L_n=\sum_{m=0}^{m_0}(-1)^m c_m\frac{\partial^{2m+1}}{\partial n^{2m+1}}, \tag{3.3}$$

$\partial/\partial n$ being the derivative normal to S.

The proof of the generalization (3.2) of the standard Green's theorem (3.1) is given in Appendix A.

The modified form of Green's theorem given by (3.2) is now employed to obtain the desired relations between different hydrodynamic quantities for a general boundary value problem in a single-layer or a two-layer fluid with higher-order derivatives in the free surface.

3.2. Single-layer fluid

Let ϕ, ψ be the solutions of the two different problems with ϕ_n, ψ_n being prescribed on the submerged body boundary B. Let the far-field form of ϕ be given by (2.9) while that for ψ be given by

$$\psi \sim (P^-, Q^-; P^+, Q^+). \tag{3.4}$$

Let S be chosen as the boundary of the fluid region as given in Fig. 1 where X and Y are arbitrarily large, S_{FS} denotes the portions of free surface between $-X$ to X. Then application of (3.2) to ϕ, ψ produces

$$\left(\int_{S_{\mathrm{FS}}} + \int_{S_2} + \int_{S_{\inf}} + \int_{S_1} + \int_B\right)(\phi L_n \psi - \psi L_n \phi)\,\mathrm{d}s = 0. \tag{3.5}$$

The first integral in (3.5) is

$$\int_{S_{\mathrm{FS}}} (\phi L_y \psi - \psi L_y \phi)(x,0)\,\mathrm{d}x = \sum_{m=0}^{m_0} c_m \int_{S_{\mathrm{FS}}} \left(\phi \frac{\partial^{2m+1}\psi}{\partial x^{2m}\partial y} - \psi \frac{\partial^{2m+1}\phi}{\partial x^{2m}\partial y}\right)(x,0)\,\mathrm{d}x.$$

Use of the free-surface condition with higher-order derivatives (2.2′) on $y=0$ makes this integral identically equal to zero for any X.

The second integral in (3.5) is

$$\int_{S_2} (\psi L_x \phi - \phi L_x \psi)(-X, y)\,\mathrm{d}y.$$

Making $X \to \infty$, $Y \to \infty$, this reduces to, after using the far-field condition (2.6a),

$$\mathrm{i}\sum_{m=0}^{m_0} c_m p_0^{2m}(Q^- A^- - P^- B^-). \tag{3.6}$$

Similarly, the fourth integral in (3.5) reduces to, as $X, Y \to \infty$,

$$-\mathrm{i}\sum_{m=0}^{m_0} c_m p_0^{2m}(P^+ B^+ - Q^+ A^+). \tag{3.7}$$

Again, the third integral in (3.5) tends to 0 as $Y \to \infty$ by using the bottom condition (2.4a). Finally, the last integral in (3.5) is

$$\int_B \left[\phi \sum_{m=0}^{m_0} (-1)^m c_m \frac{\partial^{2m+1}\psi}{\partial n^{2m+1}} - \psi \sum_{m=0}^{m_0} (-1)^m c_m \frac{\partial^{2m+1}\phi}{\partial n^{2m+1}}\right]\mathrm{d}s.$$

Let the cross-section B of the body boundaries be described parametrically by $x = X(\theta)$, $y = Y(\theta)$, $(0 \leqslant \theta \leqslant 2\pi$ for a submerged body and $\alpha \leqslant \theta \leqslant \beta$ for wetted portion of floating body, α, β may be negative) where $\theta = 0$ is chosen to be coincident with the line $x = 0$. Defining (s, n) as rectangular co-ordinates along the normal and tangent to B at any point of B, and using the relation (2.14) of Porter (2002), the harmonic functions ϕ, ψ satisfy $\nabla_1^2\phi = 0$, $\nabla_1^2\psi = 0$ where $\nabla_1^2 = \partial^2/\partial s^2 + \partial^2/\partial n^2 + \kappa(s)\partial/\partial n$, $\kappa(s)$ being the curvature as a function of the arc length s. Now

$$\frac{\partial^{2m}}{\partial n^{2m}} = (-1)^m \left(\frac{\partial^2}{\partial s^2} + \kappa(s) Q(s) \frac{\partial}{\partial s} \right)^{2m},$$

where $Q(s) = ((Y'(\theta))^2 - (X'(\theta))^2)/X'(\theta)Y'(\theta)$, and is a function of s. Using these we find the last integral in (3.5) to be

$$\int_B \left[\phi \mathscr{M}_s \frac{\partial \psi}{\partial n} - \psi \mathscr{M}_s \frac{\partial \phi}{\partial n} \right] \mathrm{d}s, \tag{3.8}$$

where

$$\mathscr{M}_s \equiv \sum_{m=0}^{m_0} c_m \left(\frac{\partial^2}{\partial s^2} + \kappa(s) Q(s) \frac{\partial}{\partial s} \right)^{2m}.$$

If $\partial\phi/\partial n$ and $\partial\psi/\partial n$ vanish on B (for diffraction problems), then this integral vanishes identically.

Collecting all the terms in (3.5) we obtain the relation

$$\mathrm{i} \sum_{m=0}^{m_0} c_m p_0^{2m} (Q^- A^- - P^- B^-) + \int_B \cdots \mathrm{d}s = \mathrm{i} \sum_{m=0}^{m_0} c_m p_0^{2m} (P^+ B^+ - Q^+ A^+), \tag{3.9}$$

where $\int_B \cdots \mathrm{d}s$ is the same as given by (3.8).

For a *diffraction problem*, $\partial\phi/\partial n = 0$ on B and the far-field form of ϕ is given by

$$\phi \sim (R, 1; T, 0), \tag{3.10}$$

where R and T are the reflection and transmission coefficients, respectively, due to an incident field propagating from the direction of $x = -\infty$. Let $\overline{\phi}$ denote the complex conjugate of ϕ. Then $\partial\overline{\phi}/\partial n = 0$ on B, and the far-field form of $\overline{\phi}$ is

$$\overline{\phi} \sim (1, \overline{R}; 0, \overline{T}). \tag{3.11}$$

Writing $\psi = \overline{\phi}$ in (3.9), we obtain

$$(|T|^2 + |R|^2 - 1) \sum_{m=0}^{m_0} c_m p_0^{2m} = 0,$$

giving

$$|T|^2 + |R|^2 = 1 \tag{3.12}$$

which is the desired *energy identity*.

For a *diffraction* potential function ϕ and a *radiation* potential function ψ, $\partial\phi/\partial n = 0$ on B and the far-field form of ϕ is given by (3.10) and $\partial\psi/\partial n = n_j$, where n_j is the component of the normal to the body in the *j*th mode of motion. The far-field form of ψ is

$$\psi \sim (\Theta_j^-, 0; \Theta_j^+, 0). \tag{3.13}$$

Then we obtain from (3.9)

$$\mathrm{i}\rho\omega \int_B \phi \mathscr{M}_s n_j \,\mathrm{d}s = -\rho\omega \sum_{m=0}^{m_0} c_m p_0^{2m} \Theta_j^-, \tag{3.14}$$

where ρ is the density of the fluid. For $m_0 = 0$ and $c_0 = 1$, i.e. when the free-surface condition has the usual form $K\phi + \phi_y = 0$, the left side of (3.14) produces $\mathrm{i}\rho\omega \int_B \phi n_j \,\mathrm{d}s$, which is the hydrodynamic force on the body B in the *j*th mode of motion. Thus we may call (3.14) as the *modified* form of Haskind–Hanaoka relation. However, it should be noted that, for $m_0 > 0$, the left side of (3.14) cannot be termed as the actual hydrodynamic force on the body since it cannot obtained by integrating over the body surface the expression of pressure from Bernoulli's equation.

Now we consider the case of two *radiation* potential functions. Let $\phi = \phi_j$ and $\psi = \phi_k$ be two *radiation* potentials whose behavior in the far-field is given by

$$\phi_j \sim (\Theta_j^-, 0; \Theta_j^+, 0),$$

$$\phi_k \sim (\Theta_k^-, 0; \Theta_k^+, 0)$$

and which satisfy the body boundary condition

$$\frac{\partial \phi_j}{\partial n} = n_j, \quad \frac{\partial \phi_k}{\partial n} = n_k \quad \text{on } B.$$

Then we obtain from (3.9)

$$\int_B (\phi_j \mathscr{M}_s n_k - \phi_k \mathscr{M}_s n_j) \,\mathrm{d}s = 0. \tag{3.15}$$

If we now use $\psi = \overline{\phi}_k$, the complex conjugate of ϕ_k, then we find from (3.9) that

$$\int_B (\phi_j \mathscr{M}_s n_k - \overline{\phi}_k \mathscr{M}_s n_j) \,\mathrm{d}s = -\mathrm{i} \sum_{m=0}^{m_0} c_m p_0^{2m} (\Theta_j^- \overline{\Theta}_k^- + \Theta_j^+ \overline{\Theta}_k^+). \tag{3.16}$$

In particular, for the case when $j = k$ Eq. (3.9) becomes

$$\int_B (\phi_j \mathscr{M}_s n_j - \overline{\phi}_j \mathscr{M}_s n_j) \,\mathrm{d}s = -\mathrm{i} \sum_{m=0}^{m_0} c_m p_0^{2m} (|\Theta_j^-|^2 + |\Theta_j^+|^2). \tag{3.17}$$

This is an extension of the modified *energy-conservation principle* to the case of a wave propagation problem in a single-layer fluid having free-surface condition with higher-order derivatives.

For the case of water of *uniform finite depth h*, we replace Y by h in (3.5). Then as $X \to \infty$, the second integral reduces to

$$\mathrm{i}\sum_{m=0}^{m_0} c_m p_0^{2m}(1-\mathrm{e}^{-2p_0h})(Q^-A^- - P^-B^-) \tag{3.18}$$

while the fourth integral reduces to

$$-\mathrm{i}\sum_{m=0}^{m_0} c_m p_0^{2m}(1-\mathrm{e}^{-2p_0h})(P^+B^+ - Q^+A^+). \tag{3.19}$$

Thus, in this case, we obtain the relation

$$\mathrm{i}\sum_{m=0}^{m_0} c_m p_0^{2m}(1-\mathrm{e}^{-2p_0h})\{(Q^-A^- - P^-B^-) - (P^+B^+ - Q^+A^+)\} + \int_B \cdots \mathrm{d}s = 0, \tag{3.20}$$

where the integral $\int_B \cdots \mathrm{d}s$ is the same as given by (3.8).

For a diffraction problem $\phi_n = 0$ on B and choosing $\psi = \overline{\phi}$ so that $\psi_n = 0$ on B, we finally obtain, after using (2.8b), the same identity (3.12).

For a diffraction potential function ϕ and a radiation potential function ψ, we obtain from (3.20) the relation

$$\mathrm{i}\rho\omega \int_B \phi \mathscr{M}_s n_j \,\mathrm{d}s = -\rho\omega \sum_{m=0}^{m_0} c_m p_0^{2m}\ (1-\mathrm{e}^{-2p_0h})\Theta_j^- \tag{3.21}$$

which is the *modified* form of Haskind–Hanaoka relation.

Again if ϕ_j and ψ_k denote solutions of two radiation problems satisfying on the body boundary B, $\partial\phi_j/\partial n = n_j$, $\partial\psi_k/\partial n = n_k$, then we find from (3.20) the relation for $j = k$

$$\int_B (\phi_j \mathscr{M}_s n_j - \overline{\phi}_j \mathscr{M}_s n_j)\,\mathrm{d}s = \mathrm{i}\sum_{m=0}^{m_0} c_m p_0^{2m}(1-\mathrm{e}^{-2p_0h})(|\Theta_j^-|^2 + |\Theta_j^+|^2). \tag{3.22}$$

This is the modified form of the energy-conservation principle.

Thus the modified forms of the energy identity for any diffraction problem, the Haskind–Hanaoka relation for any radiation and diffraction problems, the energy-conservation principle for any radiation problem are established for a single-layer fluid having free-surface boundary condition with higher-order derivatives. When we put $c_0 = 1$, $c_m = 0$ $(m = 1, 2, 3, \ldots, m_0)$ in these relations, the energy identity, the Haskind–Hanaoka relation and the energy-conservation principle for a single-layer fluid with the usual free-surface condition as given as Mei (1982) are obtained.

3.3. Two-layer fluid

Let there be situated a finite number of bodies in a two-layer fluid, some in the upper layer and some in the lower layer (cf. Fig. 2). There may be some bodies which are present in both the layers. Let the boundaries of the bodies lying in the upper layer be denoted by B_{I} and those in the lower layer by B_{II}.

Let ϕ, ψ be the solutions of two different boundary value problems with ϕ_n, ψ_n being prescribed on the boundaries B_{I} and B_{II}, and the far-field form of ϕ being given by (2.16) and that of ψ is given by

$$\psi \sim (M^-, N^-, P^-, Q^-; M^+, N^+, P^+, Q^+). \tag{3.23}$$

In this case we choose S in (3.2) first to be the boundary of the region in the upper fluid as shown in Fig. 2 and ultimately make $X \to \infty$, and next to be the boundary of the region in the lower fluid as shown in Fig. 2 and ultimately make both $X, Y \to \infty$.

For the upper layer, (3.2) produces

$$\left(\int_{S_{\mathrm{FS}}} + \int_{S_2^{(1)}} + \int_{S_{\mathrm{IF}}^{(1)}} + \int_{S_1^{(1)}} + \int_{B_{\mathrm{I}}}\right)(\phi^{\mathrm{I}} L_n \psi^{\mathrm{I}} - \psi^{\mathrm{I}} L_n \phi^{\mathrm{I}})\,\mathrm{d}s = 0. \tag{3.24}$$

The first integral in (3.24) vanishes identically due to the boundary condition (2.12) satisfied by both ϕ^{I} and ψ^{I}. The second integral in (3.24) is

$$\int_{S_2^{(1)}} (\phi^{\mathrm{I}} L_x \psi^{\mathrm{I}} - \psi^{\mathrm{I}} L_x \phi^{\mathrm{I}})(-X, y)\,\mathrm{d}y.$$

Making use of the far-field behavior of ϕ^{I}, ψ^{I} for large X, this produces

$$\begin{aligned}
&2\mathrm{i}\left[\sum_{m=0}^{m_0} c_m \left\{\lambda_1^{2m+1}(P^-A^- - M^-C^-)\int_0^h (g_1(y))^2\,\mathrm{d}y + \lambda_2^{2m+1}\right.\right.\\
&\qquad \left.\left.\times (Q^-B^- - N^-D^-)\int_0^h (g_2(y))^2\,\mathrm{d}y\right\}\right]\\
&\qquad + \mathrm{i}\left[\sum_{m=0}^{m_0} c_m\{(\lambda_1^{2m+1} - \lambda_2^{2m+1})((N^-A^- - M^-B^-)\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)X} + (Q^-C^- - D^-P^-)\mathrm{e}^{\mathrm{i}(\lambda_1+\lambda_2)X})\right.\\
&\qquad + (\lambda_1^{2m+1} + \lambda_2^{2m+1})((Q^-A^- - M^-D^-)\mathrm{e}^{-\mathrm{i}(\lambda_1-\lambda_2)X} + (B^-P^- - N^-C^-)\mathrm{e}^{\mathrm{i}(\lambda_1-\lambda_2)X})\}\\
&\qquad \left.\times \int_0^h g_1(y)g_2(y)\,\mathrm{d}y\right].
\end{aligned} \tag{3.25}$$

Similarly the fourth integral in (3.24) produces an expression which is similar to (3.25) with the subscripts minus (−) replaced by plus (+). The third integral in (3.24) is

$$\int_{S_{\mathrm{IF}}^{(1)}} (\phi^{\mathrm{I}} L_y \psi^{\mathrm{I}} - \psi^{\mathrm{I}} L_y \phi^{\mathrm{I}})(x, 0)\,\mathrm{d}x. \tag{3.26}$$

Finally, the last integral in (3.24) becomes, after using the same reasoning as used in obtaining (3.8),

$$\int_{B_{\mathrm{I}}} \left[\phi^{\mathrm{I}} \mathscr{M}_s\left(\frac{\partial \psi^{\mathrm{I}}}{\partial n}\right) - \psi^{\mathrm{I}} \mathscr{M}_s\left(\frac{\partial \phi^{\mathrm{I}}}{\partial n}\right)\right]\mathrm{d}s. \tag{3.27}$$

For the lower layer, (3.2) produces

$$\left(\int_{S_{\mathrm{IF}}^{(2)}}+\int_{S_2^{(2)}}+\int_{S_{\mathrm{inf}}}+\int_{S_1^{(2)}}+\int_{B_{\mathrm{II}}}\right)(\phi^{\mathrm{II}}L_n\psi^{\mathrm{II}}-\psi^{\mathrm{II}}L_n\phi^{\mathrm{II}})\,\mathrm{d}s=0. \tag{3.28}$$

The first integral in (3.28) is

$$\int_{S_{\mathrm{IF}}^{(2)}}(\phi^{\mathrm{II}}L_y\psi^{\mathrm{II}}-\psi^{\mathrm{II}}L_y\phi^{\mathrm{II}})(x,0)\,\mathrm{d}x. \tag{3.29}$$

The second integral in (3.28) reduces to, after using (2.15b) and making $Y\to\infty$,

$$\begin{aligned}&\mathrm{i}\left[\sum_{m=0}^{m_0}c_m\{\lambda_1^{2m}(P^-A^--M^-C^-)+\lambda_2^{2m}(Q^-B^--N^-D^-)\}\right]\\&\quad+\mathrm{i}\left[\sum_{m=0}^{m_0}c_m\{(\lambda_1^{2m+1}-\lambda_2^{2m+1})((N^-A^--M^-B^-)\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_2)X}\right.\\&\quad+(Q^-C^--D^-P^-)\mathrm{e}^{\mathrm{i}(\lambda_1+\lambda_2)X})\\&\quad+(\lambda_1^{2m+1}+\lambda_2^{2m+1})((Q^-A^--M^-D^-)\mathrm{e}^{-\mathrm{i}(\lambda_1-\lambda_2)X}+(B^-P^--N^-C^-)\mathrm{e}^{\mathrm{i}(\lambda_1-\lambda_2)X})\}\\&\quad\left.\times\int_{-\infty}^{0}\mathrm{e}^{(\lambda_1+\lambda_2)y}\,\mathrm{d}y\right].\end{aligned} \tag{3.30}$$

Similarly, the fourth integral in (3.28) reduces to the same expression (3.30) with the subscripts minus ($-$) replaced by plus ($+$).

Again, the third integral in (3.28) tends to 0 as $Y\to-\infty$, after using the conditions at infinite depth. Finally, the last integral in (3.28) becomes,

$$\int_{B_{\mathrm{II}}}\left[\phi^{\mathrm{II}}\mathscr{M}_s\left(\frac{\partial\psi^{\mathrm{II}}}{\partial n}\right)-\psi^{\mathrm{II}}\mathscr{M}_s\left(\frac{\partial\phi^{\mathrm{II}}}{\partial n}\right)\right]\mathrm{d}s. \tag{3.31}$$

Substituting all these results in (3.24) and (3.28), and using the condition at the interface given by (see Appendix B for its derivation)

$$\rho(\phi^{\mathrm{I}}L_y\psi^{\mathrm{I}}-\psi^{\mathrm{I}}L_y\phi^{\mathrm{I}})=\phi^{\mathrm{II}}L_y\psi^{\mathrm{II}}-\psi^{\mathrm{II}}L_y\phi^{\mathrm{II}}\quad\text{on } y=0 \tag{3.32}$$

and the result (see Appendix C for its derivation)

$$\rho\int_0^h g_1(y)g_2(y)\,\mathrm{d}y+\int_{-\infty}^0\mathrm{e}^{(\lambda_1+\lambda_2)y}\,\mathrm{d}y=0, \tag{3.33}$$

and making $X\to\infty$, we obtain

$$\begin{aligned}&\rho\int_{B_{\mathrm{I}}}[\;]\,\mathrm{d}s+\int_{B_{\mathrm{II}}}[\;]\,\mathrm{d}s+\mathrm{i}J_{\lambda_1}(P^+A^+-M^+C^++P^-A^--M^-C^-)\\&\quad+\mathrm{i}J_{\lambda_2}(Q^+B^+-N^+D^++Q^-B^--N^-D^-)=0,\end{aligned} \tag{3.34}$$

where $\int_{B_{\mathrm{I}}}[\ ]\,\mathrm{d}s$ is given in (3.27), $\int_{B_{\mathrm{II}}}[\ ]\,\mathrm{d}s$ is given in (3.31), and

$$J_{\lambda_j} = \sum_{m=0}^{m_0} c_m \lambda_j^{2m} \left\{ 1 + 2\rho\lambda_j \int_0^h (g_j(y))^2\,\mathrm{d}y \right\}, \quad j = 1, 2. \tag{3.35}$$

Eq. (3.34)gives a relation between the wave amplitudes of the two boundary value problems described by ϕ, ψ in terms of their values together with their normal derivatives on the body boundaries B_{I} and B_{II}.

If we now consider wave diffraction by a fixed set of bodies, then in general there are two problems to be considered. Diffraction of an incident wave at mode λ_1 is referred to as Problem 1. Problem 2 refers to diffraction of an incident wave at mode λ_2.

The notations R_{λ_j} and T_{λ_j} $(j = 1, 2)$ are used to denote the reflection and transmission coefficients, respectively, corresponding to waves of wavenumber λ_1 due to an incident wave of wavenumber λ_j $(j = 1, 2)$ while r_{λ_j} and t_{λ_j} $(j = 1, 2)$ are used to denote reflection and transmission coefficients corresponding to waves of wavenumber λ_2 due to an incident wave of wavenumber λ_j $(j = 1, 2)$. Thus the two diffraction problems are characterized by

$$\phi_1 \sim (R_{\lambda_1}, r_{\lambda_1}, 1, 0; T_{\lambda_1}, t_{\lambda_1}, 0, 0), \tag{3.36}$$

$$\phi_2 \sim (R_{\lambda_2}, r_{\lambda_2}, 0, 1; T_{\lambda_2}, t_{\lambda_2}, 0, 0). \tag{3.37}$$

Also $\partial\phi^{\mathrm{I}}/\partial n = 0$ on B_{I}, $\partial\phi^{\mathrm{II}}/\partial n = 0$ on B_{II}. Applying (3.34) to $\phi = \phi_1$ and $\psi = \overline{\phi_1}$, the complex conjugate of ϕ_1, we obtain the identity relating the reflection coefficients $R_{\lambda_1}, r_{\lambda_1}$ and the transmission coefficients $T_{\lambda_1}, t_{\lambda_1}$ as given by

$$|R_{\lambda_1}|^2 + |T_{\lambda_1}|^2 + J(|r_{\lambda_1}|^2 + |t_{\lambda_1}|^2) = 1, \tag{3.38}$$

where

$$J = J_{\lambda_2}/J_{\lambda_1}. \tag{3.39}$$

Similarly we obtain for the diffraction Problem 2

$$|R_{\lambda_2}|^2 + |T_{\lambda_2}|^2 + J(|r_{\lambda_2}|^2 + |t_{\lambda_2}|^2) = J. \tag{3.40}$$

Relations (3.38) and (3.40) are the two desired identities.

Let ψ be a radiation potential function with far-field behavior given by

$$\psi \sim (\Theta^-_{\lambda_1}, \Theta^-_{\lambda_2}, 0, 0; \Theta^+_{\lambda_1}, \Theta^+_{\lambda_2}, 0, 0) \tag{3.41}$$

and on the body boundaries $\partial\psi/\partial n = n_j$. Applying (3.34) to $\phi = \phi_1$ and ψ, we obtain the relation

$$\mathrm{i}\rho_{\mathrm{II}}\omega \left[\rho \int_{B_{\mathrm{I}}} \phi^{\mathrm{I}} \mathscr{M}_s n_j\,\mathrm{d}s + \int_{B_{\mathrm{II}}} \phi^{\mathrm{II}} \mathscr{M}_s n_j\,\mathrm{d}s \right] = -\rho_{\mathrm{II}}\omega J_{\lambda_1} \Theta^-_{\lambda_1}. \tag{3.42}$$

Similarly, for $\phi = \phi_2$ we obtain

$$\mathrm{i}\rho_{\mathrm{II}}\omega \left[\rho \int_{B_{\mathrm{I}}} \phi^{\mathrm{I}} \mathscr{M}_s n_j\,\mathrm{d}s + \int_{B_{\mathrm{II}}} \phi^{\mathrm{II}} \mathscr{M}_s n_j\,\mathrm{d}s \right] = -\rho_{\mathrm{II}}\omega J_{\lambda_2} \Theta^-_{\lambda_2}. \tag{3.43}$$

Relations (3.42) and (3.43) represent the *modified* Haskind–Hanaoka relations in a two-layer fluid having free-surface boundary condition with higher-order derivatives.

Let ϕ and ψ denote the radiation potential functions whose far-field behaviors are given by

$$\phi \sim (A^-_{\lambda_1}, A^-_{\lambda_2}, 0, 0; A^+_{\lambda_1}, A^+_{\lambda_2}, 0, 0),$$

$$\psi \sim (\Theta^-_{\lambda_1}, \Theta^-_{\lambda_2}, 0, 0; \Theta^+_{\lambda_1}, \Theta^+_{\lambda_2}, 0, 0),$$

and on the body boundaries $\partial\phi/\partial n = n_j$ and $\partial\psi/\partial n = n_k$.

Applying (3.34) to ϕ and ψ we obtain

$$\rho \int_{B_{\mathrm{I}}} (\phi^{\mathrm{I}} \mathscr{M}_s n_k - \psi^{\mathrm{I}} \mathscr{M}_s n_j)\, \mathrm{d}s + \int_{B_{\mathrm{II}}} (\phi^{\mathrm{II}} \mathscr{M}_s n_k - \psi^{\mathrm{II}} \mathscr{M}_s n_j)\, \mathrm{d}s = 0. \tag{3.44}$$

Now applying (3.34) to ϕ and $\overline{\psi}$, the complex conjugate of ψ, we obtain

$$\rho \int_{B_{\mathrm{I}}} (\phi^{\mathrm{I}} \mathscr{M}_s n_k - \overline{\psi}^{\mathrm{I}} \mathscr{M}_s n_j)\, \mathrm{d}s + \int_{B_{\mathrm{II}}} (\phi^{\mathrm{II}} \mathscr{M}_s n_k - \overline{\psi}^{\mathrm{II}} \mathscr{M}_s n_j)\, \mathrm{d}s$$
$$= -\mathrm{i}J_{\lambda_1}(A^+_{\lambda_1}\overline{\Theta}^+_{\lambda_1} + A^-_{\lambda_1}\overline{\Theta}^-_{\lambda_1}) - \mathrm{i}J_{\lambda_2}(A^+_{\lambda_2}\overline{\Theta}^+_{\lambda_2} + A^-_{\lambda_2}\overline{\Theta}^-_{\lambda_2}). \tag{3.45}$$

Thus in particular, for $\psi = \overline{\phi}$, the above relation reduces to

$$\rho \int_{B_{\mathrm{I}}} (\phi^{\mathrm{I}} \mathscr{M}_s n_j - \overline{\phi}^{\mathrm{I}} \mathscr{M}_s n_j)\, \mathrm{d}s + \int_{B_{\mathrm{II}}} (\phi^{\mathrm{II}} \mathscr{M}_s n_j - \overline{\phi}^{\mathrm{II}} \mathscr{M}_s n_j)\, \mathrm{d}s$$
$$= -\mathrm{i}J_{\lambda_1}(|A^+_{\lambda_1}|^2 + |A^-_{\lambda_1}|^2) - \mathrm{i}J_{\lambda_2}(|A^+_{\lambda_2}|^2 + |A^-_{\lambda_2}|^2). \tag{3.46}$$

This relation (3.46) is the modified form of the energy-conservation principle for a wave propagation problem in a two-layer fluid having free-surface boundary condition with higher-order derivatives.

Thus we have obtained above various hydrodynamic relations such as the energy identities, the *modified* Haskind–Hanaoka relations and the energy-conservation principle for diffraction and radiation problems in a two-layer fluid having free-surface boundary condition with higher-order derivatives. If we put $c_0 = 1$, $c_m = 0$ ($m = 1, 2, 3, \ldots, m_0$), then these reduce to the corresponding relations in a two-layer fluid with the usual free-surface (cf. Newman, 1976; Linton and McIver, 1995; Yeung and Nguyen, 2000; Ten and Kashiwagi, 2004; Kashiwagi et al., 2006).

4. Conclusion

A modified form of Green's integral theorem for harmonic functions is employed to derive the various hydrodynamic relations such as energy identity for any wave diffraction problem, *modified* Haskind–Hanaoka relation for diffraction and radiation problems and energy-conservation principle for any radiation problem involving a finite number of bodies present in water having free-surface condition with higher-order derivatives. For a two-layer fluid having free-surface condition with higher-order derivatives, two energy identities for diffraction problems and other relations such as *modified* Haskind–Hanaoka relations and energy-conservation principles for radiation problems are also derived by employing this modified Green's integral theorem. For a single-layer fluid, we have considered here both infinite depth and uniform finite depth. For a two-layer fluid, the lower layer is assumed to be of infinite depth. However, appropriate modifications can be made to derive the energy identities in the circumstance when the lower layer is of uniform finite depth, in the case of a two-fluid layer. For oblique waves the potential function satisfies Helmholtz equation and in this case the energy identities with some modifications can be derived by using the same modified Green's integral theorem. When the incident wave is incoming from the opposite direction, it has been checked that the same energy identity is obtained. For a two-layer fluid having a free-surface with first-order derivative, this has also been shown by Linton and McIver (1995).

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Appendix A. Derivation of modified Green's integral theorem

The standard Green's integral theorem is

$$\int_S (\phi\psi_n - \psi\phi_n)\,\mathrm{d}s = 0, \tag{A.1}$$

where ϕ, ψ are harmonic in a region bounded by S. Now, let the curve S have the parametric representations

$$x = X(\xi), \quad y = Y(\xi), \quad \xi \in (\alpha, \beta).$$

Let us define the local orthogonal co-ordinates (s, n) where s is along the curve S and n is normal to it, then the Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial n^2} + \frac{\partial^2}{\partial s^2} + \kappa(s)\frac{\partial}{\partial n},$$

where $\kappa(s)$ is the curvature at point (s, n) on S. At a point on S (cf. Porter, 2000)

$$\frac{\partial}{\partial n} = \frac{1}{\sigma(\xi)}\left(Y'(\xi)\frac{\partial}{\partial x} - X'(\xi)\frac{\partial}{\partial y}\right), \tag{A.2a}$$

$$\frac{\partial}{\partial s} = \frac{1}{\sigma(\xi)}\left(X'(\xi)\frac{\partial}{\partial x} + Y'(\xi)\frac{\partial}{\partial y}\right), \tag{A.2b}$$

where

$$\sigma(\xi) = \{(X'(\xi))^2 + (Y'(\xi))^2\}^{1/2}, \quad \xi \in (\alpha, \beta).$$

We also use the (s, n) co-ordinates to represent a local orthogonal system in the neighborhood of S, and apply (A.2a) and (A.2b) for points away from S (cf. Porter, 2000). A harmonic function ϕ satisfies

$$\frac{\partial^2\phi}{\partial n^2} + \frac{\partial^2\phi}{\partial s^2} + \kappa(s)\frac{\partial\phi}{\partial n} = 0 \tag{A.3}$$

and so also ϕ_n satisfies (A.3), i.e. ϕ_n is harmonic.

Let ψ be a harmonic function in a region bounded by S and let us define

$$\psi_n = \frac{\partial \psi}{\partial n} = \chi. \tag{A.4}$$

Then χ is also harmonic in the same region.

The standard Green's integral theorem for harmonic functions ϕ and χ is

$$\int_S (\phi \chi_n - \chi \phi_n)\, \mathrm{d}s = 0 \tag{A.5}$$

so that

$$\int_S (\phi \psi_{nn} - \psi_n \phi_n)\, \mathrm{d}s = 0. \tag{A.6}$$

Interchanging ψ and ϕ in (A.6) we find

$$\int_S (\psi \phi_{nn} - \phi_n \psi_n)\, \mathrm{d}s = 0. \tag{A.7}$$

Now subtracting (A.7) from (A.6), we obtain

$$\int_S (\phi \psi_{nn} - \psi \phi_{nn})\, \mathrm{d}s = 0. \tag{A.8}$$

Again let us chose

$$\psi_{nn} = \varphi. \tag{A.9}$$

It is obvious that φ is harmonic function.

Similarly Green's integral theorem for ϕ and φ is

$$\int_S (\phi \varphi_n - \varphi \phi_n)\, \mathrm{d}s = 0 \tag{A.10}$$

so that

$$\int_S (\phi \psi_{nnn} - \psi_{nn} \phi_n)\, \mathrm{d}s = 0. \tag{A.11}$$

Interchanging ψ and ϕ in (A.11) we find

$$\int_S (\psi \phi_{nnn} - \phi_{nn} \psi_n)\, \mathrm{d}s = 0. \tag{A.12}$$

Subtracting (A.12) from (A.11), we obtain

$$\int_S (\phi \psi_{nnn} - \psi \phi_{nnn})\, \mathrm{d}s + \int_S (\phi_{nn} \psi_n - \psi_{nn} \phi_n)\, \mathrm{d}s = 0. \tag{A.13}$$

But, the second integral in (A.13) is equal to zero, because of (3.1) with $\phi_n = \phi_1$ and $\psi_n = \psi_1$.

Thus we get from (A.13)

$$\int_S (\phi\psi_{nnn} - \psi\phi_{nnn})\,\mathrm{d}s = 0. \tag{A.14}$$

Repeating this procedure, we can establish the more general result of the type (A.14), where normal derivative of any order for the two harmonic functions ϕ and ψ appear.

Then, utilizing such a general result and also using the operator L_n in (3.3), we now easily establish the modified form of Green's integral theorem (3.2).

Appendix B. Derivation of (3.32)

Using interface conditions (2.11a) and (2.11b), we can write

$$\rho(\phi_y^{\mathrm{I}}\psi^{\mathrm{I}} - \psi_y^{\mathrm{I}}\phi^{\mathrm{I}}) = (\phi_y^{\mathrm{II}}\psi^{\mathrm{II}} - \psi_y^{\mathrm{II}}\phi^{\mathrm{II}}) \quad \text{on } y = 0. \tag{B.1}$$

Differentiating both sides twice with respect to x, and also using the interface conditions we find

$$\rho(\phi_{xxy}^{\mathrm{I}}\psi^{\mathrm{I}} - \psi_{xxy}^{\mathrm{I}}\phi^{\mathrm{I}}) = (\phi_{xxy}^{\mathrm{II}}\psi^{\mathrm{II}} - \psi_{xxy}^{\mathrm{II}}\phi^{\mathrm{II}}) \quad \text{on } y = 0 \tag{B.2}$$

which is equivalent to

$$\rho(\phi_{yyy}^{\mathrm{I}}\psi^{\mathrm{I}} - \psi_{yyy}^{\mathrm{I}}\phi^{\mathrm{I}}) = (\phi_{yyy}^{\mathrm{II}}\psi^{\mathrm{II}} - \psi_{yyy}^{\mathrm{II}}\phi^{\mathrm{II}}) \quad \text{on } y = 0. \tag{B.3}$$

Again, differentiating both sides of (B.3) twice with respect to x and using the interface conditions, we similarly obtain

$$\rho(\phi_{yyyyy}^{\mathrm{I}}\psi^{\mathrm{I}} - \psi_{yyyyy}^{\mathrm{I}}\phi^{\mathrm{I}}) = (\phi_{yyyyy}^{\mathrm{II}}\psi^{\mathrm{II}} - \psi_{yyyyy}^{\mathrm{II}}\phi^{\mathrm{II}}) \quad \text{on } y = 0. \tag{B.4}$$

Then utilizing such a general relations and also using the operator L_y in (3.3), we now establish the result (3.32) at the interface between two-layer.

Appendix C. Derivation of (3.33)

Let us define

$$G_1(y) = \begin{cases} g_1(y), & 0 \leqslant y \leqslant h, \\ \mathrm{e}^{\lambda_1 y}, & -\infty \leqslant y \leqslant 0, \end{cases} \tag{C.1}$$

$$G_2(y) = \begin{cases} g_2(y), & 0 \leqslant y \leqslant h, \\ \mathrm{e}^{\lambda_2 y}, & -\infty \leqslant y \leqslant 0, \end{cases} \tag{C.2}$$

where $g_1(y)$, $g_2(y)$ are defined in (2.18).

From the boundary condition (2.12) at the upper surface $y = h$, we find that

$$\sum_{m=0}^{m_0} (-1)^m c_m \lambda_1^{2m} \frac{\mathrm{d}g_1(y)}{\mathrm{d}y} - K g_1(y) = 0 \quad \text{on } y = h, \tag{C.3}$$

$$\sum_{m=0}^{m_0} (-1)^m c_m \lambda_2^{2m} \frac{\mathrm{d}g_2(y)}{\mathrm{d}y} - K g_2(y) = 0 \quad \text{on } y = h. \tag{C.4}$$

Now we consider the integral

$$I=\rho\int_0^h\left(G_2\frac{\mathrm{d}^2G_1}{\mathrm{d}y^2}-G_1\frac{\mathrm{d}^2G_2}{\mathrm{d}y^2}\right)\mathrm{d}y+\int_{-\infty}^0\left(G_2\frac{\mathrm{d}^2G_1}{\mathrm{d}y^2}-G_1\frac{\mathrm{d}^2G_2}{\mathrm{d}y^2}\right)\mathrm{d}y.$$

This integral reduces to

$$I=\rho\left[G_2\frac{\mathrm{d}G_1}{\mathrm{d}y}-G_1\frac{\mathrm{d}G_2}{\mathrm{d}y}\right]_0^h+\left[G_2\frac{\mathrm{d}G_1}{\mathrm{d}y}-G_1\frac{\mathrm{d}G_2}{\mathrm{d}y}\right]_{-\infty}^0.$$

Using the conditions (C.3) and (C.4) I further reduces

$$I=[\mathrm{e}^{(\lambda_1+\lambda_2)y}(\lambda_1-\lambda_2)]_{y=0}-\rho\left[g_2\frac{\mathrm{d}g_1}{\mathrm{d}y}-g_1\frac{\mathrm{d}g_2}{\mathrm{d}y}\right]_{y=0}.$$

The interface conditions (2.11a) and (2.11b) imply

$$\rho g_1(y)\frac{\mathrm{d}g_2(y)}{\mathrm{d}y}-\lambda_2\mathrm{e}^{(\lambda_1+\lambda_2)y}=\frac{1}{K}\left\{\frac{\mathrm{d}g_1(y)}{\mathrm{d}y}\frac{\mathrm{d}g_2(y)}{\mathrm{d}y}-\lambda_1\lambda_2\mathrm{e}^{(\lambda_1+\lambda_2)y}\right\}\quad\text{on } y=0 \tag{C.5}$$

$$\rho g_2(y)\frac{\mathrm{d}g_1(y)}{\mathrm{d}y}-\lambda_1\mathrm{e}^{(\lambda_1+\lambda_2)y}=\frac{1}{K}\left\{\frac{\mathrm{d}g_1(y)}{\mathrm{d}y}\frac{\mathrm{d}g_2(y)}{\mathrm{d}y}-\lambda_1\lambda_2\mathrm{e}^{(\lambda_1+\lambda_2)y}\right\}\quad\text{on } y=0. \tag{C.6}$$

Using (C.5) and (C.6) we finally obtain

$$I=\rho\int_0^h\left(G_2\frac{\mathrm{d}^2G_1}{\mathrm{d}y^2}-G_1\frac{\mathrm{d}^2G_2}{\mathrm{d}y^2}\right)\mathrm{d}y+\int_{-\infty}^0\left(G_2\frac{\mathrm{d}^2G_1}{\mathrm{d}y^2}-G_1\frac{\mathrm{d}^2G_2}{\mathrm{d}y^2}\right)\mathrm{d}y=0. \tag{C.7}$$

Now from (C.7) after using (2.18), we find

$$\rho\int_0^h g_1(y)g_2(y)\,\mathrm{d}y+\int_{-\infty}^0\mathrm{e}^{(\lambda_1+\lambda_2)y}\,\mathrm{d}y=0. \tag{C.8}$$

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