Sankhyā : The Indian Journal of Statistics
1997, Volume 59, Series B, Pt. 3, pp. 396-420

RELIABILITY FUNCTION OF CONSECUTIVE-k-OUT-OF-n SYSTEMS FOR THE GENERAL CASE

By K.G. RAMAMURTHY
Indian Statistical Institute, Bangalore

SUMMARY. In this paper, we characterise the coefficients in the simple form of the reliability function of Consecutive-k-out-n:G systems. We also provide a table using which the reliability function can be written down when $k \leq n \leq 6k+4$.

1. **Introduction**

We write '(C, k, n)' as a shortened form of 'Consecutive-k-out-of-n'. A $(C, k, n : G)((C, k, n : F))$ system consists of n linearly ordered components and the system functions (fails) if and only if at least k consecutive components function (fail). A $(C, k, n : F)((C, k, n : G))$ system is the dual of $(C, K, n : G)((C, k, n : F))$ system (Chao *et al* (1995, p. 123)). Let $R_{g_n}(p_1, p_2, \ldots, p_n)(R_{f_n}(p_1, p_2, \ldots, p_n))$ denote the reliability function of a $(C, k, n : G)((C, k, n : F))$ system. It is known that

$$R_{f_n}(p_1, p_2, \ldots, p_n) = 1 - R_{g_n}(1-p_1, 1-p_2, \ldots, 1-p_n).$$

for all $(p_1, p_2, \ldots p_n) \in [0, 1]^n$. The derivation of a functional form for R_{g_n} (or equivalently R_{f_n}) is the subject matter of this paper.

In a recent paper (Ramamurthy (1997)) it has been shown that

$$R_{g_n}(p, p, \ldots, p) = \sum_{r=1}^{[\frac{n+1}{k+1}]} (p-1)^{r-1} \left\{ \binom{n-rk+1}{r} p^{rk} - \binom{n-rk}{r} p^{rk+1} \right\}$$

where $[x]$ denotes the integral part of x. We now generalise this result for any $(p_1, p_2, \ldots, p_n) \in [0, 1]^n$.

Recursive equations have been developed for R_{g_n} and R_{f_n}. See for example Kuo *et al* (1990), Hwang (1982) and Shantikumar (1982). $A(C, k, n : F)$ system can be modeled as a nonhomogeneous finite discrete time Markov Chain with k-transient states and one absorbing state. R_{f_n} can then be interpreted as the

---

Paper received. March 1997; revised December 1997.
AMS (1991) *subject classification.* 62N05.
Key words and phrases. Reliability function, simple form, consecutive-k-out-of-n systems.

probability that the number of steps to absorption is more than n (Fu and Hu (1987) and also Chao and Fu (1989)). The computation of R_{f_n} here requires multiplication of n transition probability matrices. Chao et al (1995) have surveyed the literature on reliability studies of (C, k, n) systems.

In this paper we look at the problem from a different angle. Let

$$R_{g_n}(p_1, p_2, \ldots, p_n) = \sum_{S\subseteq\{1,2,\ldots,n\}} \gamma_S^{(n)} \prod_{j\in S} p_j$$

be the simple form of R_{g_n}. It is shown that $\gamma_S^{(n)} \in \{-1, 0, 1\}$ for any $S \subseteq \{1, 2, \ldots, n\}$ and the value of $\gamma_S^{(n)}$ can be determined trivially. If $\Gamma = \{S : S \subseteq \{1, 2, \ldots, n\}$ and $\gamma_S^{(n)} \neq 0\}$, then

$$R_{g_n}(p_1, p_2, \ldots, p_n) = \sum_{S\in\Gamma} \gamma_S^{(n)} \prod_{j\in S} p_j.$$

We give procedures for finding the collection Γ. Finally we provide a table using which $R_{g_n}(p_1, p_2, \ldots, p_n)$ can be written down for $k \leq n \leq 6k + 4$.

2. Notation and Preliminaries

The following notation is used throughout this paper

$[x]$: integral part of x

$\mathcal{P}(A)$: power set of the set A

$|A|$: Cardinality of the set A

A^r : Cartesian product of r copies of the set A

N : the set of positive integers

$S + (r) = \{j : j = s + r, s \in S\}$ for $S \subseteq N \cup \{0\}$ and $r \in N \cup \{0\}$, that is, the translate of the set S through r

$I(r, s) = \{j : j \in N \cup \{0\}$ and $r \leq j \leq s\}$ for $(r, s) \in (N \cup \{0\})^2$

n : the number of components

$I(1, n)$: the component set

$(x_1^S, x_2^S, \ldots, x_n^S)$: binary vector associated with each $S \subseteq I(1,n)$ defined by $x_j^S = 1$ if $j \in S$ and $x_j^S = 0$ if $j \notin S$

ψ a general structure on $I(1,n)$

ψ^D : dual of ψ, another structure on $I(1,n)$

$\mu(\psi) = \{T : T \subseteq I(1,n) \text{ and } \psi(x_1^T, x_2^T, \ldots, x_n^T) = 1\}$: the collection of path sets of the structure ψ

p_j : reliability of component j

$R_\psi(p_1, p_2, \ldots, p_n)$: reliability function ψ

$\sum_{S \subseteq I(1,n)} a_S^\psi \prod_{j \in S} p_j$: the simple form of $R_\psi(p_1, p_2, \ldots, p_n)$

$(C, k, n : G)$: Consecutive-k-out-of-n : G

$(C, k, n : F)$: Consecutive-k-out-of-n : F

k : minimum number of consecutive components required to function (fail) for a $(C,k,n:G)((C,k,n:F))$ system to function (fail), it is assumed $k \geq 2$

$\overline{k}(n) = [\frac{n+1}{k+1}]$

$A_k = \{k, 2k, +1, 3k+2, 4k+3, \ldots\}$

$B_k = \{k+1, 2k+2, 3k+3, 4k+4, \ldots\}$

$\alpha_{k:n} = \{(\ell_1, \ell_2, \ldots, \ell_m) : m \geq 1, (\ell_1, \ell_2, \ldots, \ell_m) \in (A_k \cup B_k)^m \text{ and } \sum_{j=1}^{m} (\ell_j + 1) \leq n+1\}$

$\hat{\alpha}_{k:n} = \{(\ell_1, \ell_2, \ldots, \ell_m) : (\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n} \text{ and } \ell_1 \leq \ell_2 \leq \ldots \leq \ell_m\}$

$b(\ell_1, \ell_2, \ldots, \ell_m) = |\{j : j \in I(1,m) \text{ and } \ell_j \in B_k\}|$ defined for $m \geq 1$ and $(\ell_1, \ell_2, \ldots, \ell_m) \in (A_k \cup B_k)^m$

$\delta(\ell_1, \ell_2, \ldots, \ell_m) = \{S : S = \cup_{i=1}^m (I(0, \ell_i - 1) + (u_i)), u_{i-1} + \ell_{i-1} + 1 \leq u_i \leq n+2-\sum_{j=i}^{m} (\ell_j + 1) \text{ and } i \in I(1,m)\}$ with $u_0 = \ell_0 = 0$ for each $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{kn}$

$\xi_k(r,s) = \{(\ell_1, \ell_2, \ldots, \ell_m) : m \geq 1, (\ell_1, \ell_2, \ldots, \ell_m) \in (A_k \cup B_k)^m \sum_{j=1}^{m} (\ell_j + 1)$
$= r(k+1) + s$ and $b(\ell_1, \ell_2, \ldots, \ell_m) = s\}$ for $(r,s) \in N \times (N \cup \{0\})$.

$\hat{\xi}_k(r,s) = \{(\ell_1, \ell_2, \ldots, \ell_m) : (\ell_1, \ell_2, \ldots, \ell_m) \in \xi_k(r,s)$ and $\ell_1 \leq \ell_2 \ldots \leq \ell_m\}$

$\mu(g_n) = \{S : S \subseteq I(1,n)$ and $S \supseteq I(j, j+k-1)$ for some $j \in I(1, n-k+1)\}$: the collection of path sets of a $(C, k, n : G)$ system.

$R_{g_n}(p_1, p_2, \ldots, p_n)$: the reliability function of a $(C, k, n : G)$ system.

$R_{f_n}(p_1, p_2, \ldots, p_n)$: the reliability function of a $(C, k, n : F)$ system.
$\sum_{S \subseteq I(1,n)} \gamma_S^{(n)} \prod_{j \in S} p_j$: the simple form of $R_{g_n}(p_1, p_2, \ldots, p_n)$

Consider a structure or system with component set $I(1,n)$ and $\{0,1\}^n$ being the collection of component state vectors. Let $\psi : \{0,1\}^n \to \{0,1\}$ be its structure function. Since the knowledge of the structure function is equivalent to the knowledge of the structure, we shall often use the phrase 'structure ψ' in place of 'structure having structure function ψ'. When we need to keep track of the set of components, we say 'structure ψ on $I(1,n)$'. The dual ψ^D of ψ is another structure on $I(1,n)$ defined by

$$\psi^D(x_1, x_2, \ldots, x_n) = 1 - \psi(1 - x_1, 1 - x_2, \ldots, 1 - x_n)$$

for all $(x_1, x_2, \ldots, x_n) \in \{0,1\}^n$. We note that $(\psi^D)^D = \psi$.

Let $S \subseteq I(1,n)$ and $(x_1^S, x_2^S, \ldots, x_n^S)$ be the binary vector associated with S. We call $S(I(1,n) - S)$ a path (cut) set of ψ when $\psi(x_1^S, x_2^S, \ldots, x_n^S) = 1(0)$. We note that $T \subseteq I(1,n)$ is a path (cut) set of ψ if and only if it is cut (path) set of ψ^D.

Recall that $\mu(\psi)$ denotes the collection of path sets of ψ. We call $j \in I(1,n)$ an irrelevant component of ψ if $S - \{j\}$ and $S \cup \{j\} \in \mu(\psi)$ for all $S \in \mu(\psi)$. otherwise we say that j is a relevant component of ψ. It is easy to see that j is a relevant component of ψ if and only if it is a relevant component of ψ^D.

We call ψ a coherent structure on $I(1,n)$ if all the components are relevant and also

1. $\emptyset \notin \mu(\psi)$
2. $I(1,n) \in \mu(\psi)$
3. $S \subseteq T \subseteq I(1,n)$ and $S \in \mu(\psi) \Rightarrow T \in \mu(\psi)$.

It is easy to see that ψ is coherent on $I(1,n)$ if and only if ψ^D is coherent. We refer to Barlow and Proschan (1975) or Kaufman *et al* (1977) or Ramamurthy (1990) for more details about coherent structures.

Suppose there exist constants α_S^ψ for each $S \subseteq I(1,n)$ such that

$$\psi(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq I(1,n)} a_S^\psi \prod_{j \in S} x_j \text{ for } \forall (x_1, x_2, \ldots, x_n) \in \{0,1\}^n.$$

We call the right hand side the simple form of ψ. Here we adopt the convention that $\prod_{j\in S} x_j = 1$ when S is empty. The simple form always exists and is unique (Ramamurthy (1990, p. 29)). Let $S \subseteq I(1,n)$ and $(x_1^S, x_2^S, \ldots, x_n^S)$ be the binary vector associated with S. We note that

$$\psi(x_1^S, x_2^S, \ldots, x_n^S) = \sum_{T \subseteq S} \alpha_T^\psi \prod_{j \in T} x_j^S$$

It follows from the Mobius Inversion Theorem (see Berge (1977) p. 85) or Ramamurthy (1990 p. 31) that for all $S \subseteq ((1,n)$ we have

$$\begin{aligned} \alpha_S^\psi &= \sum_{T \subseteq S} (-1)^{|S-T|} \psi(x_1^T, x_2^T, \ldots, x_n^T) \\ &= \sum_{T \in (\mathcal{P}(S) \cap \mu(\psi))} (-1)^{|S|-|T|} \end{aligned}$$

Suppose now ψ is coherent and $S \notin \mu(\psi)$. We note that $T \notin \mu(\psi)$ for all $T \subseteq S$ and hence $\mathcal{P}(S) \cap \mu(\psi) = \emptyset$. It follows that $\alpha_S^\psi = 0$. However it is possible that $\alpha_S^\psi = 0$ even when $S \in \mu(\psi)$. We refer to Ramamurthy (1990) for further details about simple forms.

Finally let $X_1, X_2, \ldots, X_n$ be independently distributed binary random variables with X_i taking values 1 and 0 with probabilities p_i and $1-p_i$, respectively. We now have

$$\begin{aligned} R_\psi(p_1, p_2, \ldots, p_n) &= Prob\{\psi(X_1, X_2, \ldots, X_n) = 1\} \\ &= E(\psi(X_1, X_2, \ldots, X_n)) \\ &= E \sum_{S \subseteq I(1,n)} a_S^\psi \prod_{j \in S} X_j \\ &= \sum_{S \subseteq I(1,n)} a_S^\psi \prod_{j \in S} p_j \end{aligned}$$

We also call the right hand side the simple form of the reliability function R_ψ. From the earlier discussion we note that the simple form is unique and in fact for $S \subseteq I(1,n)$ we note that a_S^ψ is given by

$$a_S^\psi = \sum_{T \in \mathcal{P}(S) \cap \mu(\psi)} (-1)^{|S|-|T|}$$

Furthermore when ψ is coherent then $a_S^{\psi} = 0$ whenever S is not a path set of ψ.

3. Reliability Function of a Consecutive-k-out-n : G system

A $(C, k, n : G)((C, k, n : F))$ system consists of n linearly ordered component and the system function (fails) if and only if at least k consecutive components function (fail). To avoid trivialities, we shall assume throughout this paper that $n \geq k \geq 2$. Without loss of any generality, we take the component set to be $I(1, n)$ unless otherwise specifically mentioned. A $(C, k, n : F)$ system is the dual of a $(C, k, n : G)$ system. We note that a subset S of $I(1, n)$ is a path (cut) set of $(C, k, n : G)((C, k, n : F))$ system if and only if $S \supseteq I(j, j + k - 1)$ some $j \in I(1, n - k + 1)$. It follows that $\mu(g_n)$ the collection of path sets of a $(C, k, n : G)$ system is given by

$$\mu(g_n) : \{S : S \subseteq I(1, n) \text{ and } S \supseteq I(j, j + k - 1) \text{ for some } j \in I(1, n - k + 1)\}$$

We verify that both $C, k, n : G)$ and $(C, k, n : F)$ systems are coherent. Recall that $R_{g_n}(p_1, p_2, \ldots, p_n)(R_{f_n}(p_1, p_2, \ldots, p_n))$ denotes the reliability function of a $(C, k, n : G)((C, k, n : F))$ system and

$$R_{f_n}(p_1, p_2, \ldots, p_n) = 1 - R_{g_n}(1 - p_1, 1 - p_2, \ldots, 1 - p_n)$$

for all $(p_1, p_2, \ldots, p_n) \in [0, 1]^n$. We also recall that $\gamma_S^{(n)}$ is the coefficient of $\prod_{j \in S} p_j$ in the simple form of R_{g_n}, that is

$$R_{g_n}(p_1, p_2, \ldots p_n) = \sum_{S \subseteq I(1,n)} \gamma_S^{(n)} \prod_{j \in S} p_j.$$

The coefficients $\gamma_S^{(n)}$ are given by

$$\gamma_S^{(n)} = \sum_{T \in \mu(g_n) \cap \mathcal{P}(S)} (-1)^{|S| - |T|} \text{ for all } S \subseteq I(1, n).$$

Furthermore $\gamma_S^{(n)} = 0$ whenever $S \in \mu(g_n)$ and in particular $\gamma_S^{(n)} = 0$ for $|S| < k$. We shall now characterise $\gamma_S^{(n)}$ for any $S \subseteq I(1, n)$.

THEOREM 1. *Let $S \subseteq I(1, n)$ and $r \in I(1, n)$. If $S + (r) \subseteq I(1, n)$ then $\gamma_{S+(r)}^{(n)} = \gamma_S^{(n)}$.*

PROOF. Let S and r be as in the hypothesis. Recall that

$$S + (r) = \{j : j = i + r \text{ and } i \in S\}$$

It follows that

$$\mathcal{P}(S+(r)) = \{j : j = T+(r) \text{ and } T \in \mathcal{P}(S)\}$$

$$\mathcal{P}(S+(r)) \cap \mu(g_n) = \{j : j = T+(r) \text{ and } T \in \mathcal{P}(S) \cap \mu(g_n)\}$$

We now have

$$\begin{aligned}\gamma^{(n)}_{S+(r)} &= \sum_{T\in(S+(r))\cap\mu(g_n)} (-1)^{|S+(r)|-|T|} = \sum_{T\in\mathcal{P}(S)\cap\mu(g_n)} (-1)^{|S|-|T+(r)|} \\ &= \sum_{T\in\mathcal{P}(S)\cap\mu(g_n)} (-1)^{|S|-|T|} = \gamma^{(n)}_S.\end{aligned}$$

□

THEOREM 2. *For $k \le m \le n$ and $S \subseteq I(1,m)$ we have $\gamma^{(m)}_S = \gamma^{(n)}_S$.*

PROOF. Let m and S be as in the hypothesis. For $T \subseteq I(1,m)$ we note that $T \in \mu(g_m)$ if and only if $T \in \mu(g_n)$.

It follows that

$$\gamma^{(m)}_S = \sum_{T\in\mathcal{P}(S)\cap\mu(g_m)} (-1)^{|S|-|T|} = \sum_{T\in\mathcal{P}(S)\cap\mu(g_n)} (-1)^{|S|-|T|} = \gamma^{(n)}_S$$

□

THEOREM 3. *For $m \in I(k,n)$ we have*

$$R_{g_m}(p_1,p_2,\ldots,p_m) = R_{g_n}(p_1,p_2,\ldots,p_m,0,0,\ldots,0)$$

PROOF. Let $m \in I(k,n)$. Using Theorem 2, we have

$$\begin{aligned}R_{g_n}(p_1,p_2,\ldots,p_m,0,0,\ldots,0) &= \sum_{S\subseteq I(1,m)} \gamma^{(n)}_S \prod_{j\in S} p_j \\ &= \sum_{S\subseteq I(1,m)} \gamma^{(m)}_S \prod_{j\in S} p_j \\ &= R_{g_n}(p_1,p_2,\ldots,p_m)\end{aligned}$$

□

LEMMA 1. *Let J and H be disjoint subsets of $I(1,n)$ and $\Gamma = \{S : S = J \cup T$ and $T \in \mathcal{P}(H)\}$. We then have*

$$\sum_{S\in\Gamma} (-1)^{|S|} = \begin{cases} (-1)^{|J|} & \text{if } H = \emptyset \\ 0 & \text{if } H \neq \emptyset \end{cases}$$

PROOF. Let J, H and Γ be as in the hypothesis. If $H = \emptyset$ then $\mathcal{P}(H) = \{\emptyset\}$ and the required result trivially holds. Suppose now $H \neq \emptyset$ and say $|H| = r$. We now have

$$\begin{aligned}\sum_{S\in\Gamma}(-1)^{|S|} &= (-1)^{|J|}\sum_{s=0}^{r}\binom{r}{s}(-1)^s \\ &= (-1)^{|J|}(1-1)^r = 0.\end{aligned}$$

REMARK. Note that we allow the possibility of J being empty in the above lemma.

LEMMA 2. *Let J_1, H_1, J_2, H_2, be disjoint subsets of $I(1,n)$ such that*
(i) H_1 and H_2 are both nonempty.
(ii) there exists an $r \in I(2, n-1)$ such that $J_1 \cup H_1 \subseteq I(1, r-1)$ and $J_2 \cup H_2 \subseteq I(r+1, n)$).
Further let $\Omega_i = \{S : S = J_i \cup T$ and $T \in \mathcal{P}(H_i)\}$ for $i = 1$ and 2 and $\Omega = \{S : S = P \cup Q$ and $(P,Q) \in \Omega_1 \times \Omega_2\}$. We then have

$$\sum_{S\in\Omega\cap\mu(g_n)}(-1)^{|S|} = -\sum_{P\in\Omega_i\cap\mu(g_n)}(-1)^{|P|}.\sum_{Q\in\Omega_2\cap\mu(g_n)}(-1)^{|Q|}$$

PROOF. Let the subsets J_1, J_2, H_1, H_2 of $I(1,n)$ be as in the hypothesis of the lemma. We define

$$\begin{aligned}\Gamma_1 &= \{T : T = P \cup Q \text{ and } (P,Q) \in (\Omega_1 \cap \mu(g_n)) \times \Omega_2\} \\ \Gamma_2 &= \{T : T = P \cup Q \text{ and } (P,Q) \in \Omega_1 \times (\Omega_2 \cap \mu(g_n))\} \\ \Gamma_3 &= \{T : T = P \cup Q \text{ and } (P,Q) \in (\Omega_1 \cap \mu(g_n)) \times (\Omega_2 \cap \mu(g_n))\} \\ b &= \sum_{T\in\Omega\cap\mu(g_n)}(-1)^{|T|}, b_i = \sum_{T\in\Gamma_i}(-1)^{|T|} \text{ for } i = 1,2,3. \\ c_i &= \sum_{T\in\Omega_i}(-1)^{|T|} \text{ and } d_i = \sum_{T\in\Omega_i\cap\mu(g_n)}(-1)^{|T|} \text{ for } i = 1,2\end{aligned}$$

Since H_1 and H_2 are both nonempty, we have in view of Lemma 1 that $c_1 = c_2 = 0$. We note that $\Gamma_3 = \Gamma_1 \cap \Gamma_2$. It is easy to see that

$$P \in \Omega_1 - \mu(g_n) \text{ and } Q \in \Omega_2 - \mu(g_n) \Rightarrow P \cup Q \notin \mu(g_n).$$
$$P \in \Omega_1 \cap \mu(g_n) \Rightarrow P \cup Q \in \mu(g_n) \text{ for all } Q \in \Omega_2.$$
$$Q \in \Omega_2 \cap \mu(g_n) \Rightarrow P \cup Q \in \mu(g_n) \text{ for all } P \in \Omega_1.$$

It now follows that $\Omega \cap \mu(g_n) = \Gamma_1 \cup \Gamma_2$ and hence we have $b = b_1 + b_2 - b_3$. We shall now show $b_1 = b_2 = 0$. If $\Omega_1 \cap \mu(g_n) = \emptyset$, then $\Gamma_1 = \emptyset$ and trivially $b_1 = 0$. Suppose now $\Omega_1 \cap \mu(g_n) \neq \emptyset$. In this case we have $b_1 = d_1.c_2$. Since $c_2 = 0$, it is true that $b_1 = 0$. Similarly we show that $b_2 = 0$. It follows that $b = -b_3$. It is therefore enough to show that $b_3 = d_1 d_2$. We have $b_3 = 0$ whenever $\Gamma_3 = \emptyset$. We note that for $i = 1$ and 2.

$$\Omega_i \cap \mu(g_n) = \emptyset \Rightarrow \begin{cases} d_i = 0 \\ \\ \Gamma_3 = \emptyset \end{cases}$$

It follows that $b_3 = 0 = d_1.d_2$ whenever at least one of the collections $\Omega_1 \cap \mu(g_n)$ or $\Omega_2 \cap \mu(g_n)$ is empty. Now consider the case when $\Omega_1 \cap \mu(g_n)$ and $\Omega_2 \cap \mu(g_n)$ are both nonempty. Since

$$\Gamma_3 = \{T : T = P \cup Q \text{ and } (P, Q) \in (\Omega_1 \cap \mu(g_n)) \times (\Omega_2 \cap \mu(g_n))\}$$

we verify that $b_3 = d_1 d_2$. □

LEMMA 3. *For* $k+2 \leq m \leq n$ *and* $\Omega = \{T : T \in \mathcal{P}(I(1, m))$ *and* $(m-k) \in T\}$ *we have*

$$\sum_{T \in \Omega \cap \mu(g_n)} (-1)^{|T|} = 0.$$

PROOF. For $0 \leq r \leq k$ let

$$\begin{aligned}
\Omega_r &= \{T : T \in \mathcal{P}(I(1, m) \text{ and } T \supseteq I(m-k, m-k+r)\} \\
\xi_r &= \{T : T \in \mathcal{P}(I(1, m-k-1+r)) \text{ and } T \supseteq I(m-k, m-k-1+r)\} \\
\Gamma_r &= \{T : T = P \cup Q \text{ and } (P, Q) \in \xi_r \times \mathcal{P}(I(m-k+1+r, m))\} \\
b_r &= \sum_{T \in \Omega_r \cap \mu(g_n)} (-1)^{|T|}, \\
d_r &= \sum_{T \in \Gamma_r \cap \mu(g_n)} (-1)^{|T|},
\end{aligned}$$

We note that $\Omega_0 = \Omega$ and hence we have to show that $b_0 = 0$. We also observe that $\xi_0 = \mathcal{P}(I(1, m-k-1)), \Gamma_k = \xi_k$ and also

$$\Omega_{k-1} = \{T : T \in \mathcal{P}(I(1, m)) \text{ and } T \supseteq I(m-k, m-1)\}.$$

Since $|I(m-k, m-1)| = k$, it follows that $I(m-k, m-1) \in \mu(g_n)$.

We have

$$\Omega_{k-1} \cap \mu(g_n) = \Omega_{k-1} = \{T : T = I(m-k, m-1)) \cup P \text{ and } P \in \mathcal{P}(H)\}$$

where $H = \{m\} \cup I(1, m-k-1))$. It follows from Lemma 1 that

$$b_{k-1} = \sum_{T \in \Omega_{k-1} \cap \mu(g_n)} (-1)^{|T|} = \sum_{T \in \Omega_{k-1}} (-1)^{|T|} = 0$$

If we can show that $b_{r-1} = b_r$ for $1 \le r \le k-1$, then it follows that $b_0 = 0$. To do this, we note that $\Omega_{r-1} = \Omega_r \cup \Gamma_r$ for $1 \le r \le k-1$ and also Ω_r and Γ_r are disjoint collections of subsets of $I(1, m)$. It follows that $b_{r-1} = b_r + d_r$ for $1 \le r \le k-1$. We have using Lemma 2

$$d_r = \sum_{T \in \Gamma_r \cap \mu(g_n)} (-1)^{|T|} = -\sum_{T \in \xi_r \cap \mu(g_n)} (-1)^{|T|} \sum_{T \in \mathcal{P}(I(m-k+1+r, m) \cap \mu(g_n)} (-1)^{|T|}$$

for $1 \le r \le k-1$. Since $\mathcal{P}(I(m-k+1+r, m)) \cap \mu(g_n) = \emptyset$ for $r \ge 1$, it follows that $d_r = 0$ for $1 \le r \le k-1$. Therefore it must be true that $b_{r-1} = b_r$ for $1 \le r \le k-1$. Since $b_{k-1} = 0$, we have $b_0 = 0$.

Theorem 4. *For $k+2 \le m \le n$ we have $\gamma^{(n)}_{I(l,m)} = \gamma^{(n)}_{I(1,m-k-1)}$*

Proof. We note that $I(1, m) = \Omega \cup \Gamma$ where

$$\begin{aligned} \Omega &= \{T : T \in \mathcal{P}(I(1, m)) \text{ and } m-k \in T\} \\ \Gamma &= \{T : T \in \mathcal{P}(I(1, m)) \text{ and } m-k \notin T\} \end{aligned}$$

and Ω and Γ are disjoint. We have

$$\begin{aligned} \gamma^{(n)}_{I(1,m)} &= \sum_{T \in \mathcal{P}(I(1,m)) \cap \mu(g_n)} (-1)^{m-|T|} \\ &= (-1)^m \left(\sum_{T \in \Omega \cap \mu(g_n)} (-1)^{|T|} + \sum_{T \in \Gamma \cap \mu(g_n)} (-1)^{|T|} \right) \end{aligned}$$

In view of Lemma 3, we have

$$\sum_{T \in \Omega \cap \mu(g_n)} (-1)^{|T|} = 0$$

We note that $\mathcal{P}(I(m-k+1, m)) \cap \mu(g_n) = \{I(m-k+1, m)\}$ and

$$\Gamma = \{T : T = P \cup Q \text{ and } (P, Q) \in \mathcal{P}(I(1, m-k-1)) \times \mathcal{P}(I(m-k+1, m))\}$$

Using Lemma 2, we get

$$\begin{aligned}
\sum_{T \in \Gamma \cap \mu(g_n)} (-1)^{|T|} &= -\sum_{P \in \mathcal{P}(I(1,m-k-1)) \cap \mu(g_n)} (-1)^{|P|} . \sum_{Q \in \mathcal{P}(I(m-k+1,m)) \cap \mu(g_n)} (-1)^{|Q|} \\
&= (-1)^{k+1}(-1)^{m-k-1} \sum_{P \in \mathcal{P}(I(1,m-k-1)) \cap \mu(g_n)} (-1)^{m-k-1-|P|} \\
&= (-1)^m \gamma^{(n)}_{I(1,m-k-1)}
\end{aligned}$$

It now follows that $\gamma^{(n)}_{I(1,m)} = \gamma^{(n)}_{I(1,m-k-1)}$. □

Corollary. *For $(r, s) \in (I(1, n))^2$ such that $s \geq r + k + 1$ we have $\gamma^{(n)}_{I(r,s)} = \gamma^{(n)}_{I(r,s-k-1)}$*

Proof. The case where $r = 1$ has already been proved in Theorem 4. Consider now the case where $r \geq 2$. By Theorem 1, we have $\gamma^{(n)}_{I(r,s)} = \gamma^{(n)}_{I(1,s-r+1)}$. Since $s - r + 1 \geq k + 2$, using first Theorem 4 and then Theorem 1 we get

$$\gamma^{(n)}_{I(r,s)} = \gamma^{(n)}_{I(1,s-r+1)} = \gamma^{(n)}_{I(1,s-r-k)} = \gamma^{(n)}_{I(r,s-k-1)}$$

□

Theorem 5. *Let S_1 and S_2 be two nonempty subsets of $I(1, n)$ such that $S_1 \subseteq I(1, r-1)$ and $S_2 \subseteq I(r + 1, n)$ for some $r \in I(2, n-1)$. We then have $\gamma^{(n)}_{S_1 \cup S_2} = -\gamma^{(n)}_{S_1} . \gamma^{(n)}_{S_1}$.*

Proof. Let S_1 and S_2 be as in the hypothesis. Using Lemma 2 we have

$$\begin{aligned}
\gamma^{(n)}_{S_1 \cup S_2} &= \sum_{T \in \mathcal{P}(S_1 \cup S_2) \cap \mu(g_n)} (-1)^{|S_1|+|S_2|-|T|} \\
&= -\sum_{P \in \mathcal{P}(S_1) \cap \mu(g_n)} (-1)^{|S_1|-|P|} . \sum_{Q \in \mathcal{P}(S_2) \cap \mu(g_n)} (-1)^{|S_2|-|Q|} \\
&= -\gamma^{(n)}_{S_1} . \gamma^{(n)}_{S_2}
\end{aligned}$$

Corollary. *Let $m \geq 2$ and $S_1, S_2, \ldots, S_m$ be m nonempty subsets of $I(1, n)$. Suppose there exists $(r_1, r_2, \ldots, r_{m-1}) \in (I(1, n))^{m-1}$ such that $1 <$*

$r_1 < r_2 < \ldots < r_{m-1} < n$. *and* $S_1 \subseteq I(1, r_1 - 1), S_2 \subseteq I(r_1 + 1, r_2), \ldots, S_m \subseteq I(r_{m-1} + 1, n)$. *We then have*

$$\gamma^{(n)}_{S_1 \cup S_2 \cup \ldots \cup S_m} = (-1)^{m-1} \gamma^{(n)}_{S_1} . \gamma^{(n)}_{S_2} \ldots \gamma^{(n)}_{S_m}.$$

PROOF. Repeated application of Theorem 5

THEOREM 6. *We have*
(i) $\gamma^{(n)}_{\emptyset} = 0 = \gamma^{(n)}_{I(1,s)}$ *for* $r \in (1, k-1)$ *and* $\gamma^{(n)}_{I(1,k)} = 1$
(ii) $\gamma^{(n)}_{I(1,k+1)} = -1$ *for* $n \geq k+1$.

PROOF. We note that

$$R_{g_{k+1}}(p_1, p_2, \ldots, p_k, p_{k+1}) = \prod_{j=1}^{k} p_j + \prod_{j=2}^{k+1} p_j - \prod_{j=1}^{k+1} p_j$$

The required results follow in view of Theorem 2.

THEOREM 7. *For* $(r, s) \in (I(1, n))^2$ *such that* $r \leq s$ *we have*

$$\gamma^{(n)}_{I(r,s)} = \begin{cases} 1 & \text{when } s - r + 1 \equiv k(mod(k+1)) \\ -1 & \text{when } s - r + 1 \equiv 0(mod(k+1)) \\ 0 & \text{otherwise} \end{cases}$$

PROOF. Let r and s be as in the hypothesis and note that $I(r, s)$ is not empty. Suppose $s - r + 1 \equiv k(mod(k+1))$. This implies $s - r + 1 = l(k+1) + k$ or $s = r - 1 + \ell(k+1) + k$ for some $\ell \in N \cup \{0\}$. We now have

$$\begin{aligned} \gamma^{(n)}_{I(r,s)} &= \gamma^{(n)}_{l(r, r-1+\ell(k+1)+k)} \\ &= \gamma^{(n)}_{l(1, l(k+1)+k)} \quad \textit{by Theorem 1} \\ &= \gamma^{(n)}_{I(1,k)} \quad \textit{by Theorem 4} \\ &= 1 \quad \textit{by Theorem 6} \end{aligned}$$

Consider now the case where $s - r + 1 \equiv 0(mod(k+1))$. We note that $s = r - 1 + \ell(k+1)$ for some $\ell \in N$. It follows that

$$\begin{aligned}\gamma^{(n)}_{I(r,s)} &= \gamma^{(n)}_{l(r,r-1+\ell(k+1))}\\ &= \gamma^{(n)}_{l(1,l(k+1))} \text{ by Theorem 1}\\ &= \gamma^{(n)}_{I(1,k+1)} \text{ by Theorem 4}\\ &= -1 \text{ by Theorem 6}\end{aligned}$$

Finally let $s-r+1 \equiv h(mod(k+1))$ where $h \in I(1,k-1)$. We note that $s = r-1+\ell(k+1)+h$ for some $\ell \in N \cup \{0\}$. It follows that

$$\begin{aligned}\gamma^{(n)}_{I(r,s)} &= \gamma^{(n)}_{l(r,r-1+\ell(k+1)+h)}\\ &= \gamma^{(n)}_{l(1,\ell(k+1)+h)} \text{ by Theorem 1}\\ &= \gamma^{(n)}_{I(1,h)} \text{ by Theorem 4}\\ &= 0 \text{ by Theorem 6}\end{aligned}$$

THEOREM 8. *For any nonempty subset S of $I(1,n)$ there exist an $m \in I(1,n)$ and $(r_i, s_i) \in (I(1,n))^2$ for $1 \le i \le m$ such that $1 \le r_1, s_m \le n, r_i \le s_i$ for $1 \le i \le m, r_{i+1} \ge s_i + 2$ for $1 \le i \le m-1$ and*

$$S = \bigcup_{i=1}^{m} I(r_i, s_i)$$

Furthermore

$$\gamma^{(n)}_S = (-1)^{m-1} \prod_{i=1}^{m} \gamma^{(n)}_{I(r_i,s_i)}$$

PROOF. The proof for the first part is constructive in nature. Suppose S is a nonempty subset of $I(1,n)$. Let $h = \max j$ *s.t.* $j \in S$ and put $T_1 = S$. Further let $r_1 = \min j$ *s.t.* $j \in T_1$ and $s_1 = \max j$ *s.t.* $j \in T_1$ and also $i \in T_1$ for $r_1 \le i \le j$. If $s_1 = h$ then $m = 1$ and note that $S = I(r_1, s_1)$. Otherwise put $T_2 = T_1 - I(r_1, s_1)$. Let $r_2 = \min j$ *s.t.* $j \in T_2$ and $s_2 = \max j$ *s.t.* $j \in T_2$ and $i \in T_2$ for $r_2 \le i \le j$. It is easy to verify that $r_2 \ge s_1 + 2$. If $s_2 = h$ then $m = 2$ and note that $S = I(r_1, s_1) \cup I(r_2, s_2)$. Otherwise let $T_3 = T_2 - I(r_2, s_2)$ and continue so on till termination.

The validity of the second part follows from the corollary to Theorem 5.

REMARKS. We call the nonempty collection $\{I(r_i, s_i) : i \in I(1,m)\}$ of Theorem 8 the $R-$ partition of the nonempty subset S of $I(1,n)$. Here m denotes the number of sets which constitute the partition. Since $r_i \le s_i$, we note that each one of the sets $I(r_i, s_i)$ is nonempty. It is easy to see that

$$n \geq |S| + m - 1 = \sum_{i=1}^{m}(s_i - r_i + 1) + m - 1 = \sum_{i=1}^{m}(s_i - r_i) + 2m - 1.$$

THEOREM 9. *Let S be a nonempty subset of $I(1,n)$ and $\{I(r_i, s_i) : i \in I(1,m)\}$ be its R-partition. Further let*

$$\begin{aligned} D_1 &= \{i : i \in I(1,m) \text{ and } s_i - r_i + 1 \equiv k(mod(k+1))\} \\ D_2 &= \{i : i \in I(1,m) \text{ and } s_i - r_i + 1 \equiv 0(mod(k+1))\} \\ D_3 &= \{i : i \in I(1,m) \text{ and } s_i - r_i + 1 \equiv h(mod(k+1)), h \in I(1,k-1)\} \end{aligned}$$

we then have

$$\gamma_S^{(n)} = \begin{cases} 0 & \text{when } D_3 \neq \emptyset \\ (-1)^{|D_1|-1} & \text{when } D_3 = \emptyset. \end{cases}$$

PROOF. Let $S, m, I(r_i, s_i), i \in I(1,m)$ and D_i for $i = 1,2,3$ be as in the hypothesis. Further let $z_i = |D_i|$ for $i = 1,2,3$ and note that $I(1,m) = D_1 \cup D_2 \cup D_3$ and $z_1 + z_2 + z_3 = m$. Since $r_i \leq s_i$ for $1 \leq i \leq m$, in view of Theorem 7, we have.

$$\gamma_{I(r_i,s_i)}^{(n)} = \begin{cases} 1 & \text{if } i \in D_1 \\ -1 & \text{if } i \in D_2 \\ 0 & \text{if } i \in D_3 \end{cases}$$

Using Theorem 8 we get

$$\begin{aligned} \gamma_S^{(n)} &= (-1)^{m-1} \prod_{i=1}^{m} \gamma_{I(r_i,s_i)}^{(n)} \\ &= (-1)^{z_1+z_2+z_3-1} \left(\prod_{i \in D_1} \gamma_{I(r_i,s_i)}^{(n)} \right) \left(\prod_{i \in D_2} \gamma_{I(r_i,s_i)}^{(n)} \right) \left(\prod_{i \in D_3} \gamma_{I(r_i,s_i)}^{(n)} \right) \end{aligned}$$

where we use the convention that

$$\prod_{i \in D_j} \gamma_{I(r_i,s_i)}^{(n)} = 1 \text{ when } D_j = \emptyset \text{ for } j = 1,2,3$$

It now follows that

$$D_3 \neq \emptyset \quad \Rightarrow \quad \gamma_S^{(n)} = 0$$

$$D_3 = \emptyset \quad \Rightarrow \quad z_3 = 0 \Rightarrow \gamma_S^{(n)} = (-1)^{z_1+z_2-1}(-1)^{z_2} = (-1)^{z_1-1}$$

□

We note from Theorem 9 that $\gamma_S^{(n)} \in \{-1, 0, 1\}$ for all $S \subseteq I(1,n)$. Let $\Gamma = \{S : S \subseteq I(1,n) \text{ and } \gamma_S^{(n)} \neq 0\}$. We then have

$$R_{g_n}(p_1, p_2, \ldots, p_n) = \sum_{S \in \Gamma} \gamma_S^{(n)} \prod_{j \in S} p_j.$$

If we can develop a procedure for finding Γ and $\gamma_S^{(n)}$ for each $S \in \Gamma$, the problem of finding a computationally feasible expression for the reliability function R_{g_n} is solved to a great extent. This is what we propose to do.

When we translate suitably one or more sets in the R-partition of a subset S of $I(1,n)$, we get another subset S' of $I(1,n)$ with the property $\gamma_{S'}^{(n)} = \gamma_S^{(n)}$. We make use of this concept to develop a simple procedure for generating Γ. Recall (see the list of notation) that

$$\begin{aligned}
A_k &= \{k, 2k+1, 3k+2, 4k+3, \ldots\} \\
B_k &= \{k+1, 2(k+1), 3(k+1), 4(k+1), \ldots\} \\
\alpha_{k:n} &= \{(\ell_1, \ell_2, \ldots, \ell_m) : m \geq 1, (\ell_1, \ell_2, \ldots, \ell_m) \in (A_k \cup B_k)^m \\
&\qquad \text{and } \sum_{j=1}^{m} (\ell_j + 1) \leq n+1\}
\end{aligned}$$

and also for each $(\ell_1, \ldots, \ell_m) \in (A_k \cup B_k)^m$ we define

$$b(\ell_1, \ell_2, \ldots, \ell_m) = |\{j : j \in I(1,m) \text{ and } \ell_j \in B_k\}|$$

Further we associate with each $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n}$ a collection $\delta(\ell_1, \ell_2, \ldots, \ell_m)$ of subsets of $I(1,n)$ defined by

$$\begin{aligned}
\delta(\ell_1, \ell_2, \ldots, \ell_m) &= \{S : S = \textstyle\bigcup_{i=1}^{m} (I(0, \ell_i - 1) + (u_i)), u_{i-1} + \ell_{i-1} + 1 \leq u_i \\
&\qquad \leq n + 2 - \sum_{j=i}^{m} (\ell_j + 1) \text{ and } i \in I(1,m)\}
\end{aligned}$$

where $\ell_0 = u_0 = 0$. It is now fairly straight forward to verify that

$$\Gamma = \bigcup_{(\ell_1,\ell_2,\ldots,\ell_m)\in\alpha_{k:n}} \delta(\ell_1,\ell_2,\ldots,\ell_m)$$

and note that $\gamma_S^{(n)} = (-1)^{m+1-b(\ell_1,\ell_2,\cdots,\ell_m)}$ for all $S \in \delta(\ell_1,\ell_2,\ldots,\ell_m)$. It follows that

$$R_{g_n}(p_1,p_2,\ldots,p_n) = \sum_{(\ell_1,\ell_2,\ldots,\ell_m)\in\alpha_{k:n}} (-1)^{m+1-b(\ell_1,\ell_2,\ldots,\ell_m)} \sum_{u_1=1}^{h_1} \sum_{u_2=u_1+\ell_1+1}^{h_2} \cdots \sum_{u_m=u_{m-1}+\ell_{m-1}+1}^{h_m} \prod_{i=1}^{m} \left(\prod_{j=u_i}^{u_i+\ell_i-1} p_j \right)$$

where $h_i = n+2-\sum_{j=i}^{m}(\ell_j+1)$.

We note from the definition itself that $\alpha_{k:n}$ is empty when $n<k$. We shall now investigate some more properties of $\alpha_{k:n}$ mainly from the computational point of view.

LEMMA 4. *For $\ell \in N$ we have $\ell+1-(k+1)\overline{k}(\ell) \in I(0,k)$. Furthermore $\overline{k}(\ell) \geq 1$ for $\ell \geq k$.*

PROOF. Recall that $\overline{k}(\ell)$ is the integral part of $(l+1)/(k+1)$, that is

$$\overline{k}(\ell) = \left[\frac{\ell+1}{k+1}\right]$$

It follows that $\ell+1-(k+1)\overline{k}(\ell) \in I(0,k)$. It is trivially true that $\overline{k}(\ell) \geq 1$ when $\ell \geq k$.

LEMMA 5. *Let $m \in N$ and $(\ell_1,\ell_2,\ldots,\ell_m) \in (A_k \cup B_k)^m$ be such that $n+1=\sum_{j=1}^{m}(\ell_j+1)$. We then have*

(i) $b(\ell_1,\ell_2,\ldots,\ell_m) = (k+1)[\dfrac{b(\ell_1,\ell_2,\ldots,\ell_m)}{k+1}] + (n+1-(k+1)\overline{k}(n))$

(ii) $\displaystyle\sum_{j=1}^{m}\overline{k}(\ell_j) = \frac{n+1-b(\ell_1,\ell_2,\ldots\ell_m)}{k+1} = \overline{k}(n) - \left[\frac{b(\ell_1,\ell_2,\ldots,\ell_m)}{k+1}\right]$

(iii) $\displaystyle\left[\frac{b(\ell_1,\ell_2,\ldots,\ell_m))}{k+1}\right] \leq \left[\frac{\overline{k}(n)-(n+1-(k+1)\overline{k}(n))}{k+2}\right]$

PROOF. First of all we note that $0 \leq n+1-(k+1)\overline{k}(n) \leq k$ and

$$\ell_j \in A_k \quad \Rightarrow \quad l_j + 1 = (k+1)\overline{k}(\ell_j)$$

$$\ell_j \in B_k \quad \Rightarrow \quad l_j + 1 = (k+1)\overline{k}(\ell_j) + 1.$$

We now have

$$(n+1) = \sum_{j=1}^{m}(\ell_j + 1) = (k+1)(\overline{k}(\ell_1) + \overline{k}(\ell_j) + \ldots + \overline{k}(\ell_m)) + b(\ell_1, \ell_2, \ldots, \ell_m)$$

If follows that

$$\overline{k}(n) = \left[\frac{n+1}{k+1}\right] = \overline{k}(\ell_1) + \overline{k}(\ell_2) + \ldots + \overline{k}(\ell_m) + \left[\frac{b(\ell_1, \ell_2, \ldots, \ell_m)}{k+1}\right]$$

$$n + 1 - (k+1)\overline{k}(n) = b(\ell_1, \ell_2, \ldots, \ell_m) - \left[\frac{b(\ell_1, \ell_2, \ldots, \ell_m)}{k+1}\right](k+1)$$

This proves (i) and (ii). To prove (iii) we note that

$$(k+1)\overline{k}(n) + (n+1-(k+1)\overline{k}(n)) = n+1 = \sum_{j=1}^{m}(\ell_j+1) \geq b(\ell_1, \ell_2, \ldots, \ell_m)(k+2)$$

Using (i) we get

$$\left[\frac{b(\ell_1, \ell_2, \ldots, \ell_m)}{k+1}\right](k+1)(k+2) \leq (\overline{k}(n) - (n+1-(k+1)\overline{k}(n)))(k+1)$$

It now follows that

$$\left[\frac{b(\ell_1, \ell_2, \ldots, \ell_m)}{k+1}\right] \leq \left[\frac{\overline{k}(n) - (n+1-(k+1)\overline{k}(n))}{k+2}\right]$$

This proves (iii) □

Lemma 6. *When $r(k+1) + s \geq k+1$ we have $\xi_k(r,s) \neq \emptyset$ if only if $s \leq r$.*

Proof. Recall that (see the list of notation)

$$\xi_k(r,s) = \{(\ell_1, \ell_2, \ldots, \ell_m) : m \geq 1, (\ell_1, \ell_2, \ldots, \ell_m) \in (A_k \cup B_k)^m, \sum_{j=1}^{m}(\ell_j + 1) = r(k+1) + s \text{ and } b(\ell_1, \ell_2, \ldots, \ell_m) = s\}$$

Suppose $s \geq r+1$ and also $\xi_k(r,s) \neq \emptyset$. Then there exists a vector $(\ell_1, \ell_2, \ldots, \ell_m) \in \xi_k(r,s)$ for some $m \geq 1$. We now have $r(k+1) + s = \sum_{j=1}^{m} (\ell_j + 1) \geq s(k+2) = s(k+1) + s \geq (r+1)(k+1) + s$ leading to a contradiction. Therefore it must be true that $\xi_k(r,s)$ is empty when $s \geq r$.

Suppose now $s \leq r$. We put

$$\ell_j = \begin{cases} k+1 & \text{for } j = 1 \text{ to } s \\ k & \text{for } j = s+1 \text{ to } r \end{cases}$$

and $m = r$. We now have

$$\sum_{j=1}^{m} (\ell_j + 1) = s(k+2) + (r-s)(k+1) = r(k+1) + s$$

with $b(\ell_1, \ell_2, \ldots, \ell_m) = s$. It follows that $(\ell_1, \ell_2, \ldots, \ell_m) \in \xi_k(r,s)$ and hence $\xi_k(r,s)$ is nonempty. □

LEMMA 7. *For $m \in N$ we have*
(i) $\alpha_{k:n-1} \subseteq \alpha_{k:n}$
(ii) $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n} \Rightarrow m \leq \overline{k}(n)$
(iii) $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n} - \alpha_{k:n-1} \Rightarrow \sum_{j=1}^{m} (\ell_j + 1) = n + 1.$

PROOF. Suppose $m \geq 1$ and $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n-1}$. We note that $(\ell_1, \ell_2, \ldots, \ell_m) \in (A_k \cup B_k)^m$ and also

$$\sum_{j=1}^{m} (\ell_j + 1) \leq n \leq n + 1$$

If follows that $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n}$. This establishes (i).

Suppose now $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n}$ for some $m \geq 1$. We have

$$n + 1 \geq \sum_{j=1}^{m} (\ell_j + 1) \geq m(k+1).$$

It follows that $m \leq \overline{k}(n)$. This proves (ii).

Finally let $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n} - \alpha_{k:n-1}$ for some $m \geq 1$. We have

$$(\ell_1 + 1) + (\ell_2 + 1) + \ldots + (\ell_m + 1) \leq n + 1.$$

If the strict inequality holds above then

$$(\ell_1 + 1) + (\ell_2 + 1) + \ldots + (\ell_m + 1) \leq n$$

which implies $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n-1}$ leading to a contradiction. Therefore it must be true that

$$(\ell_1 + 1) + (\ell_2 + 1) + \ldots + (\ell_{m+1}) = n + 1.$$

This completes the proof. □

THEOREM 10. *Let* $t = n + 1 - (k+1)\overline{k}(n)$ *and*

$$d = \left[\frac{\overline{k}(n) - t}{k+2}\right].$$

We then have

$$\alpha_{k:n} - \alpha_{k:n-1} = \bigcup_{i \in I(0,d)} \Gamma_i$$

where Γ_i *is the collection defined for* $i \in I(0,d)$ *by*

$$\begin{aligned} \Gamma_i &= \{(\ell_1, \ell_2, \ldots, \ell_m) : m \geq 1, (\ell_1, \ell_2, \ldots, \ell_m) \in (A_k \cup B_k)^m, \sum_{j=1}^{m} (\ell_j + 1) \\ &= n + 1 \text{ and } b(\ell_1, \ell_2, \ldots, \ell_m) = t + i(k+1)\} \end{aligned}$$

PROOF. First we note that $0 \leq t \leq k$ and also in view of Lemma 7 we have $\alpha_{k:n} - \alpha_{k:n-1} = \{(\ell_1, \ell_2, \ldots, \ell_m) : m \geq 1,\ (\ell_1, \ell_2, \ldots, \ell_m) \in (A_k \cup B_k)^m$ and $\sum_{j=1}^{m} (\ell_j + 1) = n + 1\}$. Let D and E be the sets defined by $D = \{s : s = b(\ell_1, \ell_2, \ldots, \ell_m)$ for some $m \geq 1$ and $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n} - \alpha_{k:n-1}\}$ and $E = \{s : s = t + i(k+1)$ for some $i \in I(0,d)\}$.

We shall now show that $D = E$. Suppose $s \in D$. Then there exists a vector $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n} - \alpha_{k:n-1}$ for some $m \geq 1$ such that $b(\ell_1, \ell_2, \ldots, \ell_m) = s$. In view of Lemma 5 we have

$$b(\ell_1, \ell_2, \ldots, \ell_m) = [\frac{b(\ell_1, \ell_2, \ldots, \ell_m)}{k+1}](k+1) + t$$

$$[\frac{b(\ell_1, \ell_2, \ldots, \ell_m)}{k+1}] \leq d.$$

It follows that $s \in E$ and hence $D \subseteq E$. Conversely suppose now that $s \in E$. Then there exists an $i \in I(0,d)$ such that $s = t + i(k+1)$. It must be true now that $d \geq 0$. We note that

$$i \leq [\frac{\overline{k}(n) - t}{k+2}] \leq \frac{\overline{k}(n) - t}{k+2}$$

and therefore $t+i(k+1) \leq \overline{k}(n)-i$. If $i=0$, then obviously $\overline{k}(n)-i = \overline{k}(n) > 0$. If $i \neq 0$ then also $\overline{k}_{(n)-i} \geq t+i(k+1) > 0$.

We now put $m = \overline{k}(n) - i$ and also

$$\ell_j = \begin{cases} k+1 & \text{for } j = 1 \text{ to } s \\ k & \text{for } j = s+1 \text{ to } m \end{cases}$$

We note that $(\ell_1, \ell_2, \dots \ell_m) \in (A_k \cup B_k)^m$ and also

$$\begin{aligned} \sum_{j=1}^{m} (\ell_j + 1) &= (k+2)s + (k+1)(m-s) = (k+1)m + s \\ &= (k+1)(\overline{k}(n) - i) + t + i(k+1) \\ &= (k+1)\overline{k}(n) + t = n+1 \end{aligned}$$

Since $b(\ell_1, \ell_2, \dots, \ell_m) = s$, it follows that $s \in D$ and hence $E \subseteq D$. Therefore it is true that $D = E$. Recall that

$$\alpha_{k:n} - \alpha_{k:n-1} = \{(\ell_1, \ell_2, \dots, \ell_m) : m \geq 1,\ (\ell_1, \ell_2, \dots, \ell_m) \in (A_k \cup B_k)^m$$

and

$$\sum_{j=1}^{m} (\ell_j + 1) = n+1\}$$

By conditioning the right hand side such that $b(\ell_1, \ell_2, \dots, \ell_m) = t+i(k+1)$ and considering all the possibilities for i, we get

$$\alpha_{k:n} - \alpha_{k:n-1} = \bigcup_{i \in I(0,d)} \Gamma_i.$$

This completes the proof of the theorem. □

REMARKS. We note that

$$i \in (0,d) \Leftrightarrow \overline{k}(n) - i \geq t + i(k+1).$$

Therefore in Theorem 10, we can replace the condition $i \in I(0,d)$ by the equivalent condition $\overline{k}(n) - i \geq t + i(k+1)$. We note that $I(0,d)$ is empty if and only if $d < 0$.

THEOREM 11. *Let Ω be the collection defined by*

$$\Omega = \{(r,s) : (r,s) \in (N \cup \{0\})^2,\ r \geq s \text{ and } r(k+1) + s = n+1\}.$$

We then have

$$\alpha_{k:n} - \alpha_{k:n-1} = \bigcup_{(r,s)\in\Omega} \xi_k(r,s).$$

PROOF. Let $t = n + 1 - (k+1)\overline{k}(n)$ and also

$$d = [\frac{\overline{k}(n) - t}{k+2}].$$

It is easy to see that

$$(r,s) \in \Omega \Leftrightarrow r = \overline{k}(n) - i,\ \ s = t + i(k+1) \text{ for some } i \in I(0,d).$$

Recall from Theorem 10 that

$$\alpha_{k:n} - \alpha_{k:n-1} = \bigcup_{i\in I(0,d)} \Gamma_i$$

where

$$\Gamma_i = \{(\ell_1, \ell_2, \ldots, \ell_m) : m \geq 1,\ (\ell_1, \ell_2, \ldots, \ell_m) \in (A_k \cup B_k)^m,\ \sum_{j=1}^{m}(\ell_j + 1) = n+1$$

and

$$b(\ell_1, \ell_2, \ldots, \ell_m) = t + i(n+1)\}$$

It now follows that

$$\alpha_{k:n} - \alpha_{k:n-1} = \bigcup_{i\in I(0,d)} \xi_k(\overline{k}(n) - i, t + i(k+1)).$$

Putting now $r = \overline{k}(n) - i$ and $s = t + i(k+1)$, we get

$$\alpha_{k:n} - \alpha_{k:n-1} = \bigcup_{(r,s)\in\Omega} \xi_k(r,s).$$

This completes the proof. □

We can use Theorem 11 for the computation of $\alpha_{k:n} - \alpha_{k:n-1}$ or $\alpha_{k:n}$. For this purpose, we have to compute $\xi_k(r,s)$ for the required values of r and s. We can make the vectors in the collection $\xi_k(r,s)$ independent of k and depend only on r and s by a simple trick. Suppose $(\ell_1, \ell_2, \ldots, \ell_m) \in \xi_k(r,s)$. Instead of ℓ_j, it is enough to keep the information of $r_j = \overline{k}(l_j)$ and whether $\ell_j \in A_k$ or $\ell_j \in B_k$. This we do by keeping the information on r_j and assigning a label $L_j \in \{a, b\}$ to r_j such that $L_j = a(b)$ when $\ell_j \in A_k(\ell_j \in B_k)$. We retrieve the information on ℓ_j by the relation

$$\ell_j = \begin{cases} (k+1)r_j - 1 \text{ if } L_j = a \\ (k+1)r_j \text{ if } L_j = b. \end{cases}$$

We also note that

$$r(k+1) + s = \sum_{j=1}^{m} (\ell_j + 1) = (k+1) \sum_{j=1}^{m} r_j + s$$

and thus $r = r_1 + r_2 + \ldots + r_m$. Conversely let $(r_1, r_2, \ldots, r_m) \in N^m$ and $(L_1, L_2, \ldots, L_m) \in \{a, b\}^m$ where L_j is the label of r_j for $j = 1$ to m. Further let

$$\sum_{j=1}^{m} r_j = r \text{ and } |\{j : L_j = b\}| = s$$

$$\ell_j = \begin{cases} (k+1)r_j - 1 \text{ if } L_j = a \\ (k+1)r_j \text{ if } L_j = b. \end{cases}$$

It is easy to verify that $(\ell_1, \ell_2, \ldots, \ell_m) \in \xi_k(r, s)$. To keep the notation compact, we write the label L_j just above r_j, that is $r_j^{L_j}$. We call $(r_1^{L_1}, r_2^{L_2}, \ldots, r_m^{L_m})$ the k-independent form of $(\ell_1, \ell_2, \ldots, \ell_m)$.

Recall (see list of Notation) that

$$\hat{\xi}_k(r, s) = \{(\ell_1, \ell_2, \ldots, \ell_m) : (\ell_1, \ell_2, \ldots, \ell_m) \in \xi_k(r, s) \text{ and} \ell_1 \leq \ell_2 \leq \ldots \leq \ell_m\}$$

$$\hat{\alpha}_{k:n} = \{(\ell_1, \ell_2, \ldots, \ell_m) : (\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n} \text{ and } \ell_1 \leq \ell_2 \leq \ldots, \leq \ell_m\}$$

Suppose $(\ell_1, \ell_2, \ldots, \ell_m) \in \alpha_{k:n}$ for some $m \geq 1$. We note that $(\ell_{j_1}, \ell_{j_2}, \ldots, \ell_{j_m}) \in \alpha_{k:n}$ for all permutations $j_1, j_2, \ldots, j_m$ of the integers $1, 2, \ldots, m$. Therefore it is enough to find $\hat{\alpha}_{k:n}$. We get $\alpha_{k:n}$ by permuting the components of $(\ell_1, \ell_2, \ldots, \ell_m) \in \hat{\alpha}_{k:n}$ to get all distinct vectors. The same remarks hold true for $\xi_k(r, s)$ and $\hat{\xi}_k(r, s)$. In view of Theorem 11, we have $\hat{\alpha}_{k:n}$ as the union of all $\hat{\xi}_k(r, s)$ such that $r \geq 1, s \geq 0, r \geq s$ and $r(k+1) + s \leq n + 1$.

Table 1 : k-INDEPENDENT FORM OF $\hat{\xi}_k(r,s)$

(r,s)	k-independent form of $\hat{\xi}_k(r,s)$
(1,0)	(1^a)
(1,1)	(1^b)
(2,0)	$(2^a), (1^a, 1^a)$
(2,1)	$(2^b), (1^a, 1^b)$
(2,2)	$(1^b, 1^b)$
(3,0)	$(3^a), (1^a, 2^a), (1^a, 1^a, 1^a)$
(3,1)	$(3^b), (1^a, 2^b), (1^b, 2^a), (1^a, 1^a, 1^b)$
(3,2)	$(1^b, 2^b), (1^a, 1^b, 1^b)$
(3,3)	$(1^b, 1^b, 1^b)$
(4,0)	$(4^a), (1^a, 3^a), (2^a, 2^a), (1^a, 1^a, 2^a), (1^a, 1^a, 1^a, 1^a)$
(4,1)	$(4^b), (1^a, 3^b), (1^b, 3^a), (2^a, 2^b), (1^a, 1^a, 2^b), (1^a, 1^b, 2^a), (1^a, 1^a, 1^a, 1^b)$
(4,2)	$(1^b, 3^b), (2^b, 2^b), (1^a, 1^b, 2^b), (1^b, 1^b, 2^a), (1^a, 1^a, 1^b, 1^b)$
(4,3)	$(1^b, 1^b, 2^b), (1^a, 1^b, 1^b, 1^b)$
(4,4)	$(1^b, 1^b, 1^b, 1^b)$
(5,0)	$(5^a), (1^a, 4^a), (2^a, 3^a), (1^a, 1^a, 3^a), (1^a, 2^a, 2^a), (1^a, 1^a, 1^a, 2^a), (1^a, 1^a, 1^a, 1^a, 1^a)$
(5,1)	$(5^b), (1^a, 4^b), (1^b, 4^a), (2^a, 3^b), (2^b, 3^a), (1^a, 1^a, 3^b), (1^a, 1^b, 3^a), (1^a, 2^a, 2^b)$ $(1^b, 2^a, 2^a), (1^a, 1^a, 1^a, 2^b), (1^a, 1^a, 1^b, 2^a), (1^a, 1^a, 1^a, 1^a, 1^b)$
(5,2)	$(1^b, 4^b), (2^b, 3^b), (1^a, 1^b, 3^b), (1^b, 1^b, 3^a), (1^a, 2^b, 2^b), (1^b, 2^a, 2^b),$ $(1^a, 1^a, 1^b, 2^b), (1^a, 1^b, 1^b, 2^a), (1^a, 1^a, 1^a, 1^b, 1^b)$
(5,3)	$(1^b, 1^b, 3^b), (1^b, 2^b, 2^b), (1^a, 1^b, 1^b, 2^b), (1^a, 1^a, 1^b, 1^b, 1^b)$
(5,4)	$(1^b, 1^b, 1^b, 2^b), (1^a, 1^b, 1^b, 1^b, 1^b)$
(5,5)	$(1^b, 1^b, 1^b, 1^b, 1^b)$

In Table 1, we have tabulated the k-independent form of $\hat{\xi}_k(r,s)$ for $r = 1(1)5$ and $s = 0(1)r$. We get the k-independent form of $\hat{\alpha}_{k:n}$ as the union of all $\hat{\xi}_k(r,s)$ listed in the table such that $r(k+1) + s \le n+1$ provided $k \le n \le 6k+4$.

Example. $k = 3$ and $n = 10$.

We note that $k \le n \le 6k+4$ and hence we can use Table 1. In fact we have

$$\hat{\alpha}_{3:10} = \hat{\xi}_3(1,0) \cup \hat{\xi}_3(1,1) \cup \hat{\xi}_3(2,0) \cup \hat{\xi}_3(2,1) \cup \hat{\xi}_3(2,2).$$

Using Table 1, we get

$$\begin{aligned} \hat{\alpha}_{3:10} &= \{(1^a), (1^b), (2^a), (1^a, 1^a), (2^b), (1^a, 1^b), (1^b, 1^b)\} \\ &= \{(3), (4), (7), (3,3), (8), (3,4)(4,4)\} \end{aligned}$$

It follows that

$$\alpha_{3:10} = \{(3), (4), (3,3), (7), (3,4), (4,3), (8), (4,4)\}$$

This can be verified by direct enumeration. We now have

$$\begin{aligned}
R_{g_{10}}(p_1,p_2,\ldots,p_{10}) &= \sum_{u=1}^{8}\prod_{j=u}^{u+2}p_j - \sum_{u=1}^{7}\prod_{j=u}^{u+3}p_j - \sum_{u=1}^{4}\sum_{v=u+4}^{8}\prod_{j=u}^{u+2}p_j\prod_{j=v}^{v+2}p_j \\
&+ \sum_{u=1}^{4}\prod_{j=u}^{u+6}p_j + \sum_{u=1}^{3}\sum_{v=u+4}^{7}\prod_{j=u}^{u+2}p_j\prod_{j=v}^{v+3}p_j \\
&+ \sum_{u=1}^{3}\sum_{v=u+5}^{8}\prod_{j=u}^{u+3}p_j\prod_{j=v}^{v+2}p_j - \sum_{u=1}^{3}\prod_{j=u}^{u+7}p_j. \\
&- \sum_{u=1}^{2}\sum_{v=u+5}^{7}\prod_{j=u}^{u+3}p_j\prod_{j=v}^{v+3}p_j.
\end{aligned}$$

Further for the particular case $p_1 = p_2 = \ldots = p_{10} = p$, we have

$$\begin{aligned}
R_{g_{10}}(p,p,\ldots,p) &= 8p^3 - 7p^4 - \frac{4\times 5}{2}p^6 + 4p^7 + 3.4p^7 - 3p^8 - \frac{2.3}{2}p^8 \\
&= 8p^3 - 7p^4 - 10p^6 + 16p^7 - 6p^8 \\
&= \binom{10-3+1}{1}p^3 - \binom{10-3}{1}p^4 - (1-p)\left\{\binom{10-6+1}{2}p^6\right. \\
&\left. - \binom{10-6}{2}p^7\right\}.
\end{aligned}$$

This is a particular case of the more general result

$$R_{g_n}(p,p,\ldots,p_n) = \sum_{r=1}^{\overline{k}(n)}(p-1)^{r-1}\left\{\binom{n-rk+1}{r}p^{rk}\binom{n-rk}{r}p^{rk+1}\right\}$$

in Ramamurthy (1997).

References

Barlow, R.E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing.* Holt, Rinehart and Winston, New York.

Berge, C. (1977). *Principles of Combinatorics.* Academic Press, New York and London.

CHAO, M.T. AND FU, J.C. (1989). A limit theorem of certain repairable systems. *Ann. Inst. Statist. Math.* **41**, 809-818.

CHAO, M.T., FU, J.C. AND KOUTRAS, M.V. (1995). Survey of reliability studies of consecutive-k-of-out-n: F and related systems. *IEEE Transactions on Reliability* , **14**, No.1, 120-127.

FU, J.C. AND HU., B. (1987). On reliability of large consecutive-k-out-of-n : F system with $(k-1)$ step Markov dependence. *IEEE Trans. Reliability*, **R-36**, 75-77.

HWANG, K. (1982). Fast solutions for consecutive-k-out-of-n : F system. *IEEE Trans. Reliability*, **R-31**, 447-448.

KAUFMAN, A., GROUCHKO, D. AND CRUON R. (1977). *Mathematical Models for the Study of Reliability Systems*, Academic Press. New York.

KUO, W., ZHANG, W. AND ZUO, M. (1990). A consecutive-k-out-of-n : G system : The mirror image of a consecutive-k-out-of$n$: F system. *IEEE Trans. Reliability*, **30**, 244-253.

RAMMURTHY, K.G. (1990). *Coherent Structures and Simple Games.* Kluwer Academic Publishers, Dordrecht.Boston/London.

−−−− (1997). The reliability function of consecutive-k-out-of-n systems for the i.i.d. case. *Sankhyā Ser. B.*, **59**, Part 2, 209-228.

SHANTHIKUMAR, J.G. (1982). Recursive algorithm to evaluate the reliability of a consecutive-k-out-n:F system. *IEEE Trans. Reliability*, **R-31**, 442-443.

K.G. RAMAMURTHY
16 MIG II, KHB COLONY
NANJANGUD 571301
INDIA