Pseudo-supersymmetry and third order intertwining operator

R. Roychoudhury, B. Roy *

Physics & Applied Mathematics Unit, Indian Statistical Institute, Calcutta 700035, India

Abstract

It has been shown how one can obtain an intertwining operator of order (in terms of derivatives) higher than two, linking a non-Hermitian Hamiltonian to its Hermitian conjugate partner. Explicit third order realization of this intertwining operator is derived for complex Scarf potential.

The variety of non-Hermitian interactions in quantum physics has been used in field theory and statistical mechanics for many years with applications to condensed matter, quantum optics and hadronic and nuclear physics [1]. An important subclass of non-Hermitian operators is the pseudo-Hermitian operators [2], i.e., those operators A, which satisfy

$$\chi A \chi^{-1} = A^{\dagger}, \tag{1}$$

$A^{\dagger}$ being the Hermitian conjugate of A and χ is a linear invertible operator. If χ is non-Hermitian then A is called weakly pseudo-Hermitian [3]. Many interesting properties of both pseudo-Hermitian and weakly pseudo-Hermitian operators have been examined by several authors [4,5]. Particularly, a generalization of supersymmetry [6], namely, pseudo-supersymmetry was developed in Ref. [7] that would apply for general pseudo-Hermitian Hamiltonians. In this Letter an attempt has been made to find a higher order intertwining operator linking a non-Hermitian Hamiltonian to its nonlinear pseudo-supersymmetric partner Hamiltonian [8]. Till date the intertwining method is mostly studied where the intertwining operators are taken to be Hermitian first or second order differential operators and Hamiltonians are in the standard potential forms. Recently the intertwining technique has been generalized to the case where the intertwining operator is third order in derivative [9–12] intertwining two Hermitian Hamiltonians. In [9], a particular form of third order intertwining operator was investigated, which led to a specific class of real shape invariant potentials, while Refs. [10] and [11] were devoted to the construction of real potentials with three lowest energy eigenvalues fixed. In Ref. [12], the most general solution of the intertwining relations with supercharges of third order in derivatives was derived and some particular properties of the spectrum were studied and wave functions for three energy levels were constructed. The motivation for the present work stems from the fact that compared to the studies on Hermitian intertwining operators, the literature on intertwining of Hamiltonians with complex potentials by non-Hermitian higher order intertwining operators for obtaining isospectral partners of complex potential is rather few [13].

Let us now discuss how to construct a third order intertwining operator linking H and its adjoint $H^{\dagger}$. To accomplish this we shall make use of a result obtained in Ref. [14]. There an intertwining operator η of second order was derived explicitly which links a non-Hermitian Hamiltonian to the adjoint of its pseudo supersymmetric partner. In symbols it means

$$\eta H = H_s^{\dagger} \eta, \tag{2}$$

where $H = AB$, and its pseudo supersymmetric partner Hamiltonian $H_s = BA$, A, B being first order differential operators and H is a non-Hermitian diagonalizable Hamiltonian having discrete spectrum consisting of real or complex conjugate pairs of eigenvalues and the multiplicity of complex conjugate eigenvalues are the same. More specifically H admits a complete biorthonormal system of eigenvectors $\{|\psi_n, a\rangle, |\phi_n, a\rangle\}$ which

* Corresponding author.
E-mail addresses: raj@isical.ac.in (R. Roychoudhury), barnana@isical.ac.in (B. Roy).

satisfy the following defining properties [7]:

$$H|\psi_n, a\rangle = E_n|\psi_n, a\rangle, \qquad H^\dagger|\phi_n, a\rangle = E_n^*|\phi_n, a\rangle, \tag{3}$$

$$\langle\phi_m, b|\psi_n, a\rangle = \delta_{mn}\delta_{ab}, \tag{4}$$

$$\sum_n \sum_{a=1}^{d_n} |\phi_n, a\rangle\langle\psi_n, a| = \sum_n \sum_{a=1}^{d_n} |\psi_n, a\rangle\langle\phi_n, a| = 1, \tag{5}$$

where † stands for the adjoint of the corresponding operator, d_n is the multiplicity (degree of degeneracy) of the eigenvalue E_n, n is the spectral label and a and b are degeneracy labels. Since H_s is the partner of H in the sense described above, H_s also admits a complete biorthonormal system of eigenvectors and has either a real spectrum or complex conjugate pairs of eigenvalues and the multiplicity of complex conjugate eigenvalues are the same. Then, as shown by Mostafazadeh, H_s is η_2-pseudo-Hermitian [4], i.e., there exists an operator say η_2 such that

$$\eta_2 H_s = H_s^\dagger \eta_2. \tag{6}$$

Since H and H_s are of the form $H = AB$ and $H_s = BA$, where the operators A and B are of the form $A = \frac{d}{dx} + W(x)$ and $B = -\frac{d}{dx} + W(x)$, $W(x)$ being a function of x, there exists an intertwining operator η_1 such that

$$\eta_1 H = H_s \eta_1. \tag{7}$$

An almost trivial first order solution for η_1 is B or A^{-1}, if the latter exists.

Now using the adjoint of (2) in (7) we get

$$\eta^\dagger \eta_1 H = \eta^\dagger H_s \eta_1 = H^\dagger \eta^\dagger \eta_1. \tag{8}$$

Therefore H is $\tilde{\eta} = \eta^\dagger \eta_1 = \eta_1^\dagger \eta_2^\dagger \eta_1$ pseudo-Hermitian. It is obvious that $\tilde{\eta}$ is an third order operator if η_1 and η_2 are first order operators.

At this point let us remark that since H is non-Hermitian, H_s and $H_s^\dagger$ are non-Hermitian in general, but in some particular cases they may be Hermitian [15]. The intertwining operator η_1 and η_2 may be Hermitian or non-Hermitian. In the former case we have pseudo-Hermiticity [2] while in the latter we have weak pseudo-Hermiticity [3].

Now following Mostafazadeh [7] it is possible to obtain a two-component realization of nonlinear pseudo-supersymmetry [8] in which the state vector $|\tilde{\psi}\rangle$, the nonlinear pseudo-supersymmetry generator $\mathcal{Q}$, and its pseudo-adjoint $\mathcal{Q}^\#$, the Hamiltonian H and the operator η_M are respectively represented as

$$|\tilde{\Phi}\rangle = \begin{pmatrix} |\tilde{\Phi}_+\rangle \\ |\tilde{\Phi}_-\rangle \end{pmatrix}, \qquad \mathcal{Q} = \begin{pmatrix} 0 & 0 \\ \eta & 0 \end{pmatrix}, \tag{9}$$

$$\mathcal{Q}^\# = \begin{pmatrix} 0 & (\eta^\dagger)^{-1} \\ 0 & 0 \end{pmatrix}, \qquad \tilde{H} = \begin{pmatrix} H & 0 \\ 0 & H^\dagger \end{pmatrix},$$

$$\eta_M = \begin{pmatrix} \tilde{\eta}^\dagger & 0 \\ 0 & (\tilde{\eta}^\dagger)^{-1} \end{pmatrix}, \tag{10}$$

where $\tilde{\eta} = \eta^\dagger \eta_1$, $\mathcal{Q}^\# = (\eta_M)^{-1}\mathcal{Q}^\dagger \eta_M$.

It is not difficult to see that H, $H^\dagger$ satisfy the intertwining relations

$$\tilde{\eta} H = H^\dagger \tilde{\eta}, \qquad (\tilde{\eta}^\dagger)^{-1} H^\dagger = H(\tilde{\eta}^\dagger)^{-1}. \tag{11}$$

As a consequence, H and $H^\dagger$ are isospectral, $\tilde{\eta}$ maps the eigenvectors of H to $H^\dagger$ and $(\tilde{\eta}^\dagger)^{-1}$ does the converse except for those eigenvectors that are eliminated by these operators. They also have identical degeneracy structure except possibly for the zero eigenvalue. In analogy with ordinary supersymmetric quantum mechanics they are called nonlinear pseudo-superpartner Hamiltonians.

Using the results given in (11), it can easily be shown that $\{\mathcal{Q}, \mathcal{Q}^\#\}$ commutes with $\tilde{H}$. So the anticommutator is either an identity or a function of $\tilde{H}$. So the nonlinear pseudo-superalgebra is given by

$$\mathcal{Q}^2 = \mathcal{Q}^{\#2} = 0, \qquad \{\mathcal{Q}, \mathcal{Q}^\#\} = F(\tilde{H}), \tag{12}$$

where $\mathcal{Q}^\# = (\eta_M)^{-1}\mathcal{Q}^\dagger \eta_M$ and $F(\tilde{H})$ denotes any function of $\tilde{H}$. At this point it is to be pointed out that in the above mathematical steps it is assumed that inverse of the operators like $\tilde{\eta}^\dagger$, η_M, etc., exist though to find the analytical expression for the inverse of the operators $\tilde{\eta}\dagger$, η_M and to find the explicit form of the function $F(\tilde{H})$ will be a nontrivial mathematical task itself. Our objective here to give explicit realization of third order intertwining operator, which does not need evaluation of the inverses.

The symmetry operator is given by $\eta_1^{-1}(\eta_2^\dagger)^{-1}\eta$ so that

$$\eta_1^{-1}(\eta_2^\dagger)^{-1}\eta H = H\eta_1^{-1}(\eta_2^\dagger)^{-1}\eta. \tag{13}$$

Below we give the realization for $\tilde{\eta}$ in the case of complex Scarf potential.

Complex Scarf potential. For Scarf potential

$$W(x) = -\tanh x + iV_2 \operatorname{sech} x, \tag{14}$$

where V_2 is an arbitrary parameter. Therefore

$$\begin{aligned} H = AB &= -\frac{d^2}{dx^2} + V(x) \\ &= -\frac{d^2}{dx^2} + W^2 + W' \\ &= -\frac{d^2}{dx^2} - \left(V_2^2 + 2\right)\operatorname{sech}^2(x) \\ &\quad - 3iV_2 \operatorname{sech} x \tanh x + 1, \end{aligned} \tag{15}$$

$$\begin{aligned} H_s = BA &= -\frac{d^2}{dx^2} + \tilde{V}(x) \\ &= -\frac{d^2}{dx^2} + W^2 - W' \\ &= -\frac{d^2}{dx^2} - V_2^2 \operatorname{sech}^2(x) \\ &\quad - iV_2 \operatorname{sech} x \tanh x + 1, \end{aligned} \tag{16}$$

$$\begin{aligned} H_s^\dagger &= -\frac{d^2}{dx^2} + \tilde{V}(x)^\dagger \\ &= -\frac{d^2}{dx^2} + \left(W^2 - W'\right)^\dagger \end{aligned}$$

$$
\begin{aligned}
&= -\frac{d^2}{dx^2} - V_2^2 \operatorname{sech}^2(x) \\
&\quad + iV_2 \operatorname{sech} x \tanh x + 1,
\end{aligned} \tag{17}
$$

where † denotes adjoint and

$$
\begin{aligned}
H^\dagger &= -\frac{d^2}{dx^2} + V(x)^\dagger \\
&= -\frac{d^2}{dx^2} + \left(W^2 + W'\right)^\dagger \\
&= -\frac{d^2}{dx^2} - \left(V_2^2 + 2\right) \operatorname{sech}^2(x) \\
&\quad + 3iV_2 \operatorname{sech} x \tanh x + 1.
\end{aligned} \tag{18}
$$

Now following the standard procedure to obtain second order realisation for η [16] and utilizing the factorization property of $\eta = \eta_2\eta_1$ [17] we find

$$
\begin{aligned}
\eta_1 &= -\frac{d}{dx} - \tanh x + iV_2 \operatorname{sech} x, \\
\eta_2 &= -\frac{d}{dx} + iV_2 \operatorname{sech} x.
\end{aligned} \tag{19}
$$

Consequently

$$
\begin{aligned}
\tilde{\eta} &= -\frac{d^3}{dx^3} + 3iV_2 \operatorname{sech} x \frac{d^2}{dx^2} + \left\{\left(3V_2^2 - 2\right) \operatorname{sech}^2 x \right.\\
&\quad \left. - 3iV_2 \operatorname{sech} x \tanh x + \tanh^2 x\right\} \frac{d}{dx} \\
&\quad - \left\{3\left(V_2^2 - 1\right) \operatorname{sech}^2 x \tanh x + iV_2\left(V_2^2 - 1\right) \operatorname{sech}^3 x\right\}.
\end{aligned} \tag{20}
$$

Now if we consider the intertwining relation (8) with $\tilde{\eta}$ given by

$$
\tilde{\eta} = \sum_{n=0}^{3} f_n(x)\partial^n, \qquad \partial = \frac{d}{dx}, \tag{21}
$$

where $f_n(x)$ are complex and $f_3(x) = 1$ is chosen, then one obtains a system of nonlinear differential equations for complex functions $f_n(x)$, $n = 0, 1, 2$ and complex potentials $V(x)$, $V^\dagger(x)$:

$$
\begin{aligned}
&V^\dagger - V = 2f_2', \\
&f_2^2 - f_2' - 2f_1 - 3V = 3a, \\
&2f_2'f_1 - f_1'' - 2f_0' - 3V'' - 2f_2V' = 0, \\
&f_0'' + V''' + f_2V'' + f_1V' - 2f_0f_2' = 0,
\end{aligned} \tag{22}
$$

where a is an arbitrary constant and $f' = \frac{df}{dx}$. Comparing (20) with (21) we get

$$
\begin{aligned}
f_2(x) &= 3iV_2 \operatorname{sech} x, \\
f_1(x) &= \left(3V_2^2 - 2\right) \operatorname{sech}^2 x - 3iV_2 \operatorname{sech} x \tanh x + \tanh^2 x, \\
f_0(x) &= -\left\{\left(V_2^2 - 1\right) \operatorname{sech}^2 x \tanh x \right.\\
&\quad \left. + iV_2\left(V_2^2 - 1\right) \operatorname{sech}^3 x\right\}.
\end{aligned} \tag{23}
$$

It is now straightforward to see that $f_n(x), n = 0, 1, 2$ given in Eq. (23) satisfies Eq. (22).

To summarize, we have found a third order intertwining operator linking a non-Hermitian Hamiltonian to its nonlinear pseudo-supersymmetric partner Hamiltonian. This is done using the second order intertwining operator between non-Hermitian Hamiltonians H and $H_s^\dagger$, where $H_s^\dagger$ is the adjoint of the partner Hamiltonian H_s of H. As a by product we have obtained a highly non trivial symmetry operator for H. Our result is consistent with the one obtained by a direct approach [12] in the sense that the coefficient functions $f_n(x), n = 0, 1, 2$ satisfy the nonlinear differential equations derived in the direct method. So the present method allows us to obtain a new set of solutions for third order intertwining operator in a subtle and straightforward way.

References

[1] H. Feshbach, Theoretical Nuclear Theory: Nuclear Resonance, Wiley, New York, 1992;
P.E. Hodgson, The Nucleon Optical Potential, World Scientific, Singapore, 1995;
C. Itzykson, J.M. Drouffe, Statistical Field Theory, vol. 1, Cambridge Univ. Press, Cambridge, 1989;
N. Moiseyev, Phys. Rep. 302 (1998) 211;
J.G. Muga, J.P. Palao, B. Navarro, J.L. Egusquiza, Phys. Rep. 395 (2004) 357;
A. Ramirez, B. Mielnik, Rev. Mex. Fis. 49S2 (2003) 130.

[2] A. Mostafazadeh, J. Math. Phys. 43 (2002) 205;
A. Mostafazadeh, J. Math. Phys. 43 (2002) 2814;
A. Mostafazadeh, J. Math. Phys. 43 (2002) 5439.

[3] L. Solombrino, J. Math. Phys. 43 (2002) 5439.

[4] A. Mostafazadeh, J. Math. Phys. 43 (2002) 6343;
A. Mostafazadeh, J. Math. Phys. 44 (2003) 943, Erratum;
A. Mostafazadeh, J. Phys. A 37 (2004) 10193;
A. Mostafazadeh, J. Math. Phys. 44 (2003) 974;
A. Mostafazadeh, Czech. J. Phys. 53 (2003) 1079;
A. Mostafazadeh, J. Phys. A 38 (2005) 3213;
A. Mostafazadeh, quant-ph/0506094;
A. Mostafazadeh, quant-ph/0508195;
A. Mostafazadeh, A. Batal, J. Phys. A 37 (2004) 11645.

[5] G. Scolarici, L. Solombrino, J. Math. Phys. 44 (2003) 4450;
G. Scolarici, L. Solombrino, quant-ph/0207163;
A. Blasi, G. Scolarici, L. Solombrino, quant-ph/0310106.

[6] F. Cooper, A. Khare, U. Sukhatme, Phys. Rep. 251 (1995) 267;
G. Junker, Supersymmetric Methods in Quantum and Statistical Mechanics, Springer-Verlag, Berlin, 1996.

[7] A. Mostafazadeh, Nucl. Phys. B 640 (2002) 419.

[8] A. Sinha, P. Roy, J. Math. Phys. 46 (2005) 032102;
A. Gonzalez-Lopez, T. Tanaka, J. Phys. A 39 (2006) 3715.

[9] A.A. Andrianov, F. Cannata, M.V. Ioffe, D.N. Nishnianidze, Phys. Lett. A 266 (2000) 341.

[10] T.V. Kuliy, V.M. Tkachuk, J. Phys. A 32 (1999) 2157.

[11] N. Debergh, J. Ndimubandi, B. Van den Bossche, quant-ph/0209080.

[12] M.V. Ioffe, D.N. Nishnianidze, Phys. Lett. A 327 (2004) 425.

[13] A. Andrianov, F. Cannata, M. Ioffe, D. Nishnianidze, Phys. Lett. A 266 (2000) 341;
G. Levai, F. Cannata, A. Ventura, Phys. Lett. A 300 (2002) 271;
G. Levai, M. Znojil, J. Phys. A 35 (2002) 8793;
G. Levai, M. Znojil, J. Phys. A 33 (2000) 7165;
M. Znojil, Phys. Lett. A 259 (1999) 220;
M. Znojil, J. Phys. A 33 (2000) 4561;
M. Znojil, J. Phys. A 33 (2000) L61;
M. Znojil, M. Tater, J. Phys. A 34 (2001) 1793;
A.A. Andrianov, F. Cannata, J.P. Dedonder, M.V. Ioffe, Int. J. Mod. Phys. A 14 (1999) 2675;
B. Bagchi, C. Quesne, Phys. Lett. A 300 (2002) 18;
B. Bagchi, C. Quesne, Phys. Lett. A 273 (2000) 285;
B. Bagchi, S. Mallik, C. Quesne, Int. J. Mod. Phys. A 16 (2001) 2859;
B. Bagchi, S. Mallik, C. Quesne, Int. J. Mod. Phys. A 17 (2002) 51.

[14] R. Roychoudhury, B. Roy, Phys. Lett. A 361 (2007) 291.
[15] B. Bagchi, R. Roychoudhury, J. Phys. A 33 (2000) L1.
[16] A.A. Andrianov, M.V. Ioffe, V.P. Spiridonov, Phys. Lett. A 174 (1993) 273;
A.A. Andrianov, F. Cannata, M.V. Ioffe, J. Phys. A 30 (1997) 5037;
A.A. Andrianov, A.V. Sokolov, Nucl. Phys. B 660 (2003) 25;
F. Cannata, M.V. Ioffe, D.N. Nishnianidze, J. Phys. A 35 (2002) 1389;
S. Klishevich, M. Plyushchay, Nucl. Phys. B 606 (2001) 583;
H. Aoyama, M. Sato, T. Tanaka, Nucl. Phys. B 619 (2001) 105;
R. Sasaki, K. Takasaki, J. Phys. A 34 (2001) 9533;
D.J. Fernandez, Int. J. Mod. Phys. A 12 (1997) 171;
D.J. Fernandez, E. Sallinas-Hernandez, J. Phys. A 36 (2003) 2537;
D.J. Fernandez, M.L. Glasser, L.M. Nieto, Phys. Lett. A 240 (1998) 15;
D.J. Fernandez, J. Negro, L.M. Nieto, Phys. Lett. A 275 (2000) 338;
B. Roy, R. Roychoudhury, Mod. Phys. Lett. A 19 (2004) 2279.
[17] B. Bagchi, C. Quesne, Phys. Lett. A 301 (2002) 173.