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COVOLUTION POWERS OF PROBABILITES ON STOCHASTIC MATRICES OF ORDER 3

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SUMMARY. In this note we show that for any probability on the space of 3×3 stochastic matrices, its convolution powers coverage unless some periodicities are present. This extends a result of A. Mukherjea for 2×2 case.

1. **Introduction**

There is a vast amount of literature on convergence of convolution powers of probabilities on the space of matrices. In this paper, our starting point is the following beautiful result :

Theorem 1 (A. Mukherjea (1979)). *Let μ be a probability on S, the set of stochastic matrices of order 2 and μ^n denote the nth convolution power of μ. Then the sequence μ^n converges weakly to a probability iff μ is not the point mass at* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. *Further, if $S(\mu)$, the closed support of μ contains a matrix* $\begin{bmatrix} a & 1-a \\ b & 1-b \end{bmatrix}$ *where either $0 < a < 1$ or $0 < b < 1$ then the limit probability is concentrated on K – the kernel of the semigroup S, which consists of all matrices of the form* $\begin{bmatrix} a & 1-a \\ a & 1-a \end{bmatrix}$, $0 \leq a \leq 1$.

In this paper, we shall generalize this result to the case of 3×3 stochastic matrices. Unfortunately, our book keeping is too cumbersome to be carried over to higher dimensions. It should be remarked that the space of stochastic matrices being already compact, tightness criteria (A. Mukherjea, 1991) for the sequence $\{\mu^n\}_{n \geq 1}$ are of little help. It should be noted that a necessary and sufficient condition for convergence of convolution powers is given in Lemma 3,

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p.151 of Rosenblatt (1971). However this condition involves determining the kernel of the closure of $\{\mu^n, n \geq 1\}$ which in practice is not easy to apply. In fact origins of the present problem can be traced to notes 5.4, p.159 of Rosenblatt (1971).

Our interest in the problem stems from various points of view. Firstly, each μ on S gives rise to a Markov Process on $I\!R^3$ via random iterations as follows : If we are at $x \in I\!R^3$, we select a matrix A according to μ and move to Ax (see Berger, 1992). Secondly, in the classical theory of Markov chains with three states, one knows all about the limiting behaviour of powers of the transition matrix. However, if the transition matrix is selected according to some probability law, at each step, one would like to know if the classical result still holds in some form. Thirdly, it is natural to enquire if the neat proposition of Arunava Mukherjea quoted above admits a neat generalization.

The organisation of the paper is as follows. In Section 2, we recall some preliminary results. In Section 3, we classify 3×3 stochastic matrices according to the number of recurrent and transient classes which will be helpful in our book keeping later on. Our arguments regarding convergence of $(\mu^n, n \geq 1)$ start in Section 4 where some simple cases are presented leaving nontrivial cases to Sections 5 and 6. Our main theorem is Theorem 7.1 which is proved in Section 7. The main conclusion is that :- μ^n converges unless some cyclicity is present – as in the classical case. Finally, we conclude with a few remarks in Section 8.

After writing this paper we came to know that S. Dhar and A. Mukherjea have also obtained Theorem 7.1 by different techniques. Their results have in the meanwhile appeared in Dhar and Mukherjea (1997). Their arguments are mainly algebraic in nature and depend on earlier results of Mukherjea and his coauthors. Our argument appears lengthy but is self contained, and is perhaps more probabilistic in nature with explicit computations in some cases where convergence actually occurs. We thank professor A. Mukherjea for his comments on an earlier draft as well as encouragement to publish our argument.

2. **Preliminaries**

Here we introduce some convenient notations. $S^{(d)}$ denotes the set of all $d \times d$ stochastic matrices with usual topology. $S^{(d)}$ is a semigroup with identity under multiplication. For any probability μ on $S^{(d)}, S^{(d)}(\mu)$ denotes the closed support of μ and $SS^{(d)}(\mu)$ denotes the closed semigroup generated by $S^{(d)}(\mu)$. For two probabilities μ, ν on $S^{(d)}$; $\mu \star \nu$ denotes their convolution and μ^n denotes the nth convolution power of μ. $K^{(d)}$ is the Kernel of $S^{(d)}$, i.e., it is the smallest non-empty two-sided ideal of $S^{(d)}$. Then, clearly, $K^{(d)}$ consists of all $d \times d$ stochastic matrices with identical rows. If $SS^{(d)}(\mu) \cap K^{(d)} \neq \phi$, then by a result of Rosenblatt [M. Rosenblatt (1971), p.141], every limit point ν of μ^n is concentrated on $SS^{(d)}(\mu) \cap K^{(d)}$. Observe that $xy = y$ holds for all $x, y \in K^{(d)}$.

This implies that if ν_1 and ν_2 are two probabilities concentrated on $K^{(d)}$ then $\nu_1 * \nu_2 = \nu_2$. As a consequence, if $SS^{(d)}(\mu) \cap K^{(d)} \neq \phi$ and ν_1, ν_2 are two limit points of $\{\mu^n\}_{n\geq 1}$, then $\nu_2 = \nu_1 * \nu_2 = \nu_2 * \nu_1 = \nu_1$. Thus we get

LEMMA 2.1. *Let $d \geq 2$. If $SS^{(d)}(\mu) \cap K^{(d)} \neq \phi$ then μ^n converges.*

Of course, even when $SS^{(d)}(\mu) \cap K^{(d)} = \phi$, μ^n may converge as will be seen later. Now, for $d = 2$, there are only three cases when $SS^{(d)}(\mu) \cap K^{(d)} = \phi$, namely

$$S^{(d)}(\mu) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \text{ or } S^{(d)}(\mu) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

or

$$S^{(d)}(\mu) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

In the 1st case, μ^n does not converge. In the 2nd case, $\mu^n = \mu$ for each n. In the 3rd case, μ^n converges to the probability putting equal mass at $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. But even for $d = 3$, there are a large number of cases where $SS^{(d)}(\mu) \cap K^{(d)} = \phi$. We deal with them in the following sections.

In what follows, we omit the superscripts 'd' from $S^{(d)}$, $K^{(d)}$, $SS^{(d)}(\mu)$ and $S^{(d)}(\mu)$ and write them simply as $S, K, SS(\mu)$ and $S(\mu)$ respectively with the understanding that $d = 3$.

3. Classification of Stochastic Matrices of Order 3

We divide S into certain subsets according to – following classical terminology – the number of recurrent and transient classes :

(1) All three states are recurrent and they form disjoint classes :- Identity matrix is the only matrix in this class.

(2) Two recurrent classes and no transient class :-

$$\underset{(a)}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 1-\alpha \\ 0 & \beta & 1-\beta \end{pmatrix}} \quad \underset{(b)}{\begin{pmatrix} 1-\beta & 0 & \beta \\ 0 & 1 & 0 \\ 1-\alpha & 0 & \alpha \end{pmatrix}} \quad \underset{(c)}{\begin{pmatrix} \alpha & 1-\alpha & 0 \\ \beta & 1-\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

where $0 \leq \alpha < 1, \quad 0 < \beta \leq 1$

(3) Two recurrent and one transient classes :-

$$\underset{(a)}{\begin{pmatrix} 1-\alpha-\beta & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{(b)}{\begin{pmatrix} 1 & 0 & 0 \\ \beta & 1-\alpha-\beta & \alpha \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{(c)}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & 1-\alpha-\beta \end{pmatrix}}$$

where $\alpha, \beta \geq 0$, $0 < \alpha + \beta \leq 1$.

(4) One recurrent class and no transient class :-

$$(i) \underset{(a)}{\begin{pmatrix} 0 & \alpha & 1-\alpha \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \quad \underset{(b)}{\begin{pmatrix} 0 & 1 & 0 \\ 1-\alpha & 0 & \alpha \\ 0 & 1 & 0 \end{pmatrix}} \quad \underset{(c)}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \alpha & 1-\alpha & 0 \end{pmatrix}}$$

where $0 < \alpha < 1$.

$$\text{(ii)} \underset{(a)}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}} \quad \underset{(b)}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$$

(iii) Irreducible, aperiodic matrices.

(5) One recurrent class having two states and one transient class :-

$$\text{(i)} \underset{(a)}{\begin{pmatrix} 1-\gamma-\delta & \gamma & \delta \\ 0 & \alpha & 1-\alpha \\ 0 & \beta & 1-\beta \end{pmatrix}} \quad \underset{(b)}{\begin{pmatrix} 1-\beta & 0 & \beta \\ \delta & 1-\gamma-\delta & \gamma \\ 1-\alpha & 0 & \alpha \end{pmatrix}}$$

$$\underset{(c)}{\begin{pmatrix} \alpha & 1-\alpha & 0 \\ \beta & 1-\beta & 0 \\ \gamma & \delta & 1-\gamma-\delta \end{pmatrix}}$$

where $\gamma, \delta \geq 0$, $0 < \gamma + \delta \leq 1$. Also, $0 < \alpha, \beta < 1$.

$$(ii) \underset{(a)}{\begin{pmatrix} 1-\gamma-\delta & \gamma & \delta \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} \quad \underset{(b)}{\begin{pmatrix} 0 & 0 & 1 \\ \delta & 1-\gamma-\delta & \gamma \\ 1 & 0 & 0 \end{pmatrix}}$$

$$\underset{(c)}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \gamma & \delta & 1-\gamma-\delta \end{pmatrix}}$$

where $\gamma, \delta \geq 0$, $1 \geq \gamma + \delta > 0$.

It should be noted that (i) and (ii) may be put together in the same class. But for book keeping purposes, we are listing them separately.

(6) One recurrent class with one state and other states are transient :

$$\underset{(a)}{\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1-\alpha-\beta & \beta \\ \gamma & \delta & 1-\gamma-\delta \end{pmatrix}} \quad \underset{(b)}{\begin{pmatrix} 1-\gamma-\delta & \gamma & \delta \\ 0 & 1 & 0 \\ \beta & \alpha & 1-\alpha-\beta \end{pmatrix}}$$

$$\begin{pmatrix} 1-\alpha-\beta & \beta & \alpha \\ \delta & 1-\gamma-\delta & \gamma \\ 0 & 0 & 1 \end{pmatrix}$$
$$(c)$$

where $0 \le \alpha, \beta, \gamma, \delta$ and $\alpha + \beta > 0$, $\gamma + \delta > 0$.

REMARK 1. If we consider the following three large subclasses of S, we observe that matrices from (2), (3) and (5) above have come from one of the following types :-

$$\underset{(A)}{\begin{pmatrix} 1-\gamma-\delta & \gamma & \delta \\ 0 & \alpha & 1-\alpha \\ 0 & \beta & 1-\beta \end{pmatrix}} \quad \underset{(B)}{\begin{pmatrix} 1-\beta & 0 & \beta \\ \delta & 1-\gamma-\delta & \gamma \\ 1-\alpha & 0 & \alpha \end{pmatrix}}$$
$$\underset{(C)}{\begin{pmatrix} \alpha & 1-\alpha & 0 \\ \beta & 1-\beta & 0 \\ \gamma & \delta & 1-\gamma-\delta \end{pmatrix}}$$

where only restriction is : $0 \le \gamma, \delta, \gamma+\delta, \alpha, \beta \le 1$

Observe that an appropriate renaming of the states leads from one of the types above to the other types.

REMARK 2. If we remove the restriction $0 < \alpha < 1$ and allow $0 \le \alpha \le 1$ in case of 4(i) - matrices, one gets the following larger subclasses of S :-

$$\underset{(A*)}{\begin{pmatrix} 0 & \alpha & 1-\alpha \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \quad \underset{(B*)}{\begin{pmatrix} 0 & 1 & 0 \\ 1-\alpha & 0 & \alpha \\ 0 & 1 & 0 \end{pmatrix}} \quad \underset{(C*)}{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \alpha & 1-\alpha & 0 \end{pmatrix}}$$

where $0 \le \alpha \le 1$.

In that case $(A*)$ will have, in addition to 4(i)(a) - type matrices, two other matrices – $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ from 5(ii) (b) and 5(ii) (c) respectively. Let us denote them by $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ respectively. Similarly, $(B*)$ will have, in addition to 4(i) (b) – type matrices, $\boldsymbol{M}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ from 5(ii) (c) and $\boldsymbol{M}_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ from 5(ii) (a). And $(C*)$ will have, in addition to 4(i) (c) – type matrices, $\boldsymbol{M}_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ from 5(ii) (a) and $\boldsymbol{M}_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ from 5(ii) (b).

4. **Some Simple Cases**

From now on, μ is a probability on S. We start with some preliminary observations :-

1. For a matrix A of type 4(iii), we know that A^n converges to a matrix with identical rows. So, if a matrix of type 4(iii) is in $S(\mu)$ then $SS(\mu) \cap K \neq \phi$ implying that μ^n converges.

2. If a matrix of type (6) is in $S(\mu)$ then $SS(\mu)$ has a matrix with one of the columns with unities implying that $SS(\mu) \cap K \neq \phi$ and hence μ^n converges.

3. If $S(\mu)$ contains a matrix of type 5(i) then $SS(\mu)$ contains a matrix for which one column will be zero, one column will consist of all a's and the remaining column will consist of all $(1-a)$'s for some $0 < a < 1$. This implies that once again, we have, $SS(\mu) \cap K \neq \phi$ and μ^n converges.

4. If $S(\mu) \subseteq (A_o)$ where (A_o) is the subclass of (A) [of Remark 1, Section 3] given by : $\left\{ \begin{pmatrix} 1-\gamma-\delta & \gamma & \delta \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ : \ 0 \leq \gamma, \delta, \gamma+\delta \leq 1 \right\}$
i.e. (A_o) is the union of all 5(ii) − type matrices and a specific permutation matrix (obtained by interchanging the 2nd and the 3rd rows of the identity matrix), then the supports of μ^k for k even and k odd are disjoint so that μ^n does not converge. Similar subclasses (B_o) and (C_o) of (B) and (C) respectively can also be defined with the same kind of conclusion as above.

5. Apart from the identity matrix $\boldsymbol{e}_o$, there are five more permutation matrices, namely,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\text{and} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We denote them by $\boldsymbol{e}_1, e_2, e_3, e_4$ and $\boldsymbol{e}_5$ respectively. e_1, e_2 and e_3 come from class (2) and e_4 and e_5 come from class 4(ii).

Suppose $S(\mu) \subset \{e_i, 0 \leq i \leq 5\}$. In case $\mu = \delta_{e_i} \quad 1 \leq i \leq 5$ then clearly μ^n does not converge.

If $S(\mu) \subset \{e_1, e_2, e_3\}$ then supports of μ^n & μ^{n+1} are disjoint so that μ^n does not converge.

If $S(\mu)$ is not a singleton & $S(\mu) \subset \{e_o, e_4, e_5\}$ then μ^n converges to the limit which puts equal mass at e_o, e_4 and e_5.

Finally if S_μ contains at least one of e_1, e_2, e_3 and at least one of e_o, e_4, e_5 then it is not difficult to see that μ^n conveges to the limit which puts equal mass at e_i, $0 \leq i \leq 5$.

6. If $S(\mu) \subseteq 5$(ii) (a) or $S(\mu) \subseteq 5$(ii) (b) or $S(\mu) \subseteq 5$(ii)(c) then as in (4) above, μ^n does not converge.

7. If $S(\mu) \subseteq (A*)$ or $S(\mu) \subseteq (B*)$ or $S(\mu) \subseteq (C*)$ then μ^n does not converge because supports of μ^n for n even and for n odd are disjoint.
In particular we have,

8. If $S(\mu) \subseteq 4$(i)(a) or $S(\mu) \subseteq 4$(i)(b) or $S(\mu) \subseteq 4$(i)(c), μ^n does not converge in view of (7) and Remark 2, Section 3.

9. Also, If $S(\mu) = \{M_1, M_2\}$ or $S(\mu) = \{M_3, M_4\}$ or $S(\mu) = \{M_5, M_6\}$ then again by (7) and Remark 2 of Section 3, μ^n does not converge.

10. From the above discussion, it is clear that if $S(\mu)$ contains at least one matrix from either 4(iii)-type, 5(i)-type or 6-type then $SS(\mu) \cap K \neq \phi$ and hence μ^n converges. So, to discuss the convergence of μ^n in the later sections, we shall assume that $S(\mu)$ does not include these three types of matrices.

5. Some Difficult Cases

1. $S(\mu) \subseteq \{$ 5(ii) - type matrices$\}$.

Then we already know from Section 4.6 that if $S(\mu) \subset 5$(ii)(a) - type or $S(\mu) \subset 5$(ii)(b) - type or $S(\mu) \subset 5$(ii)(c)-type, μ^n does not converge. So, let us assume that $S(\mu)$ contains at least one 5(ii)(a)-type and at least one 5(ii)(b)-type matrices. Other cases are analogous. Then, $SS(\mu) \cap K \neq \phi$ unless $S(\mu) = \{M_5, M_6\}$. From Section 4.9, we see that in the latter case, μ^n does not converge. Hence, in the case under consideration, there are only three situations where μ^n does not converge, namely, $S(\mu) = \{M_1, M_2\}$ or $S(\mu) = \{M_3, M_4\}$ or $S(\mu) = \{M_5, M_6\}$.

Remark. 4.10 and 5.1 complete the discussion of convergence of $\{\mu^n\}$ when $S(\mu)$ is contained in $\{$ 5, 6 $\}$ type matrices.

2. $S(\mu) \subseteq \{$ 5(ii) -type, 4-type matrices$\}$

If $S(\mu)$ has at least one 4(i)-type and at least one 4(ii)-type matrix, then $SS(\mu) \cap K \neq \phi$. So, μ^n converges.

If $S(\mu)$ has at least one 4(ii)-type matrix and at least one 5(ii)-type matrix then also $SS(\mu) \cap K \neq \phi$ implying that μ^n converges.

Also, we know from Section 4.5 that if $\mu = \delta_{e_4}$ or δ_{e_5}, μ^n does not converge. But if $S(\mu) = \{e_4, e_5\}, \mu^n$ converges.

So, let us assume that $S(\mu)$ does not have any of the 4(ii)-type matrices. Now, if $S(\mu)$ has at least one 4(i)-type and at least one 5(ii)-type matrix then $SS(\mu) \cap K \neq \phi$ unless

$$S(\mu) \subseteq \{\text{ 4(i)(a)-type }\} \cup \{M_1, M_2\} = (A*)$$

or

$$S(\mu) \subseteq \{\text{ 4(i)(b)-type}\} \cup \{M_3, M_4\} = (B*)$$

or

$$S(\mu) \subseteq \{\text{ 4(i)(c)-type}\} \cup \{M_5, M_6\} = (C*)$$

in which case, μ^n does not converge as argued in Section 4.7.
Remark. 4.10, 5.1 & 5.2 complete the discussion of convergence of $\{\mu^n\}$ when $S(\mu)$ is contained in { 4, 5, 6 } type matrices.

3. $S(\mu) \subseteq \{$ 3-type, 4-type & 5(ii)-type matrices $\}$.

If $S(\mu)$ has at least one 3-type and at least one 4(ii)-type matrix then $SS(\mu) \cap K \neq \phi$ implying that μ^n converges. This fact combined with (2) above allows us to assume that $S(\mu)$ does not contain any 4(ii)-type matrix.

If $S(\mu) \subseteq$ 3(a)-type (or $S(\mu) \subseteq$ 3(b)-type or $S(\mu) \subseteq$ 3(c)-type), $SS(\mu) \cap K = \phi$. But we can settle this case in the affirmative by looking at the kernel of the closed semigroup generated by all 3(a)-matrices (or 3(b)-matrices or 3(c)-matrices respectively). This will be done in Section 6, Case (I).

If $S(\mu)$ has at least one 3(a)-type and at least one 3(b)-type matrices, then $SS(\mu) \cap K \neq \phi$ unless $S(\mu) = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

But in the last case, $\mu^n = \mu$ for all n and hence trivially μ^n converges. Similarly, if $S(\mu)$ has at least one 3(b)-type and at least one 3(c)-type or if $S(\mu)$ has at least one 3(c)-type and at least one 3(a)-type matrices, μ^n converges.
($\star$) If $S(\mu)$ has a 3(a)-type and a 4(i)-type matrices, we get $SS(\mu) \cap K \neq \phi$ except in the following two cases :-

(i) $S(\mu)$ has the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ from 3(a) and at least one matrix from 4(i)(b).

(ii) $S(\mu)$ has the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ from 3(a) and at least one matrix from 4(i)(c).

In these cases, $SS(\mu) \cap K = \phi$. However we can show the convergence of μ^n, by arguing with kernels of appropriate semigroups. We do this in Section 6,

Case (II). (+) If $S(\mu)$ has a 3(a)-type and a 5(ii)-(a)-type matrix, then again $SS(\mu) \cap K = \phi$.

But we may construct the kernel of the closed semigroup generated by all 3(a)-type and 5(ii)-(a)-type matrices and use this to show that μ^n converges. This is done in Section 6, Case (III).

However, if $S(\mu)$ has a 3(a)-type and a 5(ii)-(b)-type or a 5(ii)(c)-type matrix, $SS(\mu) \cap K \neq \phi$ except in the following cases :-

$$S(\mu) = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\}$$

and

$$S(\mu) = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

However, a direct calculation shows that in the first of these two cases, μ^n converges to the limit putting equal mass at

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \& \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly, in the 2nd case also, we can show that μ^n converges to a limit putting equal masses at exactly 4 such matrices.

($\star\star$) If $S(\mu)$ has a 3(a)-type, a 4(i)-type & a 5(ii)-(a)-type matrix then $SS(\mu) \cap K \neq \phi$ except in the following two cases :-

(i) $S(\mu)$ has the 3(a)-type matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the 5(ii)(a)-type matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and at least one 4(i)(b)-type matrix

(ii) $S(\mu)$ has the 3(a)-type matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the 5(ii)(a)-type matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and at least one 4(i)(c)-type matrix

In these cases also though $SS(\mu) \cap K = \phi$, we shall show the convergence of μ^n by arguing with kernels of appropriate semigroups. This will be achieved in Section 6, Case (II). Since the kernels under consideration corresponding to (i) & (ii) of ($\star\star$) are same as those corresponding to (i) & (ii) of ($\star$) respectively, both are dealt with in Case (II) of Section 6.

The remaining cases where $S(\mu) \subset \{$ 3(b)-type, 4(i)-type, 5(ii)-type$\}$ or $S(\mu) \subset \{$ 3(c)-type, 4(i)-type, 5(ii)-type$\}$ can be treated in a similar fashion.

REMARK. 4.10, 5.1, 5.2 and 5.3 complete the discussion of convergence of μ^n when $S(\mu)$ is contained in { 3, 4, 5, 6 } type matrices.

4. $S(\mu) \subset$ { 2-type, 3-type, 4(i)-type, 4(ii)-type & 5(ii)-type matrices}.

Suppose $S(\mu) \subset$ 2(a)-type matrices. Then, state 1 being fixed, the 2×2 case proved by Arunava Mukherjee (1979) shows that μ^n converges iff $\mu \neq \delta_{e_1}$.

Similar conclusions hold if $S(\mu) \subset$ 2(b)-type or $S(\mu) \subset$ 2(c)-type.

If $S(\mu)$ has at least one 2(a)-type and at least one 2(b)-type matrix then $SS(\mu) \cap K \neq \phi$ unless $S(\mu) = \{e_1, e_2\}$. But in the latter case, μ^n does not converge from what was observed in Section 4.5.

(+ +) If $S(\mu)$ has at least one 2(a)-type and at least one 3(a)-type matrix, $SS(\mu) \cap K \neq \phi$ except when $S(\mu)$ has only e_1 from 2(a) and at least one matrix from 3(a). In that case, we construct the kernel for the semigroup generated by all 3(a)-type matrices along with e_1 and argue convergence of $\{\mu^n\}_{n \geq 1}$. Since this kernel will be the same as the one mentioned in (+) of (3), we deal with it in Case (III) of Section 6.

If $S(\mu)$ has at least one 2(a)-type and at least one 3(b)-type or at least one 3(c)-type matrices, $SS(\mu) \cap K \neq \phi$ except when,

(1) $S(\mu)$ has $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ from 3(b) and any 2(a)-type matrix or,

(2) $S(\mu)$ has $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ from 3(c) and any 2(a)-type matrix.

Here again, the 2×2 case shows that μ^n converges.

If $S(\mu)$ has a 2(a)-type and 4(i)(a)-type matrix then $SS(\mu) \cap K = \phi$ but we may construct the kernel for the semigroup generated by all 2(a)-type & 4(i)(a)-type matrices and argue that μ^n converges. We do this in Case (IV) of Section 6.

If $S(\mu)$ has a 2(a)-type and 4(i)(b)-type or 4(i)(c)-type then $SS(\mu) \cap K \neq \phi$ and μ^n converges.

If $S(\mu) \subset$ { 2-type, 4(ii)-type matrices } having at least one of each kind then $SS(\mu) \cap K \neq \phi$ except when $S(\mu) \subseteq \{e_1, e_2, e_3, e_4, e_5\}$. In this last case, in view of Section 4.5, μ^n converges – just note that $S(\mu)$ is not a singleton.

If $S(\mu) \subseteq$ { 2(a)-type, 5(ii)(a)-type matrices },. $SS(\mu) \cap K \neq \phi$ except when $S(\mu) \subseteq \{e_1,$ 5(ii)(a)-type matrices $\}$. In the latter case μ^n does not converge as observed in Section 4.

If $S(\mu)$ has a 2(a)-type and a 5(ii)(b)-type or 5(ii)(c)-type matrices, then $SS(\mu) \cap K \neq \phi$ except when

$$1.\ S(\mu) \subseteq \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 1-\alpha \\ 0 & \beta & 1-\beta \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad 0 \leq \alpha,\ \beta \leq 1 \right\}$$

$$\text{or, } 2.\ S(\mu) \subseteq \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 1-\alpha \\ 0 & \beta & 1-\beta \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad 0 \leq \alpha,\ \beta \leq 1 \right\}$$

In both the cases, $SS(\mu) \cap K = \phi$; still μ^n converges. This follows from Case (IV) in Section 6.

$(+++)$ Finally, if $S(\mu)$ has a 2(a)-type, 3(a)-type & 5(ii)(a)-type matrices, $SS(\mu) \cap K \neq \phi$ except when $S(\mu)$ has e_1 from 2(a), at least one 3(a)-type and at least one 5(ii)(a)-type matrices.

In this case, once again we use kernel argument to show that μ^n converges. It should be noted that kernels for appropriate semigroups in cases $(+)$, $(++)$ & $(+++)$ are same. So, we treat all of them in Case (III) of Section 6.

Remark. 4.10, 5.1, 5.2, 5.3 and 5.4 complete the discussion of convergence of μ^n when $S(\mu)$ is contained in { 2, 3, 4, 5, 6 } type matrices.

5. $e_o \in S(\mu)$.

In this case μ^n always converges as can be seen by going through all the previous cases successively. Firstly, in the four cases to be considered in Section 6, e_o is already allowed in Cases I, III and IV and allowing it in Case II does not cause any problem. Secondly in all the cases considered above whenever convergence holds, it continues to hold even if e_o is present in $S(\mu)$. Finally in the few cases above where convergence failed, including e_o leads to convergence either by direct computation or by observing that $SS(\mu) \cap K \neq \phi$ or by appealing to the cases in Section 6.

Remark. 4.10 and the arguments above conclude the discussion of convergence of μ^n.

6. **Exceptional Cases**

We now show that μ^n converges in the 4 cases mentioned in Section 5. It should be mentioned that these cases are not really exceptional. To avoid interrupting the arguments, we had postponed them.

Case I : $S(\mu) \subseteq$ { 3(a) - type } or $S(\mu) \subseteq$ { 3(b) - type } or $S(\mu) \subseteq$ { 3(c) - type } matrices.

In particular, we consider :-

$$S(\mu) \subseteq \left\{ \begin{pmatrix} 1-\alpha-\beta & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : 0 \leq \alpha,\ \beta \leq 1,\ \alpha+\beta > 0 \right\} = S_3^* \text{ (say).}$$

Let $S_3 = \overline{S_3^*}$. Then S_3 is the closed semigroup generated by all 3(a)-type matrices and the kernel K_3 for S_3 is given by :-

$$K_3 = \left\{ \begin{pmatrix} 0 & \alpha & 1-\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : 0 \leq \alpha \leq 1 \right\}$$

Clearly, $K_3 \cap SS(\mu) \neq \phi$. So, by Rossenblatt's result (M. Rosenblatt (1971)) mentioned in Section 2, any cluster point of $\{\mu^n\}_{n\geq 1}$ will have support $\subseteq K_3$. Since $xy = x$ for all $x, y \in K_3$, we see as in Section 2 that if ν_1 and ν_2 are two cluster points, then $\nu_1 = \nu_2$ so that μ^n converges.

Case II : $S(\mu)$ has at least one 4(i)(b)-type matrix and the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ from 3(a).

The closed semigroup generated by all 4(i)(b)-type matrices is given by

$$S_4 = \left\{ \begin{pmatrix} 1-\alpha & 0 & \alpha \\ 0 & 1 & 0 \\ 1-\alpha & 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1-\beta & 0 & \beta \\ 0 & 1 & 0 \end{pmatrix} \ : \ 0 \leq \alpha, \ \beta \leq 1 \right\}$$

$$= S_{41} \cup S_{42} \quad \text{(say)}$$

where,

$$S_{41} = \left\{ \begin{pmatrix} 1-\alpha & 0 & \alpha \\ 0 & 1 & 0 \\ 1-\alpha & 0 & \alpha \end{pmatrix} \ : \ 0 \leq \alpha \leq 1 \right\}$$

and

$$S_{42} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1-\beta & 0 & \beta \\ 0 & 1 & 0 \end{pmatrix} : 0 \leq \beta \leq 1 \right\}$$

Observe that S_4 already includes the matrices $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ from 3(a), $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ from 3(c), $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ from 5(ii)(a) and $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ from 5(ii)(c); so that the case under consideration is covered by the following claim which we shall prove

Claim : If $\mu(S_4) = 1$ then, μ^n converges as soon as $\mu(S_{41}) > 0$ and $\mu(S_{42}) > 0$.

To do this, we shall indeed show that for every Borel subset A of S_4, $\mu^n(A)$ converges – in particular μ^n converges weakly. We define a map $\phi : S_4 \longrightarrow S_4$

by

$$\phi\begin{pmatrix} 1-\alpha & 0 & \alpha \\ 0 & 1 & 0 \\ 1-\alpha & 0 & \alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1-\alpha & 0 & \alpha \\ 0 & 1 & 0 \end{pmatrix}, \quad 0 \leq \alpha \leq 1$$

and

$$\phi\begin{pmatrix} 0 & 1 & 0 \\ 1-\beta & 0 & \beta \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-\beta & 0 & \beta \\ 0 & 1 & 0 \\ 1-\beta & 0 & \beta \end{pmatrix}, \quad 0 \leq \beta \leq 1$$

Then ϕ is a bijection, $\phi = \phi^{-1}$, $\phi(S_{41}) = S_{42}$ and $\phi(S_{42}) = S_{41}$.

We start by observing that

$$\begin{aligned} x \in S_{41}, y \in S_{41} &\implies xy = y \in S_{41} \\ x \in S_{41}, y \in S_{42} &\implies xy = y \in S_{42} \\ x \in S_{42}, y \in S_{41} &\implies xy = \phi(y) \in S_{42} \\ x \in S_{42}, y \in S_{42} &\implies xy = \phi(y) \in S_{41} \end{aligned}$$

Note that, for any two probabilities μ_1 and μ_2 on S_4,

$$\begin{aligned} \mu_1 * \mu_2(A) &= \int \mu_2[y : xy \in A] \quad \mu_1(dx) \\ &= \int_{S_{41}} \mu_2[y : xy \in A] \quad \mu_1(dx) + \int_{S_{42}} \mu_2[y : xy \in A] \quad \mu_1(dx) \\ &= \int_{S_{41}} \mu_2[y : y \in A] \quad \mu_1(dx) + \int_{S_{42}} \mu_2[y : \phi(y) \in A] \quad \mu_1(dx) \\ &= \mu_2(A)\mu_1(S_{41}) + \mu_2\phi^{-1}(A)\mu_1(S_{42}) \\ &= \mu_2(A)\mu_1(S_{41}) + \mu_2\phi(A)\mu_1(S_{42}) \qquad [\text{ since } \phi = \phi^{-1}] - -(*) \end{aligned}$$

Now let μ be any probability on S_4 with $\mu(S_{41}) = c$, $0 < c < 1$. Let $A \subseteq S_{41}$. Let for $n \geq 1$, α_n and β_n denote $\mu^n(A)$ and $\mu^n(\phi(A))$ respectively.

Then, from (*), for any Borel set B,

$$\mu^{n+1}(B) = \mu * \mu^n(B) = \mu^n(B)\mu(S_{41}) + \mu^n(\phi(B))\mu(S_{42})$$

In particular, setting $B = A$, we get,

$$\begin{aligned} \alpha_{n+1} = \mu^{n+1}(A) &= \mu^n(A)\mu(S_{41}) + \mu^n(\phi(A))\mu(S_{42}) \\ &= \alpha_n c + \beta_n(1-c) \end{aligned}$$

and setting $B = \phi(A)$, we get,

$$\begin{aligned} \beta_{n+1} = \mu^{n+1}(\phi(A)) &= \mu^n(\phi(A))\mu(S_{41}) + \mu^n(A)\mu(S_{42}) \\ &= \beta_n c + \alpha_n(1-c) \end{aligned}$$

So, $$\begin{pmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{pmatrix} = \begin{pmatrix} c & 1-c \\ 1-c & c \end{pmatrix} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} c & 1-c \\ 1-c & c \end{pmatrix}^n \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \text{ as } n \to \infty$$
as $0 < c < 1$.

So $\alpha_n \to \frac{\alpha_1+\beta_1}{2}$ and $\beta_n \to \frac{\alpha_1+\beta_1}{2}$ as $n \to \infty$. Hence, $\mu^n(A)$ and $\mu^n(\phi(A))$ converges completing the proof.

The other two analogues of Case (II) concerning 4(i)(a)-type matrices as well as 4(i)(c)-type matrices can be treated similarly.

Case III : $S(\mu)$ has at least one 3(a)-type and at least one 5(ii)(a)-type matrices.

The closed semigroup generated by 3(a)-type and 5(ii)(a)-type matrices is given by $S_{35} = S_{33} \cup S_{55}$ where
$$S_{33} = \left\{ \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \geq 0, \quad a+b+c = 1 \right\}$$
$$S_{55} = \left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} : a, b, c \geq 0, \quad a+b+c = 1 \right\}.$$
The kernel of this semigroup is $K_{35} = K_{33} \cup K_{55}$ where K_{33} and K_{55} consist of all matrices in S_{33} and S_{55} respectively with $a = 0$. The case under consideration is covered by the following claim which we shall prove

Claim : If $\mu(S_{35}) = 1,\ \mu(S_{33}) > 0,\ \mu(S_{55}) > 0$, then μ^n converges.

From now on we assume that μ is as in the claim and we put $c = \mu(S_{33})$ so that $0 < c < 1$.

If Q is any limit point of μ^n then by Rosenblatt's result, $Q(K_{35}) = 1$. We now argue that $Q(K_{33}) = Q(K_{55}) = \frac{1}{2}$. First note that for $x, y \in S_{35}$ we have $xy \in S_{33}$ iff either both x, y are in S_{33} or both x, y are in S_{55}. As a consequence if we let $\alpha_n = \mu^n(S_{33})$ then
$$\begin{aligned} \alpha_{n+1} &= \int \mu^n(y : xy \in S_{33}) d\mu(x) \\ &= \alpha_n c + (1-\alpha_n)(1-c) \end{aligned}$$
Thus
$$\begin{aligned} \alpha_{n+1} &= \alpha_1 (2c-1)^n + (1-c) \sum_{k=0}^{n-1} (2c-1)^k \\ &\to \frac{1}{2} \qquad \text{as } 0 < c < 1. \end{aligned}$$

Let ϕ be the map from S_{35} to S_{35} which interchanges the last two columns. Then ϕ is a bijection, $\phi = \phi^{-1}$; $\phi(S_{33}) = S_{55}$; $\phi(K_{33}) = K_{55}$. Moreover for $x, y \in K_{35}$ we have $x.y = x$ or $\phi(x)$ according as $y \in K_{33}$ or $y \in K_{55}$. As a consequence for two probabilities Q_1, Q_2 supported on K_{35} it is easy to see that

$$Q_1 * Q_2(A) = \frac{Q_1(A) + Q_1(\phi(A))}{2}$$

Thus if Q_1, Q_2 are two limit points of (μ^n) then using the fact that $Q_1 * Q_2 = Q_2 * Q_1$ we get

$$\frac{Q_1(A) + Q_1(\phi(A))}{2} = \frac{Q_2(A) + Q_2(\phi(A))}{2}$$

If only we could show directly – a fact that emerges eventually – that $Q(A) = Q(\phi(A))$ for any limit point Q of (μ^n) we can conclude from the above equation that (μ^n) has a unique limit point and hence converges. Since we could not do this, we take a different approach to establish the convergence of (μ^n). It will be convenient to have a sequence of i.i.d. matrices $X_1, X_2, \cdots$ each having distribution μ so that $Y_n = X_1 \cdots X_n$ has distribution μ^n. First observe that if $\mu \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\} = 1$ then μ^n converges to Q which puts mass $\frac{1}{2}$ at each of these two matrices – a fact already pointed out in Section 4. From now on we assume that this is not the case. In other words if Z denotes the first entry of the random matrix X_1 then $\mu(Z < 1) > 0$. Note that as a consequence if $p = E(Z)$ then $0 \le p < 1$.

To make the arguments transparent we shall first consider the case $\mu(K_{35}) > 0$ – though the proof for the general case applies here too. In this case we show that for every Borel set $A \subset S_{35}$, $\mu^n(A)$ converges. Since $\mu(K_{35}) > 0$, almost surely $X_N \in K_{35}$ for some random integer N and then of course for all $n > N$, $Y_n \in K_{35}$. As a consequence if $A = S_{35} - K_{35}$ then $\mu^n(A) \to 0$ (as it should). Now fix any Borel set $A \subset K_{35}$. We show that $\{\mu^n(A)\}$ is a cauchy sequence. To this end fix $\epsilon > 0$. Choose an integer k so that $P(N < k) > 1 - \epsilon/4$ and also that $|\alpha_n - \frac{1}{2}| < \epsilon/4$ for $n \ge k$. Recall that $\alpha_n = \mu^n(S_{35}) \to 1/2$. Now for any $n > 2k$

$$\begin{aligned}
\mu^n(A) &= P(Y_n \in A) \\
&= P(Y_k \in A, \prod_{i=k+1}^{n} X_i \in S_{33}) \\
&\quad + P(Y_k \in \phi(A); \prod_{i=k+1}^{n} X_n \in S_{55}) \\
&\quad + P(Y_k \notin K_{35};\ Y_n \in A) \\
&= \alpha_{n-k}\mu^k(A) + (1 - \alpha_{n-k})\mu^k(\phi(A)) \\
&\quad + P(Y_k \notin K_{35},\ Y_n \in A)
\end{aligned}$$

Since $|\alpha_{n-k} - \frac{1}{2}| < \epsilon/4$ and $P(Y_k \notin K_{35}) < \epsilon/4$ we get that for $n > 2k$,

$$|\mu^n(A) - \frac{\mu^k(A) + \mu^k(\phi(A))}{2}| < \epsilon/2$$

showing that for $n, m > 2k$ $|\mu^n(A) - \mu^m(A)| < \epsilon$ to complete the proof.

We shall now consider the general case. It suffices to show that for every (bounded) continuous function f on S_{35} with bounded first derivatives $\int f d\mu^n$ converges — or that it is a cauchy sequence. Define the numbers

$$a_n = E[f(Y_n)1_{Y_n \in S_{33}}], \;\; b_n = E[f(\phi(Y_n))1_{Y_n \in S_{55}}]$$

$$c_n = E[f(Y_n)1_{Y_n \in S_{55}}], \;\; d_n = E[f(\phi(Y_n))1_{Y_n \in S_{33}}]$$

and the matrix M_n by

$$M_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

Let Z_n denote the first entry of Y_n. Explicit calculations show that if $X_{n+1} \in S_{33}$ then $Y_n X_{n+1} - Y_n$ has second and third rows null while each entry in first row is smaller then Z_n in modulus. On the other hand, if $X_{n+1} \in S_{55}$ then $Y_n X_{n+1} - \phi(Y_n)$ has second and third rows null while each entry in the first row is smaller than Z_n in modulus. This fact combined with the meanvalue theorem yields that $|f(Y_{n+1}) - f(Y_n)| \leq kZ_n$ when $X_{n+1} \in S_{33}$ and $|f(Y_{n+1}) - f(\phi(Y_n))| \leq kZ_n$ when $X_{n+1} \in S_{55}$ where k is a constant depending on the first derivates of f which were assumed bounded. Observing that

$$\begin{aligned} a_{n+1} &= E[f(Y_{n+1})1_{Y_n \in S_{33}} 1_{X_{n+1} \in S_{33}}] \\ &\quad + E[f(Y_{n+1})1_{Y_n \in S_{55}} 1_{X_{n+1} \in S_{55}}] \end{aligned}$$

we obtain

$$|a_{n+1} - [ca_n + (1-c)b_n]| \leq kE(Z_n)$$

Observing that Z_n is nothing but the product of the first entries of $X_1, \cdots, X_n$ we have $E(Z_n) = p^n$. Recall that $p = E(Z_1)$ and $0 \leq p < 1$. Thus

$$|a_{n+1} - [ca_n + (1-c)b_n]| \leq kp^n$$

Letting $\mathcal{C}$ be the matrix $\begin{pmatrix} c & 1-c \\ 1-c & c \end{pmatrix}$ and U be the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, a similar calculation with b_n, c_n, d_n gives us

$$M_n\mathcal{C} - p^n U \leq M_{n+1} \leq M_n\mathcal{C} + p^n U$$

entrywise. Noting that $\mathcal{C}^n$ converges to the matrix with all entries $\frac{1}{2}$ and the fact that Σp^n converges it is not difficult to show that entries of M_n form cauchy sequences. Observing that $E[f(Y_n)] = a_n + c_n$ we conclude that it is a cauchy sequence to complete the proof. Incidentally, notice that $E(f(Y_n))$ and $E(f(\phi(Y_n)))$ have the same limit.

This completes the proof of the claim.

Case IV :- $S(\mu)$ has at least one 2(a)-type and at least one 4(i)(a)-type matrix.

The closed semigroup generated by all 2(a)-type and all 4(i)(a)-type matrices is given by :-

$$S_{24} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 1-\alpha \\ 0 & \beta & 1-\beta \end{pmatrix}, \quad \begin{pmatrix} 0 & \delta & 1-\delta \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; : \; 0 \leq \alpha, \beta, \delta \leq 1 \right\}$$

Then the corresponding kernel is given by

$$K_{24} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 1-\gamma \\ 0 & \gamma & 1-\gamma \end{pmatrix}, \quad \begin{pmatrix} 0 & \delta & 1-\delta \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; : \; 0 \leq \gamma, \delta, \leq 1 \right\}$$

This is the same as the kernel K_4 in Case II mentioned earlier which we just get by renaming the states (1,2,3) as (2,3,1). But unlike Case II, here the closed semigroup is not itself the kernel. In that sense, it is rather like Case III.

Let us write $S_{24} = S_{22} \cup S_{44}$ where

$$S_{22} : \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 1-\alpha \\ 0 & \beta & 1-\beta \end{pmatrix} \; : \; 0 \leq \alpha, \beta \leq 1 \right\}$$

$$S_{44} : \left\{ \begin{pmatrix} 0 & \delta & 1-\delta \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; : \; 0 \leq \delta \leq 1 \right\}$$

Also write $K_{24} = K_{22} \cup K_{44}$ where

$$K_{22} : \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 1-\gamma \\ 0 & \gamma & 1-\gamma \end{pmatrix} \; : \; 0 \leq \gamma \leq 1 \right\}$$

$$K_{44} : \left\{ \begin{pmatrix} 0 & \delta & 1-\delta \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; : \; 0 \leq \delta \leq 1 \right\}$$

Then note : (1) S_{22} is the closed semigroup generated by all 2(a)-type matrices and K_{22} is its corresponding kernel. So, if $S_\mu \subseteq S_{22}$, μ^n converges unless $\mu = \delta_{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$ as explained in Section 5 and this is done via 2×2-case where the kernel is just like K_{22}.

(2) $S_{44} = K_{44}$.

Now, let $\phi : K_{24} \longrightarrow K_{24}$ be defined by

$$\phi \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 1-\gamma \\ 0 & \gamma & 1-\gamma \end{pmatrix} = \begin{pmatrix} 0 & \gamma & 1-\gamma \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\phi \begin{pmatrix} 0 & \gamma & 1-\gamma \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 1-\gamma \\ 0 & \gamma & 1-\gamma \end{pmatrix}$$

Then ϕ is a bijection, $\phi = \phi^{-1}$ and $\phi(K_{22}) = K_{44}$.

Also, observe that if $x, y \in K_{24}$, $xy = y$ or $\phi(y)$ according as $x \in K_{22}$, or K_{44}. Hence, unlike Case III, here the matrix which post multiplies determines the entries of the product matrix.

Claim : If $\mu(S_{24}) = 1$, $\mu(S_{22}) > 0$, $\mu(S_{44}) > 0$ then μ^n converges.

From now on, we assume that μ is as in the claim and we put $c = \mu(S_{33})$ so that $0 < c < 1$.

If Q is any limit point of μ^n, then once again, by Rosenblatt's result, $Q(K_{24}) = 1$. Now, denoting $\mu^n(S_{22})$ by α_n, we can prove exactly as in Case III that $\alpha_n \to \frac{1}{2}$ as $n \to \infty$. This explains : $Q(K_{22}) = Q(K_{44}) = \frac{1}{2}$.

$$\text{Again , } \quad Q_1 * Q_2(A) = \frac{Q_1(A) + Q_2(\phi(A))}{2} \quad \text{ for any Borel subset } A \text{ of } K_{24}.$$

Thus, here also, if we could show directly that $Q(A) = Q(\phi(A))$ for any limit point Q of $\{\mu^n\}_{n\geq 1}$ and for any Borel subset A of K_{24}, then since $Q_1 * Q_2 = Q_2 * Q_1$ for any two limit points Q_1 and Q_2, we can conclude that $\{\mu^n\}_{n\geq 1}$ has a unique limit point and μ^n converges. But here also, we failed to do this.

So, here also, it is convenient to have a sequence of i.i.d. random matrices $X_1, X_2, \cdots$ each having distribution μ so that $Y_n = X_n \cdots X_1$ has distribution μ^n. We shall show $\mu^n(A)$ converges for any Borel subset $A \subseteq S_{24}$. Now, since $S_{44} = K_{44}$ we have $\mu(K_{24}) > 0$ and hence almost surely, $X_N \in K_{24}$ for some random integer N. Then $\forall\ n > N$, $Y_n \in K_{24}$. Therefore for $A \subseteq S_{24} - K_{24}$, $\mu^n(A) \to 0$ as $n \to \infty$. For $A \subseteq K_{24}$, as in Case III, we can show $\{\mu^n(A)\}_{n\geq 1}$ is a Cauchy sequence. This completes the proof of the claim.

7. **Main Theorem**

From our discussions so far, it is clear that μ^n does not converge iff one of the following conditions hold :-

1. $S(\mu) \subset \{e_1,$ 5(ii)(a)-type matrices $\}$ & two other analogues.

2. $S(\mu) \subset \{$ 4(i)(a)-type matrices$\} \cup \{M_1, M_2\}$ & two other analogues.

3. $S(\mu) = \{e_4\}$ or $\{e_5\}$.

4. $S(\mu) \subset \{e_1, e_2, e_3\}$.

A clear picture will emerge if we make the following definition :-

Suppose S is a set of stochastic matrices of order 3. We say that S is a cyclic family if there are $S_1, \cdots, S_m \longrightarrow$ pairwise disjoint subsets of $\{1, 2, 3\}$ so that for any $1 \leq l \leq m$, for all $i \in S_l$, $\sum_{j \in S_{l+1}} p_{ij} = 1$ [Treat $m+1$ as 1]. Here $\cup_1^m S_i$ need not be equal to $\{1, 2, 3\}$.

Case (1) mentioned at the beginning of this section corresponds to $S_1 = \{2\}$, $S_2 = \{3\}$. Similar construction of S_1, S_2 hold for the other analogues. In case (2), $S_1 = \{1\}$, $S_2 = \{2, 3\}$. Similarly, we can write down for the other analogues. In case (3), if $S(\mu) = \{e_4\}$, $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$. & if $S(\mu) = \{e_5\}$, $S_1 = \{1\}$, $S_2 = \{3\}$, $S_3 = \{2\}$. However in Case 4 mentioned above there is no such cyclic family.

The conclusion mentioned at the beginning of the section may now be succinctly stated as follows :-

THEOREM 7.1. *Suppose μ is a probability on the set of stochastic matrices of order 3. Then μ^n does not converge to a limit if either $S(\mu)$ is cyclic or $S(\mu) \subset \{e_1, e_2, e_3\}$.*

It is interesting to note that in all the cases of nonconvergence supports of μ and μ^2 are disjoint. But of course the converse is clearly false.

8. Concluding Remarks

REMARK 1. Following a suggestion in (Rosenblatt (1971), p.160), it would be interesting to find conditions for the limit of (μ^n) – when it exists – to be discrete, singular or absolutely continuous.

REMARK 2. In all the four cases of non-convergence mentioned in Section 7, it is easy to see that we have finitely many limit points for the sequence μ^n (see Theorem 3.4 in A. Mukherjea (1979)). In fact, except case (3), we have only two limit points for the other cases. In case of (3), we have three limit points for each of the subcases :- $\mu = \delta_{e_4}$ or $\mu = \delta_{e_5}$.

REMARK 3. It is clear from Remark 2 that in any case, $\frac{1}{n} \sum_1^n \mu^k$ converges. This is of course well known Rosenblatt (1971).

Remark 4. When $d = 2$ the only case when $\{\mu^n\}_{n\geq 1}$ does not converge is given by $\mu = \delta_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$. In that case, $S_1 = \{1\}$ and $S_2 = \{2\}$ form the two cyclically moving subclasses and this is the only case when $S(\mu)$ is cyclic.

Remark 5. For μ on $S^{(d)} (d \geq 2)$, if $SS^{(d)}(\mu) \cap K^{(d)} \neq \phi$, μ^n converges weakly. This follows from our discussions in Section 2.

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