A TREATMENT OF ABSOLUTE INDICES OF POLARIZATION*

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Absolute polarization indices remain unchanged under equal absolute augmentation in all incomes. This paper identifies the class of absolute polarization indices whose orderings of alternative income distributions agree with the rankings generated by nonintersecting absolute polarization curves. We explore the possibility of using the Kolm (1976)–Blackorby-Donaldson (1980) ethical absolute inequality index in polarization measurement. We establish that although inequality and polarization are dissimilar concepts, different absolute inequality indices can be employed to design alternative absolute polarization indices. A numerical illustration is provided using Indian data and it is shown that inequality and polarization are different issues in income distribution analysis.
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1. Introduction

An index of polarization is a measure of the extent of the decline of the middle class. This issue has recently received wide attention from economists. See, for example, Esteban and Ray (1991, 1994, 1999), Foster and Wolfson (1992), Wolfson (1994, 1997), Wang and Tsui (2000) and Chakravarty and Majumder (2001).[1]

Two characteristics considered to be intrinsic to the notion of polarization are nondecreasing spread and nondecreasing bipolarity. According to nondecreasing spread, a movement of incomes from the middle position to the tail of income distribution makes the distribution at least as polarized as before. In other words, as the distribution becomes more spread out from the middle position, polarization does not diminish. On the other hand, nondecreasing bipolarity requires that a clustering of incomes below or above the median leads to a distribution at least as polarized as before. Equivalently, a reduction of gaps between any two incomes, above or below the median does not lessen polarization. Thus, polarization involves both an inequality-like constituent, the nondecreasing spread criterion, which does not decrease either inequality or polarization, and an equality-like constituent, the clustering or bunching principle, which does not lower polarization; neither does it augment any inequality measure that fulfils the Pigou-Dalton transfers principle, a requirement under which inequality is nondecreasing for a transfer of income from rich to poor. Thus, polarization and inequality are two different concepts, although

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[1] Some of these studies and several other studies have examined the extent of polarization in different countries. For instance, Morris *et al.* (1994), Wolfson (1997), Gradin (2000), Esteban *et al.* (2007), Chakravarty and Majumder (2001) and Zhang and Kanbur (2001) looked at the extent of polarization in the USA, Canada, Spain, five OECD countries (the USA, the UK, Canada, Germany and Sweden), India and China, respectively, over different periods.

there is a nice complementarity between them. (See the references cited above for further discussion.)

It is evident that a particular index of polarization will generate a complete ranking of alternative distributions of income. However, using more than one index, we may obtain different rankings of the distributions. Given the diversity of numerical indices, it is therefore reasonable to identify the class of indices that yields a similar ordering of different distributions. In this paper we address this problem for absolute polarization indices that do not alter under equal absolute changes in all incomes.[2] (In the relative case an analogous problem was considered in Foster and Wolfson, 1992.) We scale up the Foster-Wolfson polarization curve by the median to generate the absolute polarization curve. The resulting curve shows, for any cumulative population proportion, the shortfall (excess) of its income, expressed as a proportion of the total population size, from (over) the corresponding income that it would enjoy under the distribution where everybody has the median income. The area under this curve turns out to be an absolute index of polarization. We then show that of two income distributions x and y, if x is not absolute polarization inferior to y, that is, the absolute polarization curve of x lies nowhere outside that of y, then x is regarded as at least as polarized as y by all absolute, symmetric, population replication invariant polarization indices that fulfil nondecreasing spread and nondecreasing bipolarity. Furthermore, the converse is also true. The population sizes and the medians of the distributions concerned need not be the same for this general result to hold. It may be important to note that in many cases where ranking of distributions by comparison of polarization curves becomes ambiguous, ranking in terms of absolute polarization curves may be possible.

Since inequality and polarization have both similar and opposite features, it is natural to ask whether absolute indices of inequality can be utilized in constructing absolute polarization indices. We suggest a general welfare theoretic absolute index of polarization using the Kolm (1976)–Blackorby and Donaldson (1980) ethical absolute inequality index. To each social welfare function that satisfies certain regularity conditions, there corresponds an absolute inequality index and hence an absolute index of polarization. If the Gini welfare function is employed, our general index relates to the index of polarization that corresponds to the area under the absolute polarization curve. An alternative of interest arises from the Kolm (1976)–Pollak (1971) absolute inequalty index.

We then illustrate the results developed in this paper using household expenditure survey data obtained from the Indian National Sample Survey Organization (NSSO) for the rural and urban sectors. It is explicitly demonstrated that: (i) inequality and polarization are two different concepts; (ii) relative and absolute indices of polarization reflect two different notions of polarization; and (iii) many ambiguous relative polarization comparisons may become unambiguous in the absolute case. However, in some special cases the reverse could also be true.

The paper is organized as follows. Desiderata for an index of polarization are presented rigorously in Section 3. Section 4 defines the absolute polarization curve formally, discusses the ordering associated with nonintersecting absolute polarization curves and

[2] In contrast, there can be relative indices that remain invariant under equiproportionate variations in all incomes. The choice between relative and absolute indices is basically a problem of value judgement and a debate on their merits and demerits could be endless. One may want to use relative indices for some purpose and absolute indices for others. (See Kolm, 1976; Blackorby and Donaldson, 1980; Chakravarty, 1990; Amiel and Cowell, 1992 for elaborate discussions along this line.)

isolates the class of polarization indices that agrees with this ordering. Section 5 demonstrates how different absolute inequality indices can be employed to generate alternative absolute indices of polarization. A general ethical absolute inequality index is essential to the construction of these indices; we briefly discuss this general index in Section 2. Section 6 presents the numerical results. Finally, Section 7 concludes.

2. Ethical absolute inequality indices

For a population of size n, a typical income distribution is a vector $x = (x_1, x_2, \dots x_n)$, where x_i is the income of person i. Each x_i is assumed to be drawn from (a, ∞), a nondegenerate interval in the non-negative part of the real line R. The set of income distributions for this population is D^n, the n-fold Cartesian product of (a, ∞). The set of all possible income distributions is $D = \cup_{n \in N} D^n$, where N is the set of positive integers. For the sake of simplicity and convenience, the lower bound of the interval (a, ∞) has been taken to be non-negative, which in turn implies non-negativity of all incomes. Extension of our results to the situation where some incomes are negative may be an interesting investigation.

For any function $f: D \to R$, we write f^n for the restriction of f on D^n. All income distributions are assumed to be illfare ranked, that is, for all $n \in N$, $x \in D^n$, $x_1 \leq x_2 \leq \dots \leq x_n$. For any $n \in N$, $x \in D^n$, the mean and median of x are denoted respectively by $\lambda(x)$ and $m(x)$. If n is odd, $m(x)$ is the $((n+1)/2)^{\text{th}}$ observation in x. But if n is even, the arithmetic mean of the $(n/2)^{\text{th}}$ and the $(n/2+1)^{\text{th}}$ observations in x is taken as the median. Since in polarization measurement, all incomes are compared with the median, persons below the median can be called "deprived", where deprivation is a consequence of the shortfall in their incomes from the median. Similarly all persons above the median can be referred to as "satisfied". (See Runciman, 1966; Yitzhaki, 1979; Hey and Lambert, 1980; Kakwani, 1984; Berrebi and Silber, 1985; Chakravarty, 1997, 1999; Chakravarty and Mukherjee, 1999; Chakravarty and Moyes, 2003.) Let $\bar{n} = (n+1)/2$. We write x_- and x_+ for the subvectors of x that include x_i for $i < \bar{n}$ and $i > \bar{n}$, respectively. Thus, for any $n \in N$, $x \in D^n$, $x = (x_-, x_+)$ if n is even and $x = (x_-, m(x), x_+)$ if n is odd. For all $n \in N$, $x, y \in D^n$, $x \geq y$ means that $y_i \leq x_i$ for all $i = 1, 2, \dots n$. The n-coordinated vector of ones is denoted by 1^n, $n \in N$.

Some more preliminaries are necessary for the purpose at hand. For all $n \in N$, $x, y \in D^n$, x is said to be obtained from y by a simple increment if $x_j > y_j$ for some j and $x_i = y_i$ for all $i \neq j$; we write xCy to represent this. Given that income distributions are nondecreasingly ordered, the transformation C allows only rank preserving increments. For $x, y \in D^n$, where $n \in N$ is arbitrary, we say x is obtained from y by a progressive transfer, if there is a pair (i, j), such that $x_i - y_i = y_j - x_j > 0$, $x_j > x_i$ and $y_k = x_k$ for all $k \neq i, j$. That is, x and y are identical except for a positive transfer of income from the rich person j to the poor person i. This is equivalent to the statement that y can be obtained from x through a regressive transfer and we write xTy to indicate this. Note that only rank preserving transfers are allowed under the operation T.

An ethical ordering of alternative income distributions is represented by a social welfare function, $W^n : D^n \to R$, where W^n is ordinally significant and $n \in N$ is arbitrary. It is supposed that for all $n \in N$, W^n satisfies continuity, nondecreasingness, translatability and S-concavity. Translatability means that W^n can be expressed as an increasing transform of a unit-translatable function, that is, for all $x \in D^n$,

$$W^n(x) = g[L^n(x)], \tag{1}$$

where, L^n is unit translatable and g is increasing. Unit translatability of L^n is the requirement that for all $x \in D^n$,

$$L^n(x + c1^n) = L^n(x) + c, \tag{2}$$

where c is any scalar such that $x + c1^n \in D^n$. That is, equal absolute augmentation of all incomes by a scalar, augments L^n by the scalar itself.[3] According to S-concavity of W^m, a rank preserving transfer of income from a person to anyone who has a lower income does not reduce welfare.[4]

The Atkinson (1970)–Kolm (1969)–Sen (1973) representative income x_e associated with x is that level of income which, if enjoyed by everybody, makes the existing distribution of x indifferent (indifference as measured by the social welfare function). Formally,

$$W^n(x_e 1^n) = W^n(x). \tag{3}$$

Given assumptions about $W^n(x)$, Equation (3) can be solved uniquely for x_e,

$$x_e = E^n(x). \tag{4}$$

$E^n(x)$ is continuous, nondecreasing, S-concave and unit translatable. It is a particular numerical representation of W^n: for all $x, y \in D^n$,

$$E^n(x) \geq E^n(y) \leftrightarrow W^n(x) \geq W^n(y). \tag{5}$$

By S-concavity of E^n, we have $E^n(x) \leq \lambda(x)$.

The Kolm (1976)–Blackorby and Donaldson (1980) index of inequality is defined by

$$A^n(x) = \lambda(x) - E^n(x), \tag{6}$$

where $x \in D^n$, $n \in N$ are arbitrary. This index is the amount of per capita income that could be saved if society distributed income equally without any welfare loss. Alternatively, the index is the amount of income that must be given to each person to achieve a distribution that is ethically indifferent to the distribution where each person receives the current mean income. A^n is continuous, nonincreasing under a rank-preserving income transfer from rich to poor (S-convexity) and bounded from below by zero, where this bound is achieved whenever incomes are equal. Unit translatability of E^n ensures that A^n is translatable of degree zero, that is, an absolute index. Given a functional form of A^n, we can recover W^m from Equations (6), (4) and (3). Thus, the index A^n is exact in the sense that it implies and is implied by a social welfare function.

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[3] More generally, a real valued function f^n defined on D^n is called translatable of degree k if for all x in D^n, $f^n(x + c1^n) = f^n(x) + kc$, where c is any scalar such that, $x + c1^n$ is in D^n. Evidently, $k = 0$ corresponds to the case of zero degree translatability or translation invariance of f^n. On the other hand, unit translatability of f^n is obtained if $k = 1$.

[4] Equivalently, $W^n : D^n \to R$ is called S-concave if and only if for all x in D^n, $W^n(Mx) \geq W^n(x)$, where M is any bistochastic matrix of order n. An $n \times n$ non-negative matrix is a bistochastic matrix if each of its columns and rows sums to one. All S-concave functions are symmetric.

As an illustrative example, suppose that social evaluation is done with respect to the Gini welfare function. Then,

$$x_e = E_G^n(x) = \sum_{i=1}^{n}(2(n-i)+1)x_i/n^2. \tag{7}$$

Its implied inequality index is the absolute Gini index

$$A_G^n(x) = \lambda(x) - E_G^n(x) = \lambda(x) - \sum_{i=1}^{n}(2(n-i)+1)x_i/n^2. \tag{8}$$

When divided by the positive mean income, the index in Equation (8) becomes a relative index, the well-known Gini coefficient. This nice compromise property of the absolute Gini index is a consequence of unit-translatability and the linear homogeneity of E_G^n–more generally, of translatability and homotheticity of the Gini welfare function. Homotheticity of a function means that it can be written as an increasing transform of a linearly homogeneous function.

As a second example, we consider the Kolm-Pollak absolute inequality index

$$\begin{aligned} A_\alpha^n(x) = \log\Big(\sum_{i=1}^{n}\exp(\alpha(\lambda(x)-x_i)/n)\Big)\Big/\alpha, \quad \alpha > 0, \\ = 0, \quad \alpha = 0. \end{aligned} \tag{9}$$

The representative income associated with A_α^n is

$$\begin{aligned} E_\alpha^n(x) = -\log\Big(\sum_{i=1}^{n}\exp(-\alpha x_i)/n\Big)\Big/\alpha, \quad \alpha > 0, \\ = \lambda(x), \quad \alpha = 0. \end{aligned} \tag{10}$$

The parameter α determines the curvature of the social indifference surface. A rank preserving income transfer from rich to poor increases (decreases) $E_\alpha^n(A_\alpha^n)$ by a larger amount, the higher is the value of $\alpha > 0$. For a given x, A_α^n is increasing in α. For any $\alpha > 0$, A_α^n attaches greater weight to transfers lower down the scale. As specified, for $\alpha = 0$, the welfare function E_α^n becomes the symmetric linear welfare function $\lambda(x)$ producing the degenerate index. On the other hand, as $\alpha \to \infty$, $E_\alpha^n \to \min_i\{x_i\}$, the Rawlsian maximin index (Rawls, 1971), and the corresponding absolute index is the absolute maximin index $\lambda(x) - \min_i\{x_i\}$. These two particular indices share the compromise property of the absolute Gini index. However, for any finite $\alpha > 0$, A_α^n does not possess this property. Another interesting index that can be interpreted in the framework in Equation (6) is the Donaldson and Weymark (1980) single parameter generalised absolute Gini index (see Donaldson and Weymark, 1980; Chakravarty, 1990).

3. Postulates for an index of polarization

The purpose of this section is to present rigorously the postulates for an index of polarization. A polarization index I is a real valued function defined on D, that is, $I : D \to R^1$. For all $n \in N$, $x \in D^n$, the functional value $I^n(x)$ indicates the level of polarization associated with the distribution x. In view of our earlier discussion, $I : D \to R^1$ is a relative or absolute index according as I^n is homogeneous or translatable of degree zero, where $n \in N$ is arbitrary.

A polarization index $I : D \to R^1$, whether relative or absolute, should satisfy the following postulates.

Non-decreasing spread (NS): *For all* $n \in N$, if $x, y \in D^n$, *where* $m(x) = m(y)$, *are related through anyone of the following cases*,

$$\text{(a) } x_+ = y_+, \ y_- C x_-, \quad \text{(b) } x_- = y_-, \ x_+ C y_+, \quad \text{(c) } y_- C x_-, \ x_+ C y_+, \quad \text{then } I^n(x) \geq I^n(y).$$

Non-decreasing bipolarity (NB): *For all* $n \in N$, if x, $y \in D^n$, *where* $m(x) = m(y)$, *are related through anyone of the following cases*,

$$\text{(a) } x_- T y_-, \ x_+ = y_+, \quad \text{(b) } x_+ T y_+, \ x_- = y_-, \quad \text{(c) } x_- T y_-, \ x_+ T y_+, \quad \text{then } I^n(x) \geq I^n(y).$$

Symmetry (SM): *For all* $n \in N$, $x \in D^n$, $I^n(x) = I^n(Px)$, *where* P *is any* $n \times n$ *permutation matrix.*[5]

Principle of population (PP): *For all* $n \in N$, $x \in D^n$, $I^n(x) = I^{kn}(y)$, *where* y *is any k-fold replication of* x, *that is*, $y = (x^1, x^2, \ldots, x^k)$ *with each* $x^i = x$.

Normalization (NM): *For all* $n \in N$, $I^n(c1^n) = 0$, *where* $c > 0$ *is any scalar.*

Continuity (CN): *For all* $n \in N$, I^n *is a continuous function on* D^n.

The postulates **NS** and **NB** were considered among others by Foster and Wolfson (1992), Wolfson (1994, 1997), Wang and Tsui (2000) and Chakravarty and Majumder (2001). **NS** is a monotonicity condition. Since rank preserving increments (reductions) in incomes above (below) the median widen the distribution, polarization should not go down. That is, greater distancing between the groups below and above the median should not make the distribution less polarized. **NB** is a bunching or a clustering principle. Because a rank-preserving egalitarian transfer between two individuals on the same side of the median brings the individuals closer to each other, polarization should be nondiminishing. As an egalitarian transfer demands non-increasingness of inequality, **NB** explicitly establishes that inequality and polarization are two nonidentical concepts. **SM** means that polarization remains unchanged if we take a reordering of incomes. Thus, any characteristic other than income, e.g. names of the individuals, is not relevant to the measurement of polarization. One implication of **SM** is that a polarization index can be defined directly on ordered income distributions (as we have done). According to **PP**, if a population is replicated several times, the levels of polarization of the replicated and the original distributions are the same. In view of **PP**, we can regard polarization as an average concept. **PP** enables us to compare polarization across populations. Clearly, the median income remains unaltered under replications of the population. Postulate **NM** is a cardinality property of the polarization index. It says that for a perfectly equal income distribution, the level of polarization is zero. **CN** means that a polarization index will not take sudden jumps for small income changes. Thus, a polarization index will not be oversensitive to minor observational errors in incomes. Clearly, given the difficulties in measuring incomes accurately, it is reasonable to require a polarization index to vary continuously with incomes.

[5] An $n \times n$ matrix is called a permutation matrix if it is bistochastic and has exactly one positive entry in each row and column.

4. The absolute polarization ordering

The objective of this section is to develop a ranking of alternative income distributions in terms of absolute polarization indices. This ranking relies on the absolute polarization curve (**APC**). The **APC** of any income distribution shows how far the total income for any population proportion, expressed as a fraction of the population size, is from the corresponding income that it would receive under the hypothetical situation where everybody enjoys the median income.

For any $x \in D^n$, the **APC** ordinate corresponding to the population proportion

$$k/n, \quad (1 \le k \le \bar{n}),$$

is

$$AP(x,\, k/n) = \frac{1}{n}\sum\nolimits_{k \le i \le \bar{n}}(m(x) - x_i) \tag{11}$$

and corresponding to the population proportion k/n, $(\bar{n} \le k \le n)$, this ordinate is

$$AP(x,\, k/n) = \frac{1}{n}\sum\nolimits_{\bar{n} \le i \le k}(x_i - m(x)). \tag{12}$$

Note that the ordinate at $\bar{n}/n$ involves the income level $x_{\bar{n}} = m(x)$. Now if n is odd, $x_{\bar{n}}$ is one of the incomes in the distribution x. However, for even n, although $x_{\bar{n}}$ is not in x, we define the ordinate at $\bar{n}/n$, since in polarization measurement, the median is the reference income level. The **APC** of x, $AP(x; p)$ is completed by setting $AP(x; 0) = 0$ and by defining

$$AP(x; (k+\theta)/n) = (1-\theta)AP(x; k/n) + \theta AP(x; (k+1)/n) \quad \text{for all } \theta \in [0, 1] \text{ and } 1 \le k \le n, \tag{13}$$

where n is odd.

For even n, the completion of the curve is done by setting $AP(x; 0) = 0$, and by defining

$$AP(x; (k+\theta)/n) = (1-\theta)AP(x; k/n) + \theta AP(x; (k+1)/n) \quad \text{for all } \theta \in [0, 1],\ 1 \le k \le n,\ k \ne \bar{n},$$

$$AP(x; (\bar{n}-0.5+\theta)/n) = (0.5-\theta)AP(x; (\bar{n}-0.5)/n) + \theta AP(x; \bar{n}/n) \quad \text{for all } \theta \in [0, 5] \tag{14}$$

and

$$AP(x; (\bar{n}+\theta)/n) = (0.5-\theta)AP(x; \bar{n}/n) + \theta AP(x; (\bar{n}+0.5)/n) \quad \text{for all } \theta \in [0, 5].$$

This construction makes the **APC** smooth all through.

For a typical income distribution $x \in D^n$, up to $\bar{n}/n$, the midpoint of the horizontal axis, **APC** decreases monotonically, at $\bar{n}/n$ it coincides with the horizontal axis and then it increases monotonically. If x is an equal distribution, then its **APC** becomes the horizontal axis itself. For a distribution x, where x_+ is unequal but $(x_-, m(x))$ is equal, **APC** runs along the horizontal axis up to its midpoint $\bar{n}/n$ and starting from $\bar{n}/n$ the curve rises gradually. Similarly if x is such that x_- is unequal but $(m(x), x_+)$ is equal, then the **APC** is decreasing

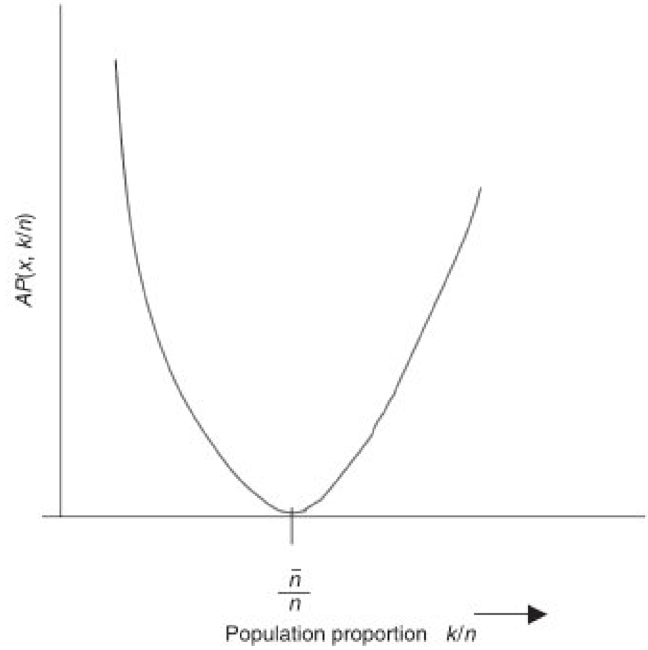

FIGURE 1. A typical absolute polarization curve (APC)

up to the midpoint of the horizontal axis, after which it runs through this axis. A typical **APC** is presented in Figure 1.

For any income distribution, $x \in D^n$, $AP(x; P)$ when divided by the median, that is, $AP(x; p)/m(x)$, becomes the Foster-Wolfson polarization curve (**PC**). Now given $x \in D^n$, the area under the Foster-Wolfson curve, associated with x is

$$F^n(x) = 2[\lambda(x_+) - \lambda(x_-) - \lambda(x)G^n(x)]/m(x), \tag{15}$$

where $G^n(x) = A^n_G(x)/\lambda(x)$ is the Gini coefficient of the income distribution x. This in turn shows that the area under $AP(x; p)$ becomes

$$Q^n(x) = 2[\lambda(x_+) - \lambda(x_-) - A^n_G(x)]. \tag{16}$$

F^n, which is a relative index, is known in the literature as the Wolfson index of polarization. In contrast, Q^n is an absolute index. Thus, while F^n involves the Gini coefficient as a component, the corresponding component of its absolute sister Q^n is the absolute Gini index. It may be important to note that these two indices satisfy all the postulates outlined in Section 3 (see the next section for a discussion).

Given any two income distributions $x, y \in D$, x is said to dominate y with respect to absolute polarization ($x \geq_{AP} y$, for short) if the **APC** of x lies nowhere outside that of y. Formally, $x \geq_{AP} y$ means that

$$APC(x; p) \geq APC(y; p) \tag{17}$$

for all $p \in [0, 1]$. Note that the relation $\geq_{AP}$ is transitive, that is, for any $x, y, z \in D$, if $x \geq_{AP} y$ and $y \geq_{AP} z$ hold, then $x \geq_{AP} z$ holds. Since for any $x \in D$, x is as polarized as itself, $\geq_{AP}$ satisfies reflexivity also. However, it is not complete, that is there may exist $x, y \in D$, such that neither $x \geq_{AP} y$ nor $y \geq_{AP} x$ holds. Clearly such a situation arises if the APCs of x and y intersect. Thus, $\geq_{AP}$ is a quasi-ordering — it is reflexive, transitive but not complete.

The following result gives an implication of the absolute polarization dominance relation $\geq_{AP}$ for income distributions over the same population size and arbitrary medians.

Theorem 1: *Let $x, y \in D^n$ be arbitrary. Then the following statements are equivalent:* (i) $x \geq_{AP} y$; *and* (ii) $I^n(x) \geq I^n(y)$ *for all absolute polarization indices* $I^n : D^n \to R$ *that satisfy* ***NS***, ***NB*** *and* ***SM***.

Proof: See Appendix.

Theorem 1 indicates that an unambiguous ranking of income distributions over a given population size by all symmetric, absolute polarization indices satisfying **NS** and **NB** can be obtained if and only if their **APC**s do not intersect. But if the two curves intersect, we can obtain two different indices with these properties that will rank the distributions in opposite directions. Clearly, for any two distributions, x and y with the same median $x \geq_{AP} y$ is equivalent to the requirement that **PC** of y is nowhere inside that of x ($x \geq_P y$, for short). But in general, the two orderings are likely to be different. In fact, in some cases, ambiguous comparison in terms of $\geq_P$ may be unambiguous by $\geq_{AP}$. For instance, suppose that of two distributions x and y, where $m(x) > m(y)$, the **PC** of x intersects that of y from above at a point below the midpoint of the horizontal axis. That is, below the point of intersection, the **PC** of x lies everywhere above but after that it lies on and below that of y. Because $m(x) > m(y)$, multiplication of these curves by the corresponding medians may generate an upward shift in the resulting curve of x at the right of the point of intersection such that $x \geq_{AP} y$ holds. Thus, higher median may be sufficient for pushing the lower curve upward to ensure absolute dominance unambiguously. In fact, our empirical illustration in the next section confirms this.

Interpopulation comparisons of polarization usually involve different population sizes, as do intertemporal comparisons. For ranking distributions with differing population sizes we have the following result.

Theorem 2: *Let $x \in D^m$, $y \in D^n$ be arbitrary. Then the following statements are equivalent:* (i) $x \geq_{AP} y$; *and* (ii) $I^m(x) \geq I^n(y)$ *for all absolute polarization indices* $I : D \to R$ *that satisfy* ***NS***, ***NB***, ***SM*** *and* ***PP***.

Proof: See Appendix.

Theorem 2 shows that an unambiguous ranking of income distributions by absolute polarization indices can be achieved through the pairwise comparisons of the **APC**s of the distributions. We can also focus our attention on fixed median arbitrary population case. In this case, the definition domain of polarization indices is $D_m = \{x \in D \mid m(x) = m\}$. For all indices on D_m that are consistent with $\geq_{AP}$, we now have the following theorem, whose proof is similar to those of theorems 1 and 2.

Theorem 3: *Let $x, y \in D_m$ be arbitrary. Then the following statements are equivalent: (i) $x \geq_{AP} y$; and (ii) $I(x) \geq I(y)$ for all polarization indices $I : D_m \to R$ that fulfil* ***NS***, ***NB***, ***SM*** *and* ***PP***.

Finally, it should be clear that if both mean income and median are fixed, indices that agree with the ordering $\geq_{AP}$ should satisfy **NB**, **SM** and **PP**. We can also have situations where mean is fixed, medians are different and population size is equal/unequal. For consistency with $\geq_{AP}$, while in the former case the absolute indices should meet **NB** and **SM**, in the latter case they are required to satisfy **PP** in addition to **NB** and **SM**. But in the practical situations, where mean, median and population size are likely to vary, theorem 2 is the result that can be applied for polarization comparison.

5. A general absolute index of polarization

We begin this section by interpreting the absolute index Q^n in Equation (16) ethically. Q^n can be rewritten as

$$\begin{aligned} Q^n(x) &= (E_G^{n/2}(x_+) + 2\lambda(x_+))/2 + (E_G^{n/2}(x_-) - 4\lambda(x_-))/2 \quad \text{if } n \text{ is even,} \\ &= 2(\bar{n}-1)^2 E_G^{\bar{n}-1}(x_+)/n^2 + 2\bar{n}\lambda(x_+)/n + 2(\bar{n}-1)^2 E_G^{\bar{n}-1}(x_-)/n^2 \\ &\quad - (2(4\bar{n}-1)(\bar{n}-1)+2)\lambda(x_-)/n^2 \quad \text{if } n \text{ is odd.} \end{aligned} \tag{18}$$

In order to interpret Q^n in terms of welfare, let us suppose for simplicity that n is even. A similar analysis will apply for odd values of n. Note that both the welfare measure $E_G^{n/2}(x_+)$ and the efficiency measure $\lambda(x_+)$ are nondecreasing under a rank preserving increase in any income in x_+. Therefore, **NS** is satisfied in the subvector x_+ because of the positive monotonic association of Q^n with these two measures. Since a rank preserving progressive income redistribution in x_+ does not decrease $E_G^{n/2}(x_+)$, keeping $\lambda(x_+)$ fixed, **NB** is also fulfilled in the region x_+. An analogous argument will apply concerning fulfilment of **NB** in x_-. **NS** is satisfied in x_- because of the particular functional form of the absolute Gini index and rest of Q^n. (As the only difference between F^n and Q^n is that the latter is obtained from the former by multiplying by the median and **NS** does not allow the median to change, satisfaction of NS by F^n in x_- implies that Q^n also verifies **NS** in x_-.) (See Wolfson, 1994, 1997, for a discussion.) This shows that for a rank preserving increase in $x_i < m(x)$, the positive impact of the welfare function $E_G^{n/2}(x_-)$ does not outweigh the negative impact of $-4\lambda(x_-)$. For all $n \geq 2$, Q^n is continuous, symmetric, population replication invariant and normalized.

Now, one way of generalizing the index Q^n is to replace the Gini representative income E_G by any arbitrary representative income. Note that the expression for Q^n appears to be simpler for even n than for odd n. Therefore, as a generalized index of polarization, we may suggest the use of

$$J^n(x) = (E^k(x_+) + 2\lambda(x_+))/2 + (E^k(x_-) - B(m(x))\lambda(x_-))/2 + H(m(x)), \tag{19}$$

$$= (3\lambda(x_+) - A^k(x_+))/2 + (\lambda(x_-) - A^k(x_-) - B(m(x))\lambda(x_-))/2 + H(m(x)), \tag{20}$$

where $k = n/2$ if n is even and $k = \bar{n} - 1$ if n is odd; $x \in D^n$, $n \in N$ are arbitrary; and the continuous specifications B and H are to be chosen so that all the properties of a polarization index are verified by J^n.

Clearly, given Equation (19), to every translatable social welfare function, there exists a different index of polarization. These indices will differ only in the manner in which the welfares of the population in the two segments x_+ and x_- are taken into account. If we use the Gini welfare function in (19) assuming that $B(m(x)) = 4$ and $H(m(x)) = 0$, then for even values of n, $J^n = Q^n$. For odd values of n, the relationship will be different. But this difference becomes negligible if $\bar{n}$ (hence n) is large.

We now have the following result concerning the use of the Kolm-Pollak inequality index given by Equation (9) in (20).

Theorem 4: *Suppose that the inequality index employed in Equation (20) is the Kolm-Pollak index, given by (9), that is, the social evaluation in (6) is done with respect to the welfare function in* (10). *Then the continuous specifications* $B(m(x)) = \exp(-\alpha(\mu - m(x)))$ *and* $H(m(x)) = m(x)\exp(-\alpha(\mu - m(x)))/2 - 2m(x)$ *are sufficient to guarantee that the resulting absolute polarization index fulfils* ***NS***, ***NB***, ***SM***, ***PP***, ***NM*** *and* ***CN***, *where the arbitrary number* $\mu \leq \alpha$ *is assumed to be measured in the unit in which incomes are measured.*

Proof: See Appendix.

Clearly, infinitely many values of μ can be chosen to illustrate J^n_α numerically. For a given distribution x, and a value of the parameter α, as μ increases J^n_α decreases. Evidently, the general index J^n depends explicitly on the welfare levels and average incomes of the two segments of the population. For suitable choices of B and H the general index can be made to satisfy translation invariance, **NS**, **NB**, **SM**, **CN** and **NM**. It meets **PP** if E^n meets so. Further, given W^n, the corresponding form of J^n can be calculated easily.

6. An empirical illustration

In this section, we numerically illustrate the results developed in Sections 4 and 5 using NSSO state-based monthly per capita expenditure data for rural and urban India for the period 1993–94. In the absence of income data, total expenditure is taken as a proxy for income. The states considered are Andhra Pradesh (**AP**), Karnataka (**KA**), Kerala (**KE**), Punjab (**PU**), Uttar Pradesh (**UP**) and West Bengal (**WB**).

In Table 1 we show the ranking of the states for rural India by the ordering $\geq_P$ and $\geq_{AP}$. In the superdiagonal part of the table, ranking generated by the criterion $\geq_{AP}$ between the states i and j is presented, where $i \neq j =$ **AP**, **KA**, **KE**, **PU**, **UP**, **WB**. On the other hand, the subdiagonal part shows the ranking results for the relation $\geq_P$. To interpret the table

TABLE 1

Ranking expenditure distributions in rural India by polarization and absolute polarization dominations

State	AP	KA	KE	PU	UP	WB	
AP		×	≤	≤	×	≥	
KA	×		≤	≤	≤	≥	
KE	≥	≥		≥	≥	≥	**APC**
PU	×	×	≤		≥	≥	
UP	×	≥	≤	×		≥	
WB	≤	≤	≤	≤	≤		
				PC			

Note: **AP**, Andhra Pradesh; **KA**, Karnataka; **KE**, Kerala; **PU**, Punjab; **UP**, Uttar Pradesh; **WB**, West Bengal.

TABLE 2
Ranking expenditure distributions in urban India by polarization and absolute polarization dominations

State	AP	KA	KE	PU	UP	WB	
AP		≤	×	×	×	≤	
KA	×		×	≤	≥	×	
KE	×	×		×	≥	×	**APC**
PU	×	≤	×		≥	×	
UP	×	≤	×	×		≤	
WB	×	×	×	≥	×		
				PC			

Note: **AP**, Andhra Pradesh; **KA**, Karnataka; **KE**, Kerala; **PU**, Punjab; **UP**, Uttar Pradesh; **WB**, West Bengal.

TABLE 3
Ranking expenditure distribution in rural and urban India by Lorenz domination

State	AP	KA	KE	PU	UP	WB	
AP		≤	×	≤	≤	≤	
KA	≤		≥	≥	×	×	
KE	≤	≤		≤	≤	≤	Rural India
PU	≥	≥	≥		×	≤	
UP	×	≥	≥	≤		×	
WB	≥	≥	≥	×	≥		
				Urban India			

Note: **AP**, Andhra Pradesh; **KA**, Karnataka; **KE**, Kerala; **PU**, Punjab; **UP**, Uttar Pradesh; **WB**, West Bengal.

we use the following notation: for any column state i and row state j, $\geq$ means that i dominates j, $\leq$ indicates that i is dominated by j and $\times$ says that the curves for the states under comparison cross. For instance, in the rural sector, **KA** dominates **WB** but is dominated by **KE** by the ordering $\geq_{AP}$. Table 2, whose format is similar to that of Table 1, gives the rankings for the urban sector.

In order to demonstrate numerically that inequality and polarization are different issues, Table 3 provides Lorenz orderings of the states for both rural and urban sectors. For any $x \in D^n$, the ordinate $L(x; i/n)$ of the Lorenz curve (**LC**) corresponding to the population proportion i/n, $1 \leq i \leq n$, is given by $\Sigma_{j=1}^{i} x_j / n\lambda(x)$. The **LC** of x, $LC(x; p)$ is then completed by taking $LC(x; 0) = 0$ and by defining

$$LC(x; (i+\theta)/n) = (1-\theta)LC(x; i/n) + \theta LC(x; (i+1)/n); \quad 0 \leq \theta \leq 1.$$

The definition of Lorenz domination ($\geq_{LC}$) is now similar to that of $\geq_{AP}$. For any x, $y \in D$, $x \geq_{LC} y$ is same as the requirement that $I(x) \geq I(y)$ for all relative inequality indices $I : D \to R^1$ that meet symmetry, population replication invariance and the Pigou-Dalton principle (Foster, 1985; Chakravarty, 1990).

Numerical estimates of absolute inequality and polarization are reported in Table 4. The first column of the table gives the names of the sectors for which calculations are done. The figures in parantheses in this column are the sectoral medians. In column 2 we show for each state the absolute Gini index A_G for the subgroups below and above the median and for the population as a whole. Columns 3–5 present analogous figures for the Kolm-Pollak index A_α for three different values of α ($\alpha = 0.10, 0.25, 0.50$). Using overall figures in column 2 we determine the values for Q for different states, which are then shown in column 6. (Since our analysis involves cross-population comparisons, in the tables we drop superscript n from all functions.) In column 7, we present polarization figures, as given by J_α, for $\alpha = 0.10, 0.25, 0.50$, respectively. All expenditures in the data set used here are positive. We have, therefore, assumed that for each state, the value of μ is 10% less than the corresponding minimum expenditure.

Table 5, which has a format similar to that of Table 4, reports relative inequality and polarization levels. It shows the Gini index G in column 2, the Atkinson (1970) inequality index β_r ($r \leq 1$) in columns 3–5, for $r = -0.5$, 0 and 0.5, the Wolfson polarization index F in column 6 and the relative polarization index K_r ($r \leq 1$) suggested by Chakravarty and Majumder (2001) for $r = -0.5$, 0 and 0.5 in the column 7.

The index $K_r\colon D \to R$ is defined by

$$\begin{aligned} K_r^n(x) = {} & (E_r^k(x) + 2\lambda(x_+))/2m(x) + (E_r^k(x_-) \\ & - \lambda(x_-)(m(x)/\mu)^{1-r})/2m(x) + (m(x)/\mu)^{1-r}/2 - 2 \end{aligned} \tag{21}$$

where $n \in N$, $x \in D^n$ and each $x_i > 0$. Here $k = n/2$ if n is even and $k = \bar{n} - 1$ if n is odd, and

$$E_r^n(x) = \left(\sum_{i=1}^{n} (x_i^r/n)\right)^{1/r} \tag{22}$$

is the symmetric mean of order r, the representative income associated with the Atkinson (1970) index

$$\beta_r^n(x) = 1 - E_r^n(x)/\lambda(x). \tag{23}$$

The parameter r represents different perceptions of welfare/inequality. For $r = 1$, E_r^n becomes the symmetric linear social welfare function and as $r \to -\infty$, E_r^n the Rawlsian maximin function. As r decreases, the social indifference curves become more convex to the origin and the implicit ethics become closer to the maximin criterion. Several interesting features emerge from Table 1. Out of ${}^6C_2 = 15$ comparable situations in the rural sector, the criterion $\geq_{AP}$ generates 13 unambiguous rankings in contrast to 10 such rankings shown by $\geq_P$. Thus, we note that three pairs (**PU**, **AP**), (**PU**, **UP**) and (**PU**, **KA**) that are noncomparable with respect to $\geq_P$ become comparable under $\geq_{AP}$. In each case the **PC** of the former intersects that of the latter at a point below the middle of the horizontal axis and the former has also a significantly higher median than the latter. Significant differences in the medians can probably be an important factor in making noncomparable situations in the relative case comparable in the absolute framework. An additional observation that may support this claim is that the medians of the states under comparison are almost equal in each of the two pairs (**AP**, **UP**) and (**AP**, **KA**) which are noncomparable under both the orderings. We note the dominance of other states over **WB** by both $\geq_P$ and $\geq_{AP}$. In view of theorem 2, we can, therefore, say that among all the states, **WB** can be considered as the least polarized state by all absolute indices that meet **NS**, **NB**, **SM** and **PP**. A similar statement can be made for the relative case. This is a consequence of low inequality in **WB** below and above the median, combined with a higher concentration of incomes around the median. In the other extreme, no state is found to dominate the others by

Table 4

Absolute inequality and polarization indices for India (1993–1994)

Sector (Median) (Rs)	Absolute Gini Index A_G			Kolm-Pollak inequality index A_α: $\alpha = 0.10$			$\alpha = 0.25$			$\alpha = 0.5$			Pol index Q	Polarization Index J_α: $\alpha = 0.10$	$\alpha = 0.25$	$\alpha = 0.50$
	BM	**AM**	**OV**	**BM**	**AM**	**OV**	**BM**	**AM**	**OV**	**BM**	**AM**	**OV**				
(1)	(2)			(3)			(4)			(5)			(6)	(7)		
(AP) Rural (263)	24.47	135.16	111.34	114.64	190.41	250.51	155.42	203.01	295.45	170.10	208.51	311.52	348.42	0.42×10^{13}	0.18×10^{30}	0.97×10^{57}
Urban (366.72)	36.91	227.21	194.99	165.27	367.46	416.26	208.18	385.00	463.33	223.12	392.64	476.66	641.72	0.11×10^{18}	0.10×10^{41}	0.18×10^{79}
(KA) Rural (262.26)	25.74	98.69	92.91	87.35	147.26	203.96	122.33	159.97	243.10	135.70	165.75	257.86	380.28	0.24×10^{12}	0.13×10^{27}	0.50×10^{51}
Urban (405.78)	44.53	255.57	218.85	158.37	401.10	438.91	194.40	418.48	479.10	207.45	425.87	493.54	711.97	0.46×10^{18}	0.25×10^{42}	0.90×10^{81}
(KE) Rural (365.40)	37.67	192.35	164.56	141.78	290.59	348.71	179.83	306.15	390.93	193.86	313.02	406.33	525.67	0.36×10^{16}	0.20×10^{37}	0.75×10^{71}
Urban (439.22)	45.72	419.25	306.68	158.20	587.91	532.16	194.37	604.02	572.48	207.96	610.73	587.46	910.16	0.88×10^{18}	0.12×10^{43}	0.21×10^{83}
(PU) Rural (395.76)	36.14	180.72	156.08	127.10	274.47	323.91	162.06	289.97	363.03	175.74	296.75	378.09	502.79	0.65×10^{15}	0.29×10^{35}	0.17×10^{68}
Urban (526.25)	52.73	255.45	230.21	179.15	410.48	478.09	217.84	427.70	520.94	231.59	435.01	536.08	761.90	0.21×10^{21}	0.83×10^{43}	0.82×10^{94}
(UP) Rural (256.27)	24.94	99.88	94.11	74.21	151.75	193.09	113.96	164.78	236.99	130.42	170.66	254.84	315.00	0.18×10^{12}	0.60×10^{23}	0.10×10^{51}
Urban (354.40)	36.08	216.31	183.45	147.92	336.09	381.63	193.65	352.64	431.52	209.07	359.70	448.32	595.53	0.27×10^{17}	0.28×10^{39}	0.14×10^{76}
(WB) Rural (260.25)	22.24	100.20	88.61	65.97	143.61	175.04	94.28	156.14	207.51	107.40	161.94	22.01	286.79	0.63×10^{10}	0.19×10^{23}	0.11×10^{44}
Urban (441.19)	46.45	219.26	204.82	174.57	368.75	444.42	214.14	386.71	488.16	228.73	394.73	504.13	697.52	0.13×10^{20}	0.96×10^{45}	0.12×10^{89}

Note: **Pol**, polarization; **BM**, below median; **AM**, above median; **OV**, overall.
AP, Andhra Pradesh; **KA**, Karnataka; **KE**, Kerala; **PU**, Punjab; **UP**, Uttar Pradesh; **WB**, West Bengal.

TABLE 5
Relative inequality and polarization indices for India (1993–1994)

Sector Median (Rs)	Gini coefficient G			Atkinson index β_r, $r = -0.5$			$r = 0$			$r = 0.5$			Wolfson index F	Polarization index K_r		
	BM	**AM**	**OV**	**BM**	**AM**	**OV**	**BM**	**AM**	**OV**	**BM**	**AM**	**OV**		$r = -0.5$	$r = 0$	$r = 0.5$
(1)	(2)			(3)			(4)			(5)			(6)	(7)		
(AP) Rural (263)	0.13	0.28	0.33	0.05	0.17	0.22	0.03	0.13	0.17	0.01	0.08	0.10	1.32	27.37	5.48	1.80
Urban (366.72)	0.15	0.30	0.38	0.06	0.17	0.28	0.04	0.13	0.21	0.02	0.07	0.12	1.75	21.02	5.26	2.19
(KA) RURAL (262.26)	0.14	0.23	0.30	0.05	0.11	0.19	0.03	0.09	0.14	0.02	0.05	0.08	1.18	3.47	1.78	1.19
Urban (405.78)	0.17	0.30	0.39	0.08	0.19	0.30	0.05	0.14	0.23	0.02	0.09	0.13	1.75	6.45	2.97	1.88
(KE) Rural (365.40)	0.14	0.28	0.35	0.06	0.16	0.24	0.04	0.12	0.18	0.02	0.07	0.10	1.44	4.13	2.17	1.51
Urban (439.22)	0.15	0.40	0.45	0.06	0.29	0.36	0.04	0.24	0.29	0.02	0.17	0.19	2.07	4.20	2.68	2.16
(PU) Rural (395.76)	0.12	0.26	0.32	0.04	0.14	0.20	0.03	0.11	0.15	0.13	0.06	0.08	1.27	2.00	1.47	1.23
Urban (526.25)	0.15	0.26	0.35	0.05	0.14	0.24	0.04	0.11	0.18	0.02	0.06	0.10	1.45	2.80	1.81	1.41
(UP) Rural (256.27)	0.14	0.23	0.30	0.05	0.11	0.19	0.03	0.08	0.14	0.02	0.05	0.08	1.23	3.82	1.92	1.26
Urban (354.40)	0.15	0.30	0.38	0.06	0.18	0.28	0.04	0.14	0.21	0.02	0.08	0.12	1.68	16.99	4.65	2.05
(WB) Rural (260.25)	0.11	0.23	0.28	0.03	0.12	0.17	0.02	0.10	0.13	0.01	0.06	0.07	1.10	1.62	1.22	1.04
Urban (441.19)	0.16	0.26	0.36	0.07	0.14	0.26	0.04	0.10	0.19	0.02	0.06	0.10	1.58	6.38	2.78	1.68

Note: **BM**, below median; **AM**, above median; **OV**, overall.
AP, Andhra Pradesh; **KA**, Karnataka; **KE**, Kerala; **PU**, Punjab; **UP**, Uttar Pradesh; **WB**, West Bengal.

either of the criteria. While in the relative case **KE** has four dominations, in the absolute case **PU** is the state with equal number dominations.

From Table 2, we see that in the urban sector, the relation $\geq_{AP}$ gives unambiguous rankings in 7 cases and its relative counterpart $\geq_P$ shows only 3 conclusive rankings. Thus, the number of unambiguous orderings by both the criteria is higher in the rural sector than in the urban sector. Once again we observe existence of some pairs that are judged inconclusive by $\geq_P$ turn out to be conclusive under $\geq_{AP}$. These are (**AP**, **KA**), (**AP**, **WB**), (**PU**, **UP**) and (**UP**, **WB**). One characteristic which is unique to the relative situation here is that none of the two states **AP** and **KE** dominates or is dominated by any other state. We also note that **UP**, which has the lowest median here, is dominated by the maximum number of states – four in the absolute case and one in the relative case. On the other hand, **PU**, the maximum median state, is dominant over only one state in each set up. What is worth noting is that for the pair (**KA**, **PU**) we have different directional rankings by the orderings concerned, that is, **PU** $\geq_{AP}$ **KA** but **KA** $\geq_P$ **PU**. This demonstrates a special feature of the concerned income distributions – higher (lower) differences from the median in the absolute case but lower (higher) differences in the relative case. Another interesting point that may be mentioned here is that the **APC** of **PU** intersects that of **WB** twice, first from above and then from below. However, in the relative case we have an unambiguous relation **WB** $\geq_P$ **PU**. In this special case the low median of **WB** may be a factor for making its **PC** steeper at the lower end so that **WB** $\geq_{AP}$ **PU** holds. These two observations along with the finding that several uncertain comparisons under $\geq_P$ may become certain under its absolute version $\geq_{AP}$ clearly establish that relative and absolute polarization are alternative conceptions and deserve to be treated separately.

Next, to show that inequality is not the same as polarization, we relate the subdiagonal part of Table 1 with the superdiagonal part of Table 3. This comparison involves only the rural dataset, in the urban case, the subdiagonal parts of Tables 2 and 3 are to be compared. (For brevity, we consider here only relative view of polarization and inequality. A similar analysis can be done for the absolute notion.) As in the context of a comparison between $\geq_{AP}$ and $\geq_P$, some states that are judged to dominate by one ordering (say $\geq_P$) become dominated by the other ($\geq_{LC}$) and vice versa. For instance, in the rural sector, for the pair (**KE**, **WB**), we have **KE** $\geq_P$ **WB** but **WB** $\geq_{LC}$ **KE**. Similarly, a clear verdict under one may become unclear under the other. This is demonstrated by the pair (**PU**, **WB**) in the urban sector, where we have **WB** $\geq_P$ **PU** but the Lorenz curves cross. Finally, as in the case between $\geq_{AP}$ and $\geq_P$, in each sector $\geq_{LC}$ gives more unambiguous rankings than $\geq_P$. These findings explicitly demonstrate that inequality and polarization are two separate aspects of income distribution analysis.

The concluding issue to be examined is a consequence of incompleteness of the orderings; that is, existence of two different indices that will rank distributions in the opposite directions when the concerned curves cross. Given the information that **APC**s of **KA** and **WB** cross in the urban sector, we note from Table 4 that Q(**WB**) < Q(**KA**) and J_α(**WB**) > J_α(**KA**) for $\alpha = 0.10$. Similarly, in the rural sector there is a cut between the **PC**s of **KA** and **PU**, and it emerges from Table 5 that F (**KA**) < F (**PU**) but K_r (**KA**) > K_r (**PU**) for $r = -0.5$. We do not go for verification of this feature in the Lorenz case because it is well known in the literature.

7. Conclusion

Absolute polarization indices, which depend on absolute income differentials, contrast with relative indices that are dependent on income shares. In this paper, we have isolated the class of all polarization indices of absolute type, whose ordering of two income distributions agrees with that generated by their non-intersecting absolute polarization curves. (An absolute polarization curve indicates, how far is the income for any population fraction, expressed as a fraction of the population size, from the median.) The latter ordering is shown to perform better (in the context of ranking distributions) than that based on non-intersecting polarization curves. (A polarization curve is obtained by dividing the absolute polarization curve by the median.) The possibility of transforming inequality indices with absolute character that rely on social welfare functions into polarization indices of the same variety is also explored. It is explicitly shown that to each translatable social welfare function, there exists a polarization index of absolute form that can be made to fulfil all the reasonable properties. We have employed several inequality indices to illustrate the framework. Which social welfare function will be adopted is essentially an issue of value judgement. A numerical illustration of the results developed has been provided using household expenditure data for rural and urban India.

One limitation remains. Throughout the paper we have assumed that income is the only indicator of welfare. However, income as the sole attribute of welfare is inappropriate and should be supplemented by other variables, e.g. housing, literacy, life expectancy, education, provision of public goods and so on. Welfare is inherently multidimensional from the view point of "capabilities" and "functionings", where functionings deal with what a person can ultimately do and capabilities indicates the freedom that a person enjoys in terms of functionings (Sen, 1985). In the capability approach, functionings are closely approximated by attributes such as literacy, life expectancy etc. and not by income alone. It will, therefore, be worth considering polarization orderings from a multidimensional perspective.

Appendix

Proof of Theorem 1:

In proving the theorem, we will assume that n is odd. A similar proof will run for the case when n takes even values.

(a) $\Rightarrow$ (b): Suppose that $m(x) > m(y)$. Define $z = y + [m(x) - m(y)] \cdot 1^n$. Since **APC** is translation invariant, $AP(y; p) = AP(z; p)$. Therefore, $x \geq_{AP} y$ is same as $x \geq_{AP} z$. Note also that $m(x) = m(z)$. Now by $x \geq_{AP} z$, we have,

$$\sum_{i=\bar{n}+1}^{k}\left(x_i - m(x)\right) \geq \sum_{i=\bar{n}+1}^{k}\left(z_i - m(z)\right), \quad \bar{n}+1 \leq k \leq n,$$

which in view of $m(x) = m(z)$ implies that

$$\sum_{i=\bar{n}+1}^{k} x_i \geq \sum_{i=\bar{n}+1}^{k} z_i, \tag{A1}$$

where $\bar{n}+1 \leq k \leq n$. Equation (A1) means that x_+ is weakly majorized by z_+ (Marshall and Olkin, 1979, p.10). That is equivalent to the condition that $x_+ = z_+S$, where $S = (s_{ij})$ is a doubly superstochastic matrix of order $n - \bar{n}$ (Marshall and Olkin, 1979, p.10). Now, S is doubly superstochastic if there exists a bistochastic matrix $M = (m_{ij})$ of order $n - \bar{n}$ such that $s_{ij} \geq m_{ij}$ for all i and j. Hence $x_+ \geq z_+M$. Since all incomes are ordered, z_+M cannot be a permutation of z_+ except z_+ itself. The relationship between z_+ and x_+ is then given by exactly one of the following:

$$\text{(i) } x_+ \neq z_+,\ z_+ = z_+M; \qquad \text{(ii) } x_+ = z_+M,\ z_+ \neq z_+M;$$
$$\text{(iii) } x_+ \neq z_+M,\ z_+ \neq z_+M; \text{ or (iv) } x_+ = z_+,\ z_+ = z_+M.$$

Condition (i) means that x_+ is obtained from z_+ by increasing some incomes above the median. If $z_+ \neq z_+M$, then z_+M is deducible from z_+ by a sequence of transfers, transferring income from rich to poor (Dasgupta *et al.*, 1973). Thus, (ii) is the condition that x_+ is obtained from z_+ by some progressive transfers in the region above the median. If (iii) holds, then x_+ is obtained from z_+ by a sequence of spread increasing movements away from the median and a sequence of egalitarian transfers for persons above the median.[6]

From $x \geq_{AP} z$ we can next establish that x_- is related to z_- by either one of the conditions that parallel (i)–(iii) or $x_- = z_-$. When $x = z$, $x_- = z_-$ holds along with (iv). Then by translation invariance of I^n, $I^n(y) = I^n(z) = I^n(x)$. If $x \neq z$ then one of (i)–(iii) ((i)–(iv)) and one of the four (first three) conditions showing relationship between x_- and z_- must hold simultaneously. This means that in either case, the overall distribution x can be acquired from the corresponding distribution z through spread augmenting movements away from the median and/or some equalizing transfers on the same side of the median.[7] Since I^n satisfies **NS** and **NB**, we have $I^n(x) \geq I^n(z)$. Note that I^n is symmetric because it has been defined on ordered distributions. As I^n is an absolute index, $I^n(y) = I^n(z)$. Hence, $I^n(x) \geq I^n(y)$.

If $m(x) \leq m(y)$, we define $t = x + [m(y) - m(x)].1^n$. Then $AP(t;\ p) = AP(x;\ p)$, so that $x \geq_{AP} y$ is the same as $t \geq_{AP} y$. Furthermore, t and y have the same median $m(y)$. The rest of the proof is analogous to the steps employed earlier and hence omitted.

(b) ⇒ (a): In proving this part, we follow Shorrocks (1983). Consider the polarization index $I^n_k(x) = \Sigma^k_{i=1}|m(x) - x_i|/n$, where $1 \leq k \leq n$. This index satisfies **NB**, **NS**, **SM** and translation invariance. Thus, $I^n_k(x) \geq I^n_k(y)$ for $1 \leq k \leq n$, which in turn implies $AP(x; p) \geq AP(y; p)$ for all $p \in [0, 1]$, that is, $x \geq_{AP} y$.

Proof of Theorem 2:

(a) ⇒ (b): Let z and t be n- and m-fold replications of x and y respectively. Since APC is population replication invariant, we have $APC(x; p) = APC(z; p)$ and $APC(y; p) = APC(t; p)$ Therefore, $x \geq_{AP} y$ gives $z \geq_{AP} t$. As z and t are two distributions over the population size mn, in view of theorem 1, we have $I^{mn}(z) \geq I^{mn}(t)$ for all absolute polarization indices I^{mn} that meet **NS**, **NB** and **SM**. By **PP**, we have, $I^{mn}(z) = I^m(x)$ and $I^{mn}(t) = I^n(y)$. Hence $I^n(x) \geq I^n(y)$.

(b) ⇒ (a): This part of the theorem follows from a construction similar to above, again using theorem 1.

Proof of Theorem 4:

The structure of the proof parallels that of proposition 1 of Chakravarty and Majumder (2001). For x_e given by Equation (10),

$$B(m(x)) = \exp(-\alpha(\mu - m(x))) \quad \text{and} \quad H(m(x)) = m(x)\exp(-\alpha(\mu - m(x)))/2 - 2m(x),$$

[6] The condition $x_+ \geq z_+$ along with $x_+ \neq z_+$, ensures that the generalized Lorenz curve of x_+ is nowhere outside that of x_+ and at some places (at least) inside the latter, where the generalized Lorenz curve of a distribution is obtained by scaling up its Lorenz curve by the mean income z_+ (see Shorrocks, 1983).

[7] Wang and Tsui (2000) provided a very brief proof of the criterion. Further, they do not relate this to ...ation in terms of polarization index.

the polarization index becomes

$$J_a^n(x) = (E_a^k(x_+) + 2\lambda(x_+))/2 + (E_a^k(x_-) - \exp(-\alpha(\mu - m(x)))\lambda(x_-))/2 \\ + m(x)\exp(-\alpha(\mu - m(x))/2 - 2m(x). \tag{A2}$$

Since x_i''s are illfare ranked, J_α^n meets **SM**. Verification of **PP** by J_α^n follows from population replication invariance of E_α^k, λ and m. Similarly, continuity of E_α^k, λ and m ensure that J_α^n meets **CN**. It can be checked easily that **NM** holds. The assumption that μ is measured in the unit in which incomes are measured, guarantee that J_α^n is translation invariant and hence an absolute index.

Fulfilment of **NB** follows from monotonic association of $E_\alpha^k(x_+)$ and $E_\alpha^k(x_-)$ with J_α^n. By a similar reasoning, we can see that J_α^n satisfies **NS** in the region above the median. To demonstrate explicitly that **NS** is satisfied in x_- also, suppose that given $x \in D^n$, y is obtained from x such that $x_+ = y_+$ and $y_- C x_-$. Assume further that $y_i = x_i + \theta$ and $y_j = x_j$ for all $j \neq i$, where x_j's are all smaller than the median and addition of θ to x_i does not change rank orders of incomes. Then it is necessary to show that

$$E_\alpha^k(y_-) - \exp(-\alpha(\mu - m(x)))\lambda(y_-) \leq E_\alpha^k(x_-) - \exp(-\alpha(\mu - m(x)))\lambda(x_-),$$

or,

$$E_\alpha^k(y_-) - E_\alpha^k(x_-) \leq \exp(-\alpha(\mu - m(x)))\theta/n(x_-) \tag{A3}$$

or,

$$(E_\alpha^k(y_-) - E_\alpha^k(x_-))/\theta \leq \exp(-\alpha(\mu - m(x)))/n(x_-), \tag{A4}$$

where $n(x_-)$ is the number of incomes in x_-. Since the right hand side of equation (A4) is independent of θ, we can choose any $\theta > 0$ on the left hand side. Taking the limit as θ tends to zero, we get,

$$\frac{\delta E_\alpha^k(x_-)}{\delta x_i} \leq \exp(-\alpha(\mu - m(x)))/n(x_-),$$

or,

$$\exp(-\alpha(x_i - E_\alpha^k(x_-))) \leq \exp(-\alpha(\mu - m(x))). \tag{A5}$$

(We have shown above that (A3) implies (A5). To show that the reverse implication is true as well, we partially integrate the left hand side of (A5) with respect to the i^{th} argument from lower limit x_i to the upper limit $x_i + \theta$ to get the left-hand side of (A3). By a similar integration, we get the right hand side of (A3) from the right hand side of (A5). Hence (A3) and (A5) are equivalent.)

In view of S-concavity of E^k, $E_\alpha^k(x_-) \leq \lambda(x_-)$. Therefore, the inequality in Equation (A5) follows from the observations that $\lambda(x_-) \leq m(x)$ and $\mu \leq \alpha \leq x_i$. This completes the proof of the theorem.

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