UNIQUENESS AND INDETERMINACY OF THE EQUILIBRIUM GROWTH PATH IN THE UZAWA-LUCAS MODEL WITH SECTOR SPECIFIC EXTERNALITIES*

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We introduce sector specific external effects of human capital on production in an otherwise Uzawa-Lucas model of endogenous growth; and show that the problem of indeterminacy of the competitive equilibrium growth path does not exist even if the production function satisfies the increasing returns to scale at the social level.
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1. Introduction

The models developed by Uzawa (1965) and Lucas (1988) (hereafter the Uzawa-Lucas model) are well known in the theoretical literature on endogenous growth. The Lucas (1988) model extends the Uzawa (1965) model by introducing the aggregate external effect of human capital into the final good production sector, which means that all the skilled workers employed in all the sectors produce the same external effect on production. However, that external effect may be sector specific, in which case only the skilled workers employed in the production sector produce an external effect on production. Gomez (2004) introduces sector specific externality in an otherwise identical Uzawa-Lucas model and shows that the competitive equilibrium steady-state growth path is socially optimal. This result is interesting because the Uzawa-Lucas model with aggregate external effect shows the competitive equilibrium steady-state solution to be suboptimal. Gomez (2004) assumes social constant returns to scale production technology in both the sectors; hence, it is a special case of the models examined by Benhabib *et al*. (2000) and Mino (2001) in which both sectors use human as well as physical capital.

However, Gomez (2004) does not analyse the transitional dynamic properties of the Uzawa-Lucas model with sector specific externalities. In this paper, we want to derive the conditions for uniqueness and indeterminacy of the competitive equilibrium growth path converging to the steady-state equilibrium in the Uzawa-Lucas model with sector specific externalities. We assume that the production technology satisfies CRS at the private level and IRS at the social level. Benhabib and Perli (1994) have undertaken a similar exercise in the Lucas model with aggregate externalities. We want to examine how different are the conditions derived by Benhabib and Perli (1994) from the corresponding conditions in the case of production function with sector specific externalities.

We obtain an interesting result from the present model. If only sector specific externalities are present in the production function satisfying social IRS then either the competitive equilibrium steady-state point is a saddle point with a unique saddle path or is an unstable

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equilibrium point. This rules out the possibility of multiple equilibria; hence this is in contrast to the Benhabib-Perli result that indeterminacy of the equilibrium growth path may emerge when the external effect of human capital is very high.[1] Also the conditions for unique saddle path in this case are different from those derived in Benhabib and Perli (1994).

This paper is organized in the following way. Section 2 presents the basic model and derives the equations of motion. In Section 3, we derive the conditions for uniqueness and indeterminacy of the competitive equilibrium growth path; we then compare them to those derived by Benhabib and Perli (1994). Concluding remarks are made in Section 4.

2. The model

The size of the labour force is normalized to unity. Otherwise the basic Uzawa-Lucas model is so well known that it needs no description. The dynamic optimization problem of the representative individual in the competitive economy is given by the following.

Maximize

$$\int_0^\infty U(C)e^{\rho t}dt$$

subject to $U(C) = (C^{1-\sigma} - 1)/(1 - \sigma)$ [utility function] $Y = AK^{\beta}(uh)^{1-\beta}.\ E$ [production function] $\dot{K} = Y - C$ [budget constraint] $\dot{h} = \delta(1 - u)h$ [human capital accumulation] where C = level of consumption; Y = level of output; K = level of capital stock; h = level of human capital; u = fraction of labour allocated to production; δ = productivity parameter in the human capital accumulation function; $U(\cdot)$ = utility function; ρ = rate of discount; E = external effect; σ = elasticity of marginal utility with respect to consumption; and β = capital elasticity of output.

In Lucas (1988), Benhabib and Perli (1994), Xie (1994), etc. external effect is derived from all skilled workers employed in both production sector and human capital accumulation sector. So we have $E = h^{\varepsilon}$. However, in Gomez (2004), external effect is derived from those workers who are employed in the production sector. Since u fraction of the identical workers is employed there, we have $E = (uh)^{\gamma}$. Here $\varepsilon(\gamma) > 0$ represents the elasticity of output with respect to the aggregate (sector specific) external effect of human capital. In our model, we consider both sector specific and aggregative external effect. So, $E = (uh)^{\gamma}h^{\varepsilon}$.

We analyse the transitional dynamic properties of the model. However the present model, without aggregative external effect is a modified version of Gomez (2004). In Gomez (2004) the production function exhibits diminishing returns to scale at the private level and constant returns to scale at the social level. Our production function satisfies the property of constant returns to scale at the private level and increasing returns to scale at the social level. This property is common to that in the Uzawa-Lucas model.

The representative individual solves this optimization problem with C and u being the control variables; we follow the style adopted by Benhabib and Perli (1994). Defining the appropriate current value Hamiltonian, maximizing it with respect to C and u, assuming interior solution and then using the first order optimality conditions[2] we derive the following equations of motion in the case where $E = (uh)^{\gamma}h^{\varepsilon}$.

[1] Xie (1994) also proves the similar result.

[2] The optimization exercise is done in the appendix which is available from the authors on request.

$$\dot{x} = Ax^{\beta}u^{1-\beta+\gamma} + \frac{\delta(1-\beta+\gamma+\varepsilon)}{\beta-1}(1-u)x - qx, \tag{1}$$

$$\dot{q} = \left(\frac{\beta}{\sigma} - 1\right)Ax^{\beta-1}u^{1-\beta+\gamma} \cdot q - \frac{\rho}{\sigma}q + q^2 \tag{2}$$

and

$$\dot{u} = \delta\frac{(\beta-\gamma-\varepsilon)}{\beta-\gamma}u^2 + \frac{\delta(1-\beta+\gamma+\varepsilon)}{\beta-\gamma}u - \frac{\beta}{\beta-\gamma}qu, \tag{3}$$

where x and q are two ratio variables given by

$$q = (C/K) \qquad \text{and} \qquad x = \frac{K}{h^{\frac{1-\beta+\gamma+\varepsilon}{1-\beta}}}.$$

Using these 3×3 dynamic systems we can examine how far the transitional dynamics properties in the case of only aggregate externalities are different from those in the case of only sector specific externalities.

3. The results

In the steady-state equilibrium, $\dot{x} = \dot{q} = \dot{u} = 0$. We denote the steady-state equilibrium values of x, q and u by x^*, q^* and u^*.

From Equation (1) setting $\dot{x} = 0$ we have,

$$x^* = \frac{1}{A}\left\{q^* + \frac{\delta(1-\beta+\gamma+\varepsilon)}{1-\beta}(1-u^*)\right\}^{\frac{1}{\beta-1}} \cdot (u^*)^{\frac{1-\beta+\gamma}{1-\beta}}.$$

Then using Equations (1) and (2) and setting $\dot{x} = \dot{q} = 0$, we have

$$q^* = \frac{(\sigma-\beta)}{\beta}\left[\frac{\rho}{(\sigma-\beta)} + \frac{(1-\beta+\gamma+\varepsilon)}{(1-\beta)}\delta(1-u)\right].$$

From Equation (3), with $\dot{u} = 0$, we have,

$$u^* = 1 - \frac{(1-\beta)(\delta-\rho)}{\delta[\sigma(1-\beta+\gamma+\varepsilon) - (\gamma+\varepsilon)]}.$$

For u^* to be less than one we need

$$\frac{(\delta-\rho)}{[\sigma(1-\beta+\gamma+\varepsilon) - (\gamma+\varepsilon)]} > 0$$

and for positive u^* we need

$$\frac{(1-\beta)(\delta-\rho)}{\delta[\sigma(1-\beta+\gamma+\varepsilon) - (\gamma+\varepsilon)]} < 1.$$

The Jacobian matrix which is evaluated at the steady-state equilibrium point corresponding to the dynamic system described by Equations (1), (2) and (3) is given by

$$J^* = w\begin{bmatrix} J^*_{xx} & \dfrac{x^*}{(\beta-1)u^*}((1-\beta+\gamma)J^*_{xx} - (1-\beta+\gamma+\varepsilon)\delta u^*) & -x^* \\ 0 & \dfrac{(\gamma-\beta+\varepsilon)}{(\gamma-\beta)}\delta u^* & -\dfrac{\beta}{(\beta-\gamma)}u^* \\ \dfrac{J^*_{xx}}{x^*}\left(\dfrac{\beta}{\sigma}-1\right)q^* & J^*_{xx}\dfrac{q^*}{u^*}\dfrac{(1-\beta+\gamma)}{(\beta-1)}\left(\dfrac{\beta}{\sigma}-1\right) & q^* \end{bmatrix},$$

where $J^*_{xx} = (\beta-1)Ax^{*\beta-1}u^{*1-\beta+\gamma} < 0$.

Here,

Trace of $J^* = \delta u^*\{[2(\gamma-\beta)+\varepsilon]/(\gamma-\beta)\} > 0$;

$$BJ^* = \frac{(\gamma+\varepsilon-\beta)}{(\gamma-\beta)}(\delta u^*)^2 + J^*_{xx}\cdot q^*\left[\frac{\sigma(1-\beta+\gamma)-\gamma}{\sigma(1-\beta)(\beta-\gamma)}\right]\beta;$$

and

$$\text{Det. } J^* = \frac{\delta u^* q^*}{\sigma(1-\beta)}\left[\frac{\sigma(1-\beta+\gamma+\varepsilon)-(\gamma+\varepsilon)}{\beta-\gamma}\right]\cdot\beta\cdot J^*_{xx}.$$

Let us first consider the case of only sector specific external effects on production, i.e. $\varepsilon = 0$ and $\gamma > 0$. Here Trace of $J^* > 0$; this implies that at least one latent root of J matrix is positive. It is a 3×3 system. So if Det. $J^* < 0$, then there are one negative latent root and two positive latent roots of J^* matrix; and in that case, the equilibrium path is locally unique. If Det. $J^* > 0$, then either all the three latent roots are positive which makes the steady-state equilibrium unstable, or two latent roots are negative and one latent root is positive which will lead to indeterminacy of growth paths. Note that

$$\text{Det. } J^* \underset{<}{>} 0 \qquad \text{if} \qquad \frac{\sigma(1-\beta+\gamma)-\gamma}{\beta-\gamma} \underset{>}{<} 0.$$

When

$$\frac{\sigma(1-\beta+\gamma)-\gamma}{\beta-\gamma} > 0,$$

Det J^* is negative, and hence the system involves one stable root. So, there exists a unique saddle path converging to the steady-state growth equilibrium.

Note that, if $\sigma = \beta$, then $[\sigma(1-\beta+\gamma)-\gamma]/(\beta-\gamma) = 1-\beta > 0$. In this case, the condition for uniqueness of growth path is always satisfied. Note that Xie (1994) assumes $\beta = \sigma$ from the viewpoint of technical simplicity while analysing the transitional dynamic properties of the Uzawa-Lucas model and shows that the equilibrium solution is unique (indeterminate) when $\gamma < (>)\ \beta$. Benhabib and Perli (1994) do not assume $\beta = \sigma$ in their analysis. However, if $\beta = \sigma$ is assumed in their model, their result is identical to that in Xie (1994). In the present case with only sector specific external effects, equilibrium growth path is always unique when $\sigma = \beta$ and the magnitude of the external effect parameter has no role to play in this context.

If $\sigma = 1$ then the condition for uniqueness of the equilibrium growth path is $\gamma < \beta$ in the case of a sector specific external effect; and $\varepsilon \in (\beta, 2\beta)$ in the case of an aggregative external effect.

Now we analyse the case where the solution is not unique. In the case of only a sector specific external effect, this non-uniqueness problem arises when Det. $[\sigma(1 - \beta + \gamma) - \gamma]/(\beta - \gamma) < 0$. Here we apply the Routh (Gantmacher, 1959) criterion according to which the number of positive roots of J^* matrix should be equal to the number of variations of sign in the scheme

$$\left[-1, \text{ Trace of } J^*, -BJ^* + \frac{\text{Det. } J^*}{\text{Tr. } J^*}, \text{ Det. } J^*\right].$$

Here Trace of J^* is positive and Det. $J^* > 0$.

Now,

$$-BJ^* + \frac{\text{Det. } J^*}{\text{Tr. } J^*} = -(\delta u^*)^2 - J^*_{xx} \cdot q^* \cdot \beta\left[\frac{\sigma(1-\beta+\gamma)-\gamma}{\sigma(1-\beta)(\beta-\gamma)}\right] + \frac{1}{2} \cdot J^*_{xx} \cdot q^* \cdot \beta\left[\frac{\sigma(1-\beta+\gamma)-\gamma}{\sigma(1-\beta)(\beta-\gamma)}\right]$$

$$= -(\delta u^*)^2 - \frac{q^*}{2}\beta \cdot J^*_{xx} \cdot \left[\frac{\sigma(1-\beta+\gamma)-\gamma}{\sigma(1-\beta)(\beta-\gamma)}\right]$$

and this is also negative when $[\sigma(1 - \beta + \gamma) - \gamma]/(\beta - \gamma) < 0$.

So the number of variations of sign in the scheme is three. Hence all the three latent roots of J^* matrix are positive when $[\sigma(1 - \beta + \gamma) - \gamma]/(\beta - \gamma) < 0$. So the intertemporal equilibrium point (x^*, q^*, u^*) is unstable. We do not have a case of two negative roots and one positive root, which leads to the indeterminacy of the solution.

The case of Benhabib and Perli (1994) is identical to the case of $\varepsilon > 0$ and $\gamma = 0$. In this case, Trace of J^* is negative if $\varepsilon > 2\beta$ and Determinant of J^* is positive if $\varepsilon > [((1-\beta)\sigma)/(1-\sigma)]$. If these two conditions are satisfied then there are two negative roots and one positive root in the system that leads to indeterminacy. In this case, if $\sigma = 1$ Determinant of J^* is negative and $\varepsilon < 2\beta$ is the necessary and sufficient condition for unique growth path. When this condition is violated then Trace of $J^* < 0$, Det. $J^* < 0$, $BJ^* < 0$. Hence, in this case with $\varepsilon > 2\beta$ and with $\sigma = 1$ there is only one variation in sign in the scheme,

$$\left[-1, \text{ Tr. } J^*, -BJ^* + \frac{\text{Det. } J^*}{\text{Tr. } J^*}, \text{ Det. } J^*\right].$$

This means that only one latent root of J^* matrix is positive and its other two latent roots are negative; and this leads to indeterminacy in solution. Thus, we have shown instances when indeterminacy of the equilibrium growth path may arise in our model with aggregative external effect, whereas it does not arise in the case of a sector specific external effect. Hence, we have the following proposition.

Proposition 1: *If the production technology in the Uzawa-Lucas economy is subject to the sector specific external effect only, then the steady-state equilibrium is either a saddle point with a unique saddle path or is unstable. The possibility of indeterminacy of solution arises only in the presence of an aggregative external effect.*

While understanding the intuition for the above mentioned result, let us first consider an arbitrary equilibrium growth path and another with a higher investment rate. For the second to be an equilibrium growth path, the rate of return from the physical capital must be sufficiently increased, otherwise it cannot justify its higher accumulation rate. Marginal productivity of physical capital varies positively with the stock of human capital. The human capital accumulation can be accelerated by reallocating more labour time from the production sector to the human capital accumulation sector. However, this can raise the rate of return on physical capital despite its accumulation only if the magnitude of the external effect of human capital is strong enough to ensure the high complementarity between physical capital and human capital. In the case of an aggregative external effect, the reallocation of workers from the production sector to the human capital accumulation sector does not lower the magnitude of the external effect because then the external effect is derived from all the workers in the economy. Thus, the external effect is very strong there. However, in the case of a sector specific external effect only the workers employed in the production sector contribute to the external effect. The reallocation of workers from the production sector lowers the magnitude of the external effect, and this weak external effect cannot ensure the high complementarity between physical capital and human capital.

4. Conclusion

This note gives an example where the consideration of sector specific externality of human capital shows the nonexistence of the indeterminacy of equilibrium growth path. This is an interesting result because many authors such as Benhabib and Farmer (1996), Mino (2001), Benhabib and Nishimura (1998), Meng (2003), Weder (2001), Nishimura and Venditti (2002) have attempted to show that sector-specific externalities can explain indeterminacy of equilibrium even when the magnitude of the external effect is very small. However, those authors have considered the sector-specific external effect of physical capital and have not used the Uzawa-Lucas framework. In this paper, a sector specific external effect comes from human capital and not from physical capital. So not only the nature of the external effect but also its source are important in explaining indeterminacy of equilibrium.

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