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On Linear Fractional Programming Problem and its Computation Using a Neural Network Model

S. K. Neogy · A. K. Das · P. Das

Abstract In this paper we consider linear fractional programming problem and look at its linear complementarity formulation. In the literature, uniqueness of solution of a linear fractional programming problem is characterized through strong quasiconvexity. We present another characterization of uniqueness through complementarity approach and show that the solution set of a fractional programming problem is convex. Finally we formulate the complementarity condition as a set of dynamical equations and prove certain results involving the neural network model. A computational experience is also reported.

Keywords KKT condition · Complementarity condition · Uniqueness of solutions · Convex set · Nonlinear dynamic system · Linear projection equation

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S. K. Neogy (✉)
Indian Statistical Institute, New Delhi 110 016, India
e-mail: skn@isid.ac.in

A. K. Das · P. Das
Indian Statistical Institute, Kolkata 700 108, India

1 Introduction

Given a matrix $A \in R^{m\times n}$, vectors $b \in R^m$, c, $d \in R^n$ and $\alpha, \beta \in R$, the *linear fractional programming problem* (*LFPP*) is the following:

$$\text{minimize } f(x) = \frac{c^t x + \alpha}{d^t x + \beta} \tag{1.1}$$

subject to

$$Ax \le b\,,\ -x \le 0, \tag{1.2}$$

where $d^t x + \beta > 0\ \forall\ x \in S$ and S denotes the feasible region defined by the constraint (1.2).

The above form of LFPP is called *general form* [4, p. 46]. For recent book on LFPP and its algorithms see Bajalinov [4]. See also [3] and [12].

Neural networks in optimization were introduced in early 1980s (see [1, 2]). The neural network approach in optimization is basically to establish a nonnegative energy function and a dynamic system that represents an artificial neural network. Normally, the dynamic system is in the form of first order differential equation. The concept behind the neural network based optimization techniques is that the objective function and constraints are mapped into a closed-loop network so that when a constraint violation occurs, the magnitude and direction of the violation are fed back to adjust the states of the neurons in the network. The energy function of the network decreases until it attains a minimum and the states of the neurons of the network are taken to be the minimizer of the original problem. The neural network approach seems to be promising for constrained optimization problems. On the other hand, classical methods for solving such problems involve an iterative procedure but large computational time limits their usage.

The organization of the paper is as follows. In Section 2, we present the required definitions, the notations, and some results used in this paper. A complementarity formulation of linear fractional programming problem using Karush–Kuhn–Tucker (KKT) conditions of optimality is considered in Section 3 and a sufficient condition for uniqueness of solutions of linear fractional programming problem is obtained. The solution set of LFPP is also shown convex. In Section 4, we propose a neural network model for solving LFPP described by the nonlinear dynamic system. We prove the sufficient condition of the dynamics for convergence to the global optimal point of LFPP. In Section 5, simulated results are presented for solving LFPP to show the effectiveness of the proposed dynamics. Finally, the proposed dynamics is compared with the dynamics of Xia and Wang [7] in Section 6.

2 Preliminaries

We denote the n-dimensional real space by R^n. We consider matrices and vectors with real entries. Any vector $x \in R^n$ is a column vector unless otherwise specified and x^t denotes the row transpose of x. For any two vectors $x, y \in R^n$, we define $\max(x, y)$ as the vector whose ith coordinate is $\max(x_i, y_i)$. By writing $A \in R^{m\times n}$, we denote that A is a matrix of real entries with m rows and n columns. For any matrix $A \in R^{m\times n}$, a_{ij} denotes its ith row and jth column entry. A is said to be a skew symmetric if $a_{ij} = -a_{ji}$ $\forall i, j$ and $a_{ii} = 0$ $\forall i$. A is a merely positive semidefinite matrix (MPSD) if it is a positive semidefinite matrix but not a positive definite matrix.

We require the following theorems and lemma in the next sections.

Lemma 2.1 ([3, Lemma 11.4.1]) *Let $f(x) = (c^t x + \alpha)/(d^t x + \beta)$, and let S be a convex set in R^n such that $d^t x + \beta \neq 0$ over S. Then, f is both pseudoconvex and pseudoconcave over S.*

Theorem 2.1 ([3, Theorem 3.5.11]) *Let S be a nonempty open convex set in R^n, and let $f : S \to R$ be a differentiable pseudoconvex function on S. Then, f is both strictly quasiconvex and quasiconvex.*

Theorem 2.2 ([3, Theorem 3.5.6]) *Let $f : S \to R$ be strictly quasiconvex. Consider the problem to minimize $f(x)$ subject to $x \in S$, where S is a nonempty convex set in R^n. If $\bar{x}$ is a local optimal solution, then $\bar{x}$ is also a global optimal solution.*

3 Complementarity and Linear Fractional Programming Problem

In this section, we describe complementary principles of mathematical programming problems.

3.1 Complementary Slackness Principle

An important aspect of the primal–dual relationship in linear programming (LP) is explained using complementary slackness principle. However, the complementary slackness principle holds not only for the linear programming problem; it also holds for more general programming problems. In particular, this principle is useful for minimizing a linear fractional function, in which the denominator does not vanish for any feasible x, and the constraints are linear. The complementary slackness principle for a programming problem is based on the KKT condition of optimality. A statement of this condition for a linear fractional programming problem with linear constraints in nonnegative variables is as follows:

Let $f : R^n_+ \to R$ be a pseudoconvex and pseudoconcave function. Let $A \in R^{m\times n}$ be a matrix and $b \in R^m$ be a vector. Consider the problem

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{subject to} \\ &Ax \leq b \\ &x \geq 0. \end{aligned}$$

Let $S = \{x | x \geq 0, Ax \leq b\}$. The KKT condition of optimality states that $\bar{x}$ is an optimal solution to the above problem if and only if there exist vectors $\bar{u} \in R^m$, $\bar{v} \in R^n$ such that

$$\begin{aligned} \nabla f(\bar{x}) + A^t \bar{u} - \bar{v} &= 0 \\ A\bar{x} &\leq b \\ \bar{x} &\geq 0, \quad \bar{u} \geq 0, \ \bar{v} \geq 0, \quad \bar{v}^t \bar{x} = 0 \\ \text{and} \ \ \bar{u}^t (b - A\bar{x}) &= 0 \end{aligned}$$

Note that the complementary slackness property $\bar{u}^t(b - A\bar{x}) = 0$, $\bar{v}^t \bar{x} = 0$ holds. For details see [3, Theorem 4.3.8].

3.2 Linear Complementarity Problem

The linear complementarity problem (LCP) is a combination of linear and nonlinear system of inequalities and equations. It is an important problem in mathematical programming and in other fields. The problem may be stated as follows:

Given a matrix $M \in R^{n \times n}$ and a vector $q \in R^n$, find $z \in R^n$ such that $Mz + q \geq 0$, $z \geq 0$ and $z^t(Mz + q) = 0$ (or prove that such a z does not exist).

Alternatively, the problem may be restated as follows:

Given a square matrix M of order n with real entries and an n dimensional vector q, find n dimensional vectors w and z satisfying

$$w - Mz = q, \quad w \geq 0, \ z \geq 0, \tag{3.1}$$

$$w^t z = 0 \tag{3.2}$$

or show that no solution exists.

This problem is denoted as LCP(q, M). If a pair of vectors (w, z) satisfies Eq. 3.1, then the problem LCP(q, M) is said to have a feasible solution. The Eq. 3.2 is known as *complementarity condition.* A pair of vectors (w, z) satisfying Eqs. 3.1 and 3.2 is called a solution to the LCP(q, M). LCP is normally identified as a part of optimization theory and equilibrium problems. The problems which can be posed as an LCP include linear programming, convex quadratic programming and bimatrix game. For recent books on this problem and applications see Cottle et al. [6] and Murty [11].

3.3 Linear Fractional Programming Problem

Consider the linear fractional programming problem given in Section 1. Suppose $S = \{x | Ax \leq b, x \geq 0\}$ denotes the set of *feasible solution* of LFPP and S^* denotes the set of optimal solution. We say that an LFPP is *solvable* if $S \neq \emptyset$. An LFPP is said to be *unbounded* if the problem has no finite lower bound. In what follows we assume that $d^t x + \beta \neq 0 \ \forall \ x \in S$. Without loss of generality, we assume that $d^t x + \beta > 0 \ \forall \ x \in S$. With this assumption the function $f(x)$ is both pseudoconvex and pseudoconcave. See [3]. It is easy to see that the problem of minimizing a linear fractional function subject to linear inequality conditions leads to a linear complementarity problem via the KKT conditions. See also [10].

Theorem 3.1 *Suppose D is an $n \times n$ matrix whose ith row and jth column element is given by $c_i d_j - c_j d_i$ for $i = 1, \ldots, n$; $j = 1, \ldots, n$. Then $\bar{x}$ is an optimal solution to Eqs. 1.1 and 1.2 iff $\bar{z} = \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}$ solves LCP(q, M) where $M = \begin{bmatrix} D & A^t \\ -A & 0 \end{bmatrix}$ and $q = \begin{bmatrix} \beta c - \alpha d \\ b \end{bmatrix}$.*

Proof Note that $f(x)$ is both pseudoconvex and pseudoconcave. Hence, the KKT optimality conditions are both necessary and sufficient for a point $\bar{x}$ to be a solution to Eqs. 1.1 and 1.2. Thus $\bar{x}$ is a solution to Eqs. 1.1 and 1.2 iff there exist vectors $\bar{y}, \bar{u}, \bar{v} \geq 0$ (where $\bar{y}, \bar{u} \in R^m$, and $\bar{v} \in R^n$) such that

$$\begin{aligned} \nabla f(\bar{x}) + A^t\bar{u} - \bar{v} &= 0 \\ A\bar{x} + \bar{y} &= b \\ \bar{x}^t\bar{v} + \bar{y}^t\bar{u} &= 0 \\ \bar{x} \geq 0, \bar{u} &\geq 0 \\ \bar{v} \geq 0, \bar{y} &\geq 0 \end{aligned}$$

Now for the LFPP we can easily calculate $\nabla f(\bar{x})$. This is given by

$$\nabla f(\bar{x}) = (d^t\bar{x} + \beta)^{-2}[(d^t\bar{x} + \beta)c - (c^t\bar{x} + \alpha)d]$$

This can be further written as

$$\nabla f(\bar{x}) = (d^t\bar{x} + \beta)^{-2}[D\bar{x} + \beta c - \alpha d]$$

where D is an $n \times n$ matrix whose ith row and jth column element is given by $c_i d_j - c_j d_i$ for $i = 1, \ldots, n$; $j = 1, \ldots, n$. We see that $\bar{x}$ is a solution to Eqs. 1.1 and 1.2 iff there exist vectors $\bar{y} \in R^m$, $\bar{u} \in R^m$ and $\bar{v} \in R^n$ such that

$$\begin{aligned} D\bar{x} + \beta c - \alpha d + A^t\bar{u} - \bar{v} &= 0 \\ A\bar{x} + \bar{y} &= b \\ \bar{x}^t\bar{v} + \bar{y}^t\bar{u} &= 0 \\ \bar{x} \geq 0, \bar{u} &\geq 0 \\ \bar{v} \geq 0, \bar{y} &\geq 0 \end{aligned}$$

Now it is easy to see that the above leads to the LCP

$$\begin{bmatrix} \bar{v} \\ \bar{y} \end{bmatrix} - \begin{bmatrix} D & A^t \\ -A & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} \beta c - \alpha d \\ b \end{bmatrix}, \begin{bmatrix} \bar{v} \\ \bar{y} \end{bmatrix} \geq 0, \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \geq 0$$

and the complementarity condition

$$\bar{v}^t\bar{x} + \bar{y}^t\bar{u} = 0$$

Hence the result. □

Remark 3.1 We note that the diagonal elements of M are 0 and $M = -M^t$. Such a matrix is PSD and it is processable by Lemke's algorithm. The algorithm presente by Lemke and Howson [9] to compute an equilibrium pair of strategies to a bimatrix game was extended later by Lemke [8] to solve an LCP(q, M). For a description on Lemke's algorithm see [6]. Note that the matrix M obtained from LFPP is a MPSD matrix.

Given a matrix M and a vector q we define the feasible set of LCP(q, M) as $F(q, M) = \{z \geq 0 \mid Mz + q \geq 0\}$ and the solution set $S(q, M) = \{z \in F(q, M) \mid z^t(Mz + q) = 0\}$ where $z = \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}$ and $M = \begin{bmatrix} D & A^t \\ -A & 0 \end{bmatrix}$. A solution $z \in S(q, M)$ is said to be *nondegenerate* if $z + (q + Mz) > 0$.

Note that f is both pseudoconvex and pseudoconcave (see Lemma 2.1). Therefore f is strictly quasiconvex and quasiconcave (see Theorem 2.1). However, if f is strongly quasiconvex then LFPP has a unique global optimal solution. See [3, Theorem 3.5.9]. The following theorem presents an alternative characterization regarding the uniqueness of the solution which is new in the literaure.

Theorem 3.2 *Consider the LFPP given by Eqs. 1.1 and 1.2. Suppose LCP(q, M) is the corresponding linear complementarity problem. Let $\hat{z}$ be nondegenerate and $\hat{z} = \begin{bmatrix} \hat{x} \\ \hat{u} \end{bmatrix} \in S(q, M)$. Further assume that $M_{\alpha\alpha}$ is nonsingular where $\emptyset \neq \alpha = \{i \mid \hat{z}_i > 0\}$. Then $\hat{x}$ is a unique solution of LFPP.*

Proof We look at the LCP(q, M) corresponding to LFPP. Let $\bar{z} = \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}$ be any solution to LCP(q, M). So, $\bar{z}$ and $\hat{z}$ solve LCP(q, M). Note that M is a MPSD matrix. By the positive semidefiniteness of M, it follows that

$$\begin{aligned} 0 \leq (\bar{z} - \hat{z})^t M(\bar{z} - \hat{z}) &= (\bar{z} - \hat{z})^t[(q + M\bar{z}) - (q + M\hat{z})] \\ &= -\bar{z}^t(q + M\hat{z}) - \hat{z}^t(q + M\bar{z}) \leq 0 \end{aligned}$$

Therefore,

$$\bar{z}^t(q + M\hat{z}) = \hat{z}^t(q + M\bar{z}) = 0 \tag{3.3}$$

Note that $\alpha = \{i \mid \hat{z}_i > 0\}$. So, by Eq. 3.3 we have $(q + M\bar{z})_i = 0$ for $i \in \alpha$. Since $\hat{z}$ is nondegenerate, $(q + M\hat{z})_i > 0$ for $i \notin \alpha$. Using Eq. 3.3, it follows that $\bar{z}_i = 0$ for $i \notin \alpha$. Therefore,

$$q_\alpha + M_{\alpha\alpha}\bar{z}_\alpha = 0 \tag{3.4}$$

The uniqueness of the solution of LCP(q, M) follows from Eq. 3.4 and the nonsingularity assumption of $M_{\alpha\alpha}$. Now by Theorem 3.1, it follows that the solution of LFPP is unique. □

Theorem 3.3 *S^* is a convex set where S^* is the set of optimal solution of LFPP.*

Proof Consider the LFPP given by Eqs. 1.1 and 1.2 and let LCP(q, M) be the corresponding linear complementarity problem. Suppose $\bar{z}$, $\hat{z} \in S(q, M)$. To show that the solution set is convex we show that the vector $z = \theta\bar{z} + (1 - \theta)\hat{z}$ is also a solution which belongs to $S(q, M)$ for any $\theta \in (0, 1)$. From Eq. 3.3, it is easy to see that $[\theta\bar{z} + (1-\theta)\hat{z}]^t[\theta(q + M\bar{z}) + (1-\theta)(q + M\hat{z})] = \theta(1-\theta)[\bar{z}^t(q + M\hat{z}) + \hat{z}^t(q + M\bar{z})] = 0$. Therefore $S(q, M)$ is a convex set. By Theorem 3.1, it follows that S^* is also a convex set. □

In 1962, Charnes and Cooper [5] showed that by a simple transformation, an LFPP can be reduced to an LP that can be solved using a simplex method. Swarup [14] instead of converting the LFPP into an equivalent linear program, attacked the problem directly and gave a very efficient Simplex type algorithm under the assumption that the denominator of the objective function is positive. In [13], Swarup derived certain characteristics of LFPP and developed an algorithm for LFPP (under certain limitations). Now we introduce another equivalent LP formulation of the LFPP under the assumption that the denominator of the objective function is positive.

Lemma 3.1 *The LFPP (Eqs. 1.1 and 1.2) considered in Section 1, can be written as the following LP*

$$\text{minimize } f(x) = (\beta c - \alpha d)^t x + b^t u \tag{3.5}$$

subject to

$$Dx + A^t u + \beta c - \alpha d \geq 0 \tag{3.6}$$

$$b - Ax \geq 0, \; -x \leq 0, \; -u \leq 0 \tag{3.7}$$

Proof Consider the following LCP(q, M) where

$$M = \begin{bmatrix} D & A^t \\ -A & 0 \end{bmatrix} \text{and } q = \begin{bmatrix} \beta c - \alpha d \\ b \end{bmatrix}.$$

It is easy to see that LCP(q, M) is equivalent to the following quadratic program (QP).

$$\text{minimize } f(x) = (\beta c - \alpha d)^t x + b^t u + \frac{1}{2}x^t Dx \tag{3.8}$$

subject to

$$Dx + A^t u + \beta c - \alpha d \geq 0 \tag{3.9}$$

$$b - Ax \geq 0, \; -x \leq 0, \; -u \leq 0 \tag{3.10}$$

Note that $D + D^t = 0$. Therefore the above QP is equivalent to the LP as stated in Eqs. 3.5, 3.6 and 3.7. □

4 Proposed Neural Network Model

We propose a recurrent neural network model which is described by the following nonlinear dynamic system. From Theorem 3.1, we observe that x is an optimal solution to Eqs. 1.1 and 1.2 iff there exist vectors $x \in R^n$ and $u \in R^m$ such that

$$\alpha d - \beta c - Dx - A^t u \leq 0 \tag{4.1}$$

$$Ax \leq b \tag{4.2}$$

$$x^t(\alpha d - \beta c - Dx - A^t u) = 0 \tag{4.3}$$

$$u^t(Ax - b) = 0 \tag{4.4}$$

$$x \geq 0, u \geq 0 \tag{4.5}$$

where D is an $n \times n$ matrix whose ith row and jth column element is given by $c_i d_j - c_j d_i$ for $i = 1, \ldots, n;\ \ j = 1, \ldots, n$.

The proposed neural network model can now be described by the following nonlinear dynamic system.

$$\frac{dx}{dt} = \alpha d - \beta c - D\left(x + k\frac{dx}{dt}\right) - A^t\left(u + k\frac{du}{dt}\right),\ x \geq 0 \tag{4.6}$$

$$\frac{du}{dt} = -b + A\left(x + k\frac{dx}{dt}\right),\ u \geq 0 \tag{4.7}$$

In the above neural network dynamics the coefficient k is some positive constant.

Theorem 4.1 *If the neural network whose dynamics is described by the differential Eqs. 4.6 and 4.7 converges to a stable state then the convergence state is the optimal solution for LFPP.*

Proof Equation 4.6 can be written as

$$\frac{dx_i}{dt} = \left[\alpha d - \beta c - D\left(x + k\frac{dx}{dt}\right) - A^t\left(u + k\frac{du}{dt}\right)\right]_i,\ \ \text{if } x_i > 0 \tag{4.8}$$

$$\frac{dx_i}{dt} = \max\left\{\left[\alpha d - \beta c - D\left(x + k\frac{dx}{dt}\right) - A^t\left(u + k\frac{du}{dt}\right)\right]_i, 0\right\}\ \text{if } x_i = 0 \tag{4.9}$$

Note that Eq. 4.9 ensures that x will be bounded from below by 0. Let $\lim_{t\to\infty} x(t) = x^*$ and $\lim_{t\to\infty} u(t) = u^*$. By stability of convergence, we have $\frac{dx^*}{dt} = 0$ and $\frac{du^*}{dt} = 0$. So, Eqs. 4.8 and 4.9 become,

$$[\alpha d - \beta c - Dx^* - A^t u^*]_i = 0,\ \text{if } x_i^* > 0. \tag{4.10}$$

$$\max\{[\alpha d - \beta c - Dx^* - A^t u^*]_i, 0\},\ \ \text{if } x_i^* = 0 \tag{4.11}$$

In other words:

$$[\alpha d - \beta c - Dx^* - A^t u^*]_i \leq 0,\ \forall\, i \tag{4.12}$$

$$x_i^*[\alpha d - \beta c - Dx^* - A^t u^*]_i = 0,\ \forall\, i \tag{4.13}$$

Therefore,

$$[\alpha d - \beta c - Dx^* - A^t u^*] \leq 0 \tag{4.14}$$

$$x^*[\alpha d - \beta c - Dx^* - A^t u^*] = 0 \tag{4.15}$$

Similarly, by taking the limit in Eq. 4.7 we get

$$[-b + Ax^*] \leq 0 \tag{4.16}$$

$$u^*[-b + Ax^*] = 0 \tag{4.17}$$

Hence we get the inequalities (4.1), (4.2), (4.3), (4.4), and (4.5). Therefore, (x^*, u^*) is a solution to LCP(q, M). By Theorem 3.1, x^* is an optimal solution of LFPP. □

Theorem 4.2 *Suppose x_0 is any seed point and the proposed dynamics converges to a stable state with a sufficiently small positive ϵ and $d^t x + \beta \neq 0$. Then the proposed dynamics always converges to the global optimal.*

Proof Suppose x_0 is a seed point and dynamics converges to a point $\bar{x}_0$ with sufficiently small positive ϵ which is local optimal. Note that pseudoconvexity of $f(x)$ implies strict quasiconvexity by Theorem 2.1. Now by Theorem 2.2 it follows that $\bar{x}_0$ is a global optimal. □

Theorem 4.3 *Assume the proposed neural network dynamics for LFPP converges to a stable state. Suppose S_1 : set of equilibrium points of the proposed dynamics, $S_2 = S(q, M)$: solution set of equivalent LCP(q, M) and $S_3 = S^*$: optimal solution set of LFPP are the three sets. Then $S_1 = S_2 = S_3$.*

Proof From Theorem 3.1, it follows that $S_2 = S_3$. We now prove $S_1 = S_2$. As the proposed dynamics converges then we get the system of inequalities given by Eqs. 4.1, 4.2, 4.3, 4.4, and 4.5. This implies that the solution set of inequalities is same as the equilibrium points of the proposed dynamics with sufficiently small positive ϵ. So $S_1 = S_2$. Hence $S_1 = S_2 = S_3$. □

Remark 4.1 In order to solve the differential equations 4.6 and 4.7, the Euler's method may be used. The following Matlab code describes the discrete implementation of our neural network. Coefficient k is set to equal the time step dt to simplify the calculations.

$$\begin{aligned}
\text{For } i &= 1:n;\\
du &= dt * (-b + A * (x + dx));\\
du &= \max(u + du, 0) - u;\ \ \%\,(\text{to make } u \geq 0)\\
u &= u + du;\\
dx &= dt * (\alpha d - \beta c - D * (x + dx) - A^t * (u + du));
\end{aligned}$$

$$
\begin{aligned}
dx &= max(x+dx, 0) - x; \ \ \%(\text{to make } x \geq 0) \\
x &= x + dx; \\
i &= i + 1; \\
&end;
\end{aligned}
$$

5 Simulation Results

Numerical experiments are conducted to demonstrate the effectiveness and efficiency of the proposed neural network dynamics. The simulation is carried out on Matlab to solve the differential equations using Euler's method. To start with, we initialize x, u, and dx at $t = 0$. We take small positive values for step length dt and ϵ. We run the dynamics and compute $\|dx\|$ and $\|du\|$. The dynamics stops if $\|dx\| < \epsilon$ and $\|du\| < \epsilon$. The simulation runs on a Compaq PC with intel pentium 4 processor 1.99 GHz 248 MB of RAM. Following examples are used for experimental purpose.

Example 5.1 Consider the following LFPP:

$$
\begin{aligned}
&\text{minimize} \quad \frac{-2x_1 + x_2 + 2}{x_1 + 3x_2 + 4} \\
&\text{subject to} \\
&\qquad -x_1 + x_2 \leq 4 \\
&\qquad x_2 \leq 6 \\
&\qquad 2x_1 + x_2 \leq 14 \\
&\qquad x_1,\ x_2 \geq 0
\end{aligned}
$$

In this case the solution $(6.9926, 0)$ is very close to the optimal solution $(7, 0)$ with only 127 iterations against the upper bound of the norm as 0.01.

Example 5.2 Consider the following LFPP:

$$
\begin{aligned}
&\text{minimize} \quad \frac{-5x_1 + 3x_2 + 6}{x_1 + x_2 + 5} \\
&\text{subject to} \\
&\qquad x_1 + 2x_2 \leq 10 \\
&\qquad 2x_1 + x_2 \leq 10 \\
&\qquad x_1,\ x_2 \geq 0
\end{aligned}
$$

The neural network dynamics for this LFPP generates solution $(5.0412, 0)$ with a step of 0.1 and ϵ as 0.01 after 137 iterations. The optimal solution of this problem is $(5, 0)$.

Example 5.3 Consider the following LFPP:

$$\begin{aligned} &\text{minimize} \quad \frac{-3x_1 + 5x_2 - 6x_3 + 24}{x_1 + x_2 + 2x_3 + 7} \\ &\text{subject to} \\ &\qquad -x_1 + x_2 + x_3 \le 4 \\ &\qquad 2x_1 + x_2 + x_3 \le 14 \\ &\qquad x_2 + x_3 \le 6 \\ &\qquad x_1,\ x_2,\ x_3 \ge 0 \end{aligned}$$

In this case, the dynamics converges to the optimal solution $(3.995, 0, 6.0082)$ after 757 iterations with ϵ equal to 0.01.

Example 5.4 Consider the following LFPP:

$$\begin{aligned} &\text{maximize} \quad \frac{3x_1 + 5x_2 - 3x_3}{x_1 - 4x_2 + 2x_3 + 3} \\ &\text{subject to} \\ &\qquad 2x_1 + x_2 + 6x_3 \le 3 \\ &\qquad -5x_1 + 4x_2 - x_3 \ge 5 \\ &\qquad x_1,\ x_2,\ x_3 \ge 0 \end{aligned}$$

Here $f(x)$ is both pseudoconvex and pseudoconcave. The dynamics converges to optimal solution $(0,\ 3,\ 0)$. The solution $(0, 3.0958, 0)$ is obtained here just after 378 iterations with a step of 0.1 and ϵ as 0.01.

Example 5.5 Consider the following LFPP:

$$\begin{aligned} &\text{minimize} \quad \frac{-2x_1 - 6x_2 - 2x_3 - 6x_4 - 5x_5 - 10}{4x_1 + 2x_2 + 2x_3 + 2x_4 + 5x_5 + 5} \\ &\text{subject to} \\ &\qquad x_1 \le 10 \\ &\qquad x_2 \le 0.1 \\ &\qquad x_3 \le 0.2 \\ &\qquad x_4 \le 20 \\ &\qquad x_5 \le 0.05 \\ &\qquad x_1,\ x_2,\ x_3,\ x_4,\ x_5 \ge 0 \end{aligned}$$

In this example, the dynamics converges to the point $(0, 0.10072, 0, 19.998, 0)$ after 5352 iterations with a step of 0.04 and ϵ equal to 0.0001. The optimal solution for this problem is $(0, 0.1, 0, 20, 0)$.

6 Comparison of the Method of Xia and Wang with the Proposed Method

Many optimization problems can be modeled as linear projection equations. Xia and Wang [7] proposed a recurrent neural network to solve linear projection equations in real time. In [7], it is claimed that if the matrix involved in the model is PSD, the model can converge globally to the solution set of the problem. As an application, it is shown that this model can be used directly to solve the linear and convex quadratic programming problems and linear complementarity problems with PSD matrices. We use Xia and Wang's model for comparison with our proposed model for linear fractional programming problem. The model of Xia and Wang can be described as follows.

$$\frac{du}{dt} = W\{P_{\Omega}(u - \beta(Mu + q)) - u\} \tag{6.1}$$

where β is a positive constant and W is a PD matrix.

Xia and Wang proved the following result in case of PSD matrices.

Theorem 6.1 ([7, Theorem 3]) *Suppose $W = B(I + \beta M^T)$ where B is a symmetric PD. If M is PSD the dynamic system (6.1) is stable in the sense of Lyapunov and globally converges to the solution subset of the linear projection equation.*

Here, for the purpose of comparison, we use the examples considered earlier. Initially, we set the step dt and the norm same for both the dynamics. Also, the seed point to operate the dynamics is chosen as null vector. Now, while comparing the rate of convergence towards meeting the requirements of stipulated norm, we note that the proposed dynamics takes uniformly less number of iterations for all the examples and reaches very close to global optimal solution than the dynamics of Xia and Wang. Computational experience on the performance of Xia and Wang model and the proposed model is reported in the following tables and figures (Tables 1, 2 and Figs. 1, 2, 3 and 4).

Table 1 Performance of Xia and Wang model

Example no.	dt	Norm	Iteration	Solution	cpu time	Optimal solution
5.1	0.1	0.01	802	(7.4327, 0.0080)	1.7813	(7,0)
5.2	0.1	0.01	991	(5.3655, -0.0406)	2.2656	(5,0)
5.3	0.1	0.01	4,504	(3.4286, -0.0003, 6.8584)	10.9531	(4,0,6)
5.4	0.1	0.01	2,075	(0.0342, 3.5188, -0.0105)	5.7344	(0,3,0)
5.5	0.04	0.0001	26,446	(0.0008, 0.1015, 0, 19.9999, 0)	88.5313	(0,0.1,0,20,0)

Table 2 Performance of the proposed model

Example no.	dt	Norm	Iteration	Solution	cpu time	Optimal solution
5.1	0.1	0.01	127	(6.9926,0)	0.1094	(7,0)
5.2	0.1	0.01	137	(5.0412,0)	0.1250	(5,0)
5.3	0.1	0.01	757	(3.995,0,6.0082)	0.7031	(4,0,6)
5.4	0.1	0.01	378	(0,3.0958,0)	0.3594	(0,3,0)
5.5	0.04	0.0001	5,352	(0,0.10072,0,19.998,0)	18.0938	(0,0.1,0,20,0)

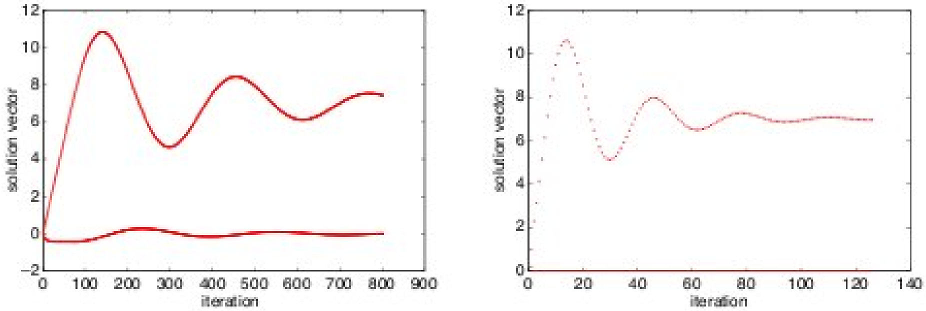

Fig. 1 Example no. 5.1

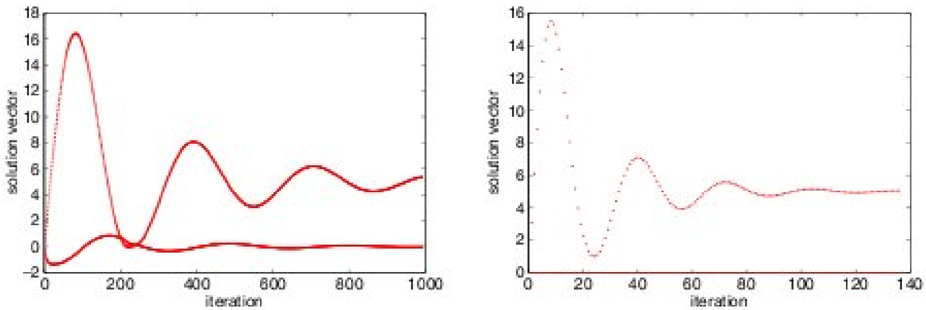

Fig. 2 Example no. 5.2

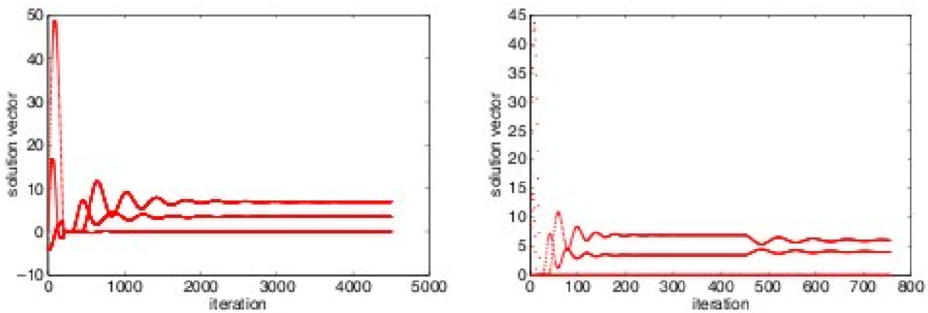

Fig. 3 Example no. 5.3

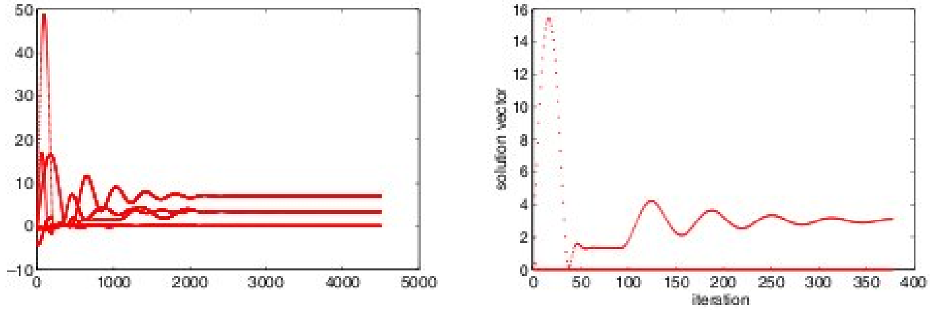

Fig. 4 Example no. 5.4

From the above tables and figures, we observe that the proposed model computes a global optimal point. In fact, for the examples considered here, we see that the proposed neural network model has the faster convergence to reach the global optimal point which is very encouraging.

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