Invariance and the twisted Chern character : a case study

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Abstract

We give details of the proof of the remark made in [7] that the Chern characters of the canonical generators on the K homology of the quantum group $SU_q(2)$ are not invariant under the natural $SU_q(2)$ coaction. Furthermore, the conjecture made in [7] about the nontriviality of the twisted Chern character coming from an odd equivariant spectral triple on $SU_q(2)$ is settled in the affirmative.

1 Introduction

Noncommutative geometry (NCG) (a la Connes, see [2]) and the C^*-algebraic theory of quantum groups (see, for example, [11], [10]) are two well-developed mathematical areas which share the basic idea of 'noncommutative mathematics', namely, to view a general (noncommutative) C^* algebra as noncommutative analogue of a topological space, equipped with additional structures resembling and generalizing those in the classical (commutative) situation, e.g. manifold or Lie group structure. A lot of fruitful interaction between these two areas is thus quite expected. However, such an interaction was not very common until recently, when a systematic effort by a number of mathematicians for understanding C^*-algebraic quantum groups as noncommutative manifolds in the sense of Connes triggered a rapid and interesting development to this direction. However, quite surprisingly, such an effort was met with a number of obstacles even in the case of the simplest non-classical quantum group, namely $SU_q(2)$ and it was not so clear for some time whether this (and other standard examples of quantum groups) could be nicely fitted into the framework of Connes' NCG (see [6] and the discussion and references therein). The problem of finding a nontrivial equivariant spectral triple for $SU_q(2)$ was finally settled in the affirmative in the papers by Chakraborty and Pal ([4], see also [3] and [5] for subsequent development), which increased the hope for a happy marriage between NCG and quantum group theory. However, even in the case of $SU_q(2)$, a few puzzling questions remain to be answered. One of them is the issue of invariance of the Chern character, which we have addressed in [7] and attempted to suggest a solution through the twisted version of the entire cyclic cohomology theory, building on the ideas of [8]. In that paper, we also made an attempt to study the connection between twisted and the conventional NCG following a comment in [3]. The present article is a follow-up of [7], and we mainly concentrate on $SU_q(2)$, considering it as a test-case for comparing the twisted and conventional formulation of NCG.

2 Notation and background

Let $\mathcal{A} = SU_q(2)$ (with $0 < q < 1$) denote the C^*-algebra generated by two elements α, β satisfying

$$\alpha^*\alpha + \beta^*\beta = I, \quad \alpha\alpha^* + q^2\beta\beta^* = I, \quad \alpha\beta - q\beta\alpha = 0,$$

$$\alpha\beta^* - q\beta^*\alpha = 0, \quad \beta^*\beta = \beta\beta^*.$$

We also denote the $*$-algebra generated by α and β (without taking the norm completion) by $\mathcal{A}^\infty$. There is a Hopf $*$ algebra structure on $\mathcal{A}^\infty$, as can be seen from, for example, [10]. We denote the canonical coproduct on $\mathcal{A}^\infty$ by Δ. We shall also use the so-called Sweedler convention, which we briefly explain now. For $a \in \mathcal{A}^\infty$, there are finitely many elements $a_i^{(1)}, a_i^{(2)}$, $i = 1, 2, ..., p$ (say), such that $\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$. For notational convenience, we abbreviate this as $\Delta(a) = a^{(1)} \otimes a^{(2)}$. For any positive integer m, let $\mathcal{A}_m^\infty$ be the m-fold algebraic tensor product of $\mathcal{A}^\infty$. There is a natural coaction of $\mathcal{A}^\infty$ on $\mathcal{A}_m^\infty$ given by

$$\Delta_{\mathcal{A}}^m(a_1 \otimes a_2 \otimes ... \otimes a_m) := ({a_1}^{(1)} \otimes ...{a_m}^{(1)}) \otimes ({a_1}^{(2)}...{a_m}^{(2)}),$$

using the Sweedler notation, with summation being implied. Let us recall the convolution $*$ defined in [7]. If $\phi : \mathcal{A}_m^\infty \to \mathbb{C}$ is an m-linear functional, and $\psi : \mathcal{A}^\infty \to \mathbb{C}$ is a linear functional, we define their convolution $\phi * \psi : \mathcal{A}_m^\infty \to \mathbb{C}$ by the following :

$$(\phi * \psi)(a_1 \otimes ... \otimes a_m) := \phi(a_1^{(1)} \otimes ... \otimes a_m^{(1)})\psi(a_1^{(2)}...a_m^{(2)}),$$

using the Sweedler convention. We say that an m-linear functional ϕ is *invariant* if $\phi * \psi = \psi(1)\phi$ for every functional ψ on $\mathcal{A}^\infty$.

In [9], the K-homology $K^*(\mathcal{A}^\infty)$ has been explicitly computed. It has been shown there that $K^0(\mathcal{A}^\infty) = K^1(\mathcal{A}^\infty) = \mathbb{Z}$, and the Chern characters (in cyclic cohomology) of the generators of these K-homology groups, denoted by $[\tau_{\text{ev}}]$ and $[\tau_{\text{odd}}]$ respectively, are also explicitly written down.

3 Main results

3.1 Chern characters are not invariant

In this subsection, we give detailed arguments for a remark made in [7] about the impossibility of having an invariant Chern character for $\mathcal{A}^\infty$ under the conventional (non-twisted) framework of NCG. To make the notion of invariance precise, we give the following definition (motivated by a comment by G. Landi, which is gratefully acknowledged).

Definition 3.1 *We say that a class $[\phi] \in HC^n(\mathcal{A}^\infty)$ is invariant if there is an invariant $n+1$-linear functional ϕ' such that ϕ' is a cyclic cocycle and $\phi' \sim \phi$ (i.e. $[\phi] = [\phi']$).*

It is easy to see that the Chern chracter $[\tau_{\mathrm{ev}}]$ cannot be invariant. Had it been so, it would follow from the uniqueness of the Haar state (say h) on $SU_q(2)$ that τ_{ev} must be a scalar multiple of h. Since τ_{ev} is a nonzero trace, it would imply that h is a trace too. But it is known (see [10]) that h is not a trace.

However, proving that $[\tau_{\mathrm{odd}}]$ is not invariant requires little bit of detailed arguments. We begin with the following observation.

Lemma 3.2 *If τ is a trace on $\mathcal{A}^\infty$, i.e. $\tau \in HC^0(\mathcal{A}^\infty)$, then we have that*

$$(\partial\xi) * \tau = \partial(\xi * \tau)$$

for every functional ξ on $\mathcal{A}^\infty$, where the Hochschild coboundary operator ∂ is defined by

$$(\partial\xi)(a,b) = \xi(ab) - \xi(ba).$$

Proof :
We shall use the Swedler notation. We have that for $a_0, a_1 \in \mathcal{A}^\infty$,

$$\begin{aligned}
&(\partial\xi * \tau)(a_0,a_1)\\
&\quad= (\partial\xi)(a_0^{(0)} \otimes a_1^{(0)})\tau(a_0^{(1)}a_1^{(1)})\\
&\quad= \xi(a_0^{(0)}a_1^{(0)})\tau(a_0^{(1)}a_1^{(1)}) - \xi(a_1^{(0)}a_0^{(0)})\tau(a_0^{(1)}a_1^{(1)})\\
&\quad= \xi(a_0^{(0)}a_1^{(0)}\tau(a_0^{(1)}a_1^{(1)}) - \xi(a_1^{(0)}a_0^{(0)})\tau(a_1^{(1)}a_0^{(1)}) \quad \text{(since } \tau \text{ is a trace)}\\
&\quad= (\xi * \tau)(a_0a_1) - (\xi * \tau)(a_1a_0)\\
&\quad= \partial(\xi * \tau)(a_0,a_1).
\end{aligned}$$

□

The above lemma allows us to define the multiplication $*$ at the level of cohomology classes. More precisely, for $[\phi] \in HC^1(\mathcal{A}^\infty)$ and $[\eta] \in HC^0(\mathcal{A}^\infty)$, we set $[\phi] * [\eta] := [\phi * \eta] \in HC^1(\mathcal{A}^\infty)$, which is well-defined by the Lemma 3.2. Similarly $[\eta] * [\phi]$ and $[\eta] * [\eta']$ (where $[\eta'] \in HC^0(\mathcal{A}^\infty)$) can be defined. We now recall from [9] that

$$[\tau_{\text{ev}}] * [\tau_{\text{ev}}] = [\tau_{\text{ev}}], \quad [\tau_{\text{ev}}] * [\tau_{\text{odd}}] = [\tau_{\text{odd}}] * [\tau_{\text{ev}}] = 0.$$

We also note that $\tau_{\text{ev}}(1) = 1$ and that τ_{ev} is a trace, i.e. $\tau_{\text{ev}}(ab) = \tau_{\text{ev}}(ba)$.

Using this observation, we are now in a position to prove that the Chern character of the generator of $K^1(\mathcal{A}^\infty)$ is not an invariant class.

Theorem 3.3 $[\tau_{\text{odd}}]$ *is not invariant.*

Proof :
Suppose that there is $\phi \sim \tau_{\text{odd}}$ such that ϕ is invariant. Then we have

$$[\phi * \tau_{\text{ev}}] = [\phi] * [\tau_{\text{ev}}] = [\tau_{\text{odd}}] * [\tau_{\text{ev}}] = 0.$$

However, since we have $\phi * \tau_{\text{ev}} = \tau_{\text{ev}}(1)\phi = \phi$ by the invariance of ϕ, it follows that $[\phi] = [\phi * \tau_{\text{ev}}] = 0$, that is, $[\tau_{\text{odd}}] = 0$, which is a contradiction. $\square$

3.2 Nontrivial pairing with the twisted Chern character

As already mentioned in the introduction, in [7] we have made an attempt to recover the desirable property of invariance by making a departure from the conventional NCG and using the twisted entire cyclic cohomology. We briefly recall here some of the basic concepts from that paper and refer the reader to [7] and the references therein for more details of this approach. We shall use the results derived in that paper wihout always giving a specific reference.

Let us give the definition of twisted entire cyclic cohomology for Banach algebras for simplicity, but note that the theory extends to locally convex algebras, which we actually need. The extension to the locally convex algebra case follows exactly as remarked in [1, page 370]. So, let $\mathcal{A}$ be a unital Banach algebra, with $\|.\|_*$ denoting its Banach norm, and let σ be a continuous automorphism of $\mathcal{A}$, $\sigma(1) = 1$. For $n \geq 0$, let C^n be the space of continuous $n+1$-linear functionals ϕ on $\mathcal{A}$ which are σ-invariant, i.e. $\phi(\sigma(a_0), ..., \sigma(a_n)) = \phi(a_0, ..., a_n) \forall a_0, ..., a_n \in \mathcal{A}$; and $C^n = \{0\}$ for

$n < 0$. We define linear maps $T_n, N_n : C^n \to C^n$, $U_n : C^n \to C^{n-1}$ and $V_n : C^n \to C^{n+1}$ by,

$$(T_n f)(a_0, ..., a_n) = (-1)^n f(\sigma(a_n), a_0, ..., a_{n-1}), N_n = \sum_{j=0}^{n} T_n^j,$$

$$(U_n f)(a_0, ..., a_{n-1}) = (-1)^n f(a_0, ..., a_{n-1}, 1),$$

$$(V_n f)(a_0, ..., a_{n+1}) = (-1)^{n+1} f(\sigma(a_{n+1}) a_0, a_1, ..., a_n).$$

Let $B_n = N_{n-1} U_n (T_n - I)$, $b_n = \sum_{j=0}^{n+1} T_{n+1}^{-j-1} V_n T_n^j$. Let B, b be maps on the complex $C \equiv (C^n)_n$ given by $B|_{C^n} = B_n, b|_{C^n} = b_n$. It is easy to verify (similar to what is done for the untwisted case , e.g. in [2]) that $B^2 = 0$, $b^2 = 0$ and $Bb = -bB$, so that we get a bicomplex ($C^{n,m} \equiv C^{n-m}$) with differentials d_1, d_2 given by $d_1 = (n - m + 1)b : C^{n,m} \to C^{n+1,m}$, $d_2 = \frac{B}{n-m} : C^{n,m} \to C^{n,m+1}$. Furthermore, let $C^e = \{(\phi_{2n})n \in I\!N; \phi_{2n} \in C^{2n} \forall n \in I\!N\}$, and $C^o = \{(\phi_{2n+1})n \in I\!N; \phi_{2n+1} \in C^{2n+1} \forall n \in I\!N\}$. We say that an element $\phi = (\phi_{2n})$ of C^e is a σ-twisted even entire cochain if the radius of convergence of the complex power series $\sum \|\phi_{2n}\| \frac{z^n}{n!}$ is infinity, where $\|\phi_{2n}\| := \sup_{\|a_j\|_* \leq 1} |\phi_{2n}(a_0, ..., a_{2n})|$. Similarly we define σ-twisted odd entire cochains, and let $C^e_\epsilon(\mathcal{A}, \sigma)$ ($C^o_\epsilon(\mathcal{A}, \sigma)$ respectively) denote the set of σ-twisted even (respectively odd) entire cochains. Let $\tilde{\partial} = d_1 + d_2$, and we have the short complex $C^e_\epsilon(\mathcal{A}, \sigma) \underset{\tilde{\partial}}{\overset{\tilde{\partial}}{\rightleftarrows}} C^o_\epsilon(\mathcal{A}, \sigma)$. We call the cohomology of this complex the σ-twisted entire cyclic cohomology of $\mathcal{A}$ and denote it by $H^*_\epsilon(\mathcal{A}, \sigma)$. Let $\mathcal{A}_\sigma = \{a \in \mathcal{A} : \sigma(a) = a\}$ be the fixed point subalgebra for the automorphism σ. There is a canonical pairing $< .,. >_{\sigma,\epsilon}: K_*(\mathcal{A}_\sigma) \times H^*_\epsilon(\mathcal{A}, \sigma) \to \mathbb{C}$. We shall need the pairing for the odd case, which we write down :

$$< [u], [\psi] > \equiv < [u], [\psi] >_{\sigma,\epsilon} = \frac{1}{\sqrt{2\pi i}} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{(2n+1)!} \psi_{2n+1}(u^{-1}, u, ..., u^{-1}, u),$$

where $[u] \in K_1(\mathcal{A}_\sigma)$ and $[\psi] \in H^1_\epsilon(\mathcal{A}, \sigma)$.

Definition 3.4 *Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{A}^\infty$ be a $*$ subalgebra (not necessarily complete) of $\mathcal{B}(\mathcal{H})$, R be a positive (possibly unbounded) operator on $\mathcal{H}$, D be a self-adjoint operator in $\mathcal{H}$ with compact resolvents such that the following hold :*
(i) $[D, a] \in \mathcal{B}(\mathcal{H})$ $\forall a \in \mathcal{A}^\infty$,
(ii) R commutes with D,

(iii) For any real number s and $a \in \mathcal{A}^\infty$, $\sigma_s(a) := R^{-s} a R^s$ is bounded and belongs to $\mathcal{A}^\infty$. Furthermore, for any positive integer n, $\sup_{s \in [-n,n]} \|\sigma_s(a)\| < \infty$.
Then we call the quadruple $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ an odd R-twisted spectral data. We say that the odd twisted spectral data is Θ-summable if Re^{-tD^2} is trace-class for all $t > 0$.

Let us now recall the construction of twisted Chern character from a given odd twisted spectral data $(\mathcal{A}^\infty, \mathcal{H}, D, R)$. Let $\mathcal{B}$ denote the set of all $A \in \mathcal{B}(\mathcal{H})$ for which $\sigma_s(A) := R^{-s} A R^s \in \mathcal{B}(\mathcal{H})$ for all real number s, $[D, A] \in \mathcal{B}(\mathcal{H})$ and $s \mapsto \|\sigma_s(A)\|$ is bounded over compact subsets of the real line. In particular, $\mathcal{A}^\infty \subseteq \mathcal{B}$. We define for $n \in I\!\!N$ an $n+1$-linear functional F_n on $\mathcal{B}$ by the formula

$$F_n(A_0, ..., A_n) = \int_{\Sigma_n} Tr(A_0 e^{-t_0 D^2} A_1 e^{-t_1 D^2} ... A_n e^{-t_n D^2} R) dt_0 ... dt_n,$$

where $\Sigma_n = \{(t_0, ..., t_n) : t_i \geq 0, \sum_{i=0}^n t_i = 1\}$.

Let us now equip $\mathcal{A}^\infty$ with the locally convex topology given by the family of Banach norms $\|.\|_{*,n}, n = 1, 2, ...$, where $\|a\|_{*,n} := \sup_{s \in [-n,n]} (\|\sigma_s(a)\| + \|[D, \sigma_s(a)]\|)$. Let $\mathcal{A}$ denote the completion of $\mathcal{A}^\infty$ under this topology, and thus $\mathcal{A}$ is Frechet space. We can construct the (twisted) Chern character in $H^o_\epsilon(\mathcal{A}, \sigma)$, where $\sigma = \sigma_1$, which extends on the whole of $\mathcal{A}$ by continuity.

Theorem 3.5 *Let $\phi^o \equiv (\phi_{2n+1})_n$ be defined by*

$$\phi_{2n+1}(a_0, ..., a_{2n+1}) = \sqrt{2i} F_{2n+1}(a_0, [D, a_1], ..., [D, a_{2n+1}]), a_i \in \mathcal{A}.$$

Then we have $(b + B)\phi^o = 0$, hence $\psi^o \equiv ((2n+1)! \phi_{2n+1})_n \in H^o_\epsilon(\mathcal{A}, \sigma)$.

We shall also need some results from the theory of semifinite spectral triples and the corresponding JLO cocycles and index formula, as discussed in, for example, [1]. An odd semifinite spectral triple is given by $(\mathcal{C}, \mathcal{N}, \mathcal{K}, D)$, where $\mathcal{K}$ is a separable Hilbert space, $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ is a von Neumann algebra with a faithful semifinite normal trace (say τ), D is a self-adjoint operator affiliated to $\mathcal{N}$, $\mathcal{C}$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{K})$ such that $[D, c] \in \mathcal{B}(\mathcal{K})$ for all $c \in \mathcal{C}$. In the terminology of [1], $(\mathcal{N}, D)$ is also called an odd, unbounded Breuer-Fredholm module for the norm-closure of $\mathcal{C}$. It is called Θ-summable if $\tau(e^{-tD^2}) < \infty$ for all $t > 0$. For a Θ-summable semifinite spectral triple, there is a canonical construction of JLO cocycle and index theorem (see [1]), which are very similar to their counterparts in the conventional framework of NCG.

Let us now settle in the affirmative conjecture made in [7] about the nontriviality of the twisted Chern character of a natural twisted spectral data obtained from the equivariant spectral triple of [4]. For reader's convenience, we briefly recall the construction of this equivariant spectral triple. Let us index the space of irreducible (co-)representations of $SU_q(2)$ by half-integers, i.e. $n = 0, \frac{1}{2}, 1, ...$; and index the orthonormal basis of the corresponding $(2n+1)^2$ dimensional subspace of $L^2(SU_q(2), h)$ by $i, j = -n, ..., n$, instead of $1, 2, ..., (2n+1)$. Thus, let us consider the orthonormal basis $e^n_{i,j}, n = 0, \frac{1}{2}, ...; i, j = -n, -n+1, ..., n$ in the notation of [4]. We consider any of the equivariant spectral triples constructed by the authors of [4] and in the associated Hilbert space $\mathcal{H} = L^2(SU_q(2), h)$ define the following positive unbounded operator R :

$$R(e^n_{i,j}) = q^{-2i-2j} e^n_{i,j},$$

$n = 0, \frac{1}{2}, , 1, ...; i, j = -n, -n+1, ..., n$. Let us choose a spectral triple given by the Dirac operator D on $\mathcal{H}$, defined by

$$D(e^n_{i,j}) = d(n, i) e^n_{i,j},$$

where $d(n, i)$ are as in (3.12) of [4], i.e. $d(n, i) = 2n + 1$ if $n = i$, $d(n, i) = -(2n+1)$ otherwise. It can easily be seen that $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is an odd R-twisted spectral data and furthermore, the fixed point subalgebra $SU_q(2)_\sigma$ for $\sigma(.) = R^{-1} \cdot R$ is the unital $*$-algebra generated by β, so it contains $u = I_1(\beta^*\beta)(\beta - I) + I$ which can be chosen to be a generator of $K_1(SU_q(2)) = \mathbb{Z}$ (see [4]). It is easily seen that the map from $K_1(C^*(u))$ to $K_1(SU_q(2))$, induced by the inclusion map, is an isomorphism of the K_1-groups (where $C^*(u)$ denotes the unital C^*-algebra generated by u). Thus, we can consider the pairing of the twisted Chern character with $K_1(C^*(u))$, and in turn with $K_1(SU_q(2))$ using the isomorphism noted before. The important question raised in [7] is whether we recover the nontrivial pairing obtained in [4] in our twisted framework, and in what follows, we shall give an affirmative answer to this question.

Theorem 3.6 *The pairing between $K_1(SU_q(2)_\sigma) \cong K_1(SU_q(2))$ and the (twisted) Chern character of the above twisted spectral data coincides with the pairing between $K_1(SU_q(2))$ and the Chern character of the (non-twisted) spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$. In particular, this pairing is nontrivial.*

Proof :

Let $\mathcal{N}$ be the von Neumann algebra in $\mathcal{B}(\mathcal{H})$ generated by β and $f(D)$ for all bounded measurable functions $f : \mathbb{R} \to \mathbb{C}$. Since R commutes with both β and D, it is easy to see that the functional $\mathcal{N} \ni X \mapsto \tau(X) := Tr(XR)$ defines a faithful, normal, semifinite trace on the von Neumann algebra $\mathcal{N}$. Moreover, $(\mathcal{N}, D)$ is an unbounded Θ-summable Breuer-Fredholm module for the norm-closure of the unital $*$-algebra (say $\mathcal{C}$) generated by β.

Moreover, it follows from the fact that R commutes with D and u that the pairing of $[u]$ with the twisted Chern character (say $\psi^o \equiv (\psi_{2n+1})$) coming from the twisted spectral data $(\mathcal{A}^\infty, \mathcal{H}, D, R)$ is given by

$$
\begin{aligned}
&< [u], [\psi^o] > \\
&= \frac{1}{\sqrt{2\pi i}} \sum_{n=0}^{\infty} (-1)^n \frac{n!}{(2n+1)!} \psi_{2n+1}(u^{-1}, u, ..., u^{-1}, u) \\
&\quad \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n n! \int_{\Sigma_{2n+1}} Tr(u^{-1} e^{-t_0 D^2} [D, u] e^{t_1 D^2} ... [D, u] e^{t_{2n+1} D^2} R) dt_0 ... dt_{2n+1}, \\
&= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n n! \int_{\Sigma_{2n+1}} \tau(u^{-1} e^{-t_0 D^2} [D, u] e^{t_1 D^2} ... [D, u] e^{t_{2n+1} D^2}) dt_0 ... dt_{2n+1}
\end{aligned}
$$

which is nothing but the pairing between $[u] \in K_1(\mathcal{C})$ and the Breuer-Fredholm module $(\mathcal{N}, D)$ mentioned before. By Theorem 10.8 of [1] and a straightforward but somewhat lengthy calculation along the lines of index computation in [4], we can show that the value of this pairing is equal to $-ind_\tau(A) \equiv -(\tau(P_A) - \tau(Q_A))$ for the following operator $A : \mathcal{H}_0 \to \mathcal{H}_0$, where $\mathcal{H}_0$ is the closed subspace spanned by $\{e^n_{n,j}, n = 0, \frac{1}{2}, ..., j = -n, -n+1, ..., n\}$, P_A, Q_A are the orthogonal projections onto the kernel of A and the kernel of A^* respectively and where r is a positive integer such that $q^{2r} < \frac{1}{2} < q^{2r-2}$:

$$
Ae^n_{n,j} = -q^{(n+j)(2r+1)}(1-q^{2(n-j)})^r(1-q^{2(n-j-1)})^{\frac{1}{2}} e^{n+\frac{1}{2}}_{n+\frac{1}{2}, j-\frac{1}{2}} + (1-q^{2r(n+j)}(1-q^{2(n-j)})^r) e^n_{n,j}.
$$

It can be verified by computations as in [4] that $Ker(A) = \{0\}$ and $Ker(A^*)$ is the one dimensional subspace spanned by the vector $\xi = \sum_{n=\frac{1}{2},\frac{3}{2},...} p_n e^n_{n,-n}$, where $p_{\frac{1}{2}} = 1$ and for $n \geq \frac{3}{2}$,

$$
p_n = \frac{1-(1-q^{4n-2})^r}{(1-q^{4n})^{\frac{1}{2}}(1-q^{4n-2})^r} ... \frac{1-(1-q^2)^r}{(1-q^4)^{\frac{1}{2}}(1-q^2)^r}.
$$

Clearly, since $Re^n_{-n,n} = e^n_{n,-n}$, we have $R\xi = \xi$ and thus

$$
-ind_\tau(A) = \frac{1}{\|\xi\|^2} \tau(|\xi >< \xi|) = \frac{1}{\|\xi\|^2} Tr(R|\xi >< \xi|) = 1,
$$

which is the same as the value of the pairing between $[u] \in K_1(SU_q(2))$ and the conventional Chern character corresponding to the spectral triple constructed in [4]. $\Box$

Thus we see that both the conventional and twisted frameworks of NCG give essentially the same results for the example we considered, namely $SU_q(2)$. The aparent weakness of the twisted NCG arising from the fact that the twisted cyclic cohomology can be paired naturally with only the K theory of the invariant subalgebra and not of the whole algebra, does not seem to pose any essential difficulty for studying the noncommutative geometric aspects of $SU_q(2)$, since by a suitable choice of the twisting operator R as we did one could make sure that the K theory of the corresponding invariant subalgebra is isomorphic with the K theory of the whole, and also the pairing between the Chern character and the generator of the K theory in the twisted framework is equal to the similar pairing in the ordinary (non-twisted) framework of NCG. It will be important and interesting to investigate whether a similar fact remains true for a larger class of quantum groups, and we hope to pursue this in the future.

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