Invariants for normal completely positive maps on the hyperfinite II_1 factor

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Dedicated to Professor K B Sinha

Abstract. We investigate certain classes of normal completely positive (CP) maps on the hyperfinite II_1 factor $\mathcal{A}$. Using the representation theory of a suitable irrational rotation algebra, we propose some computable invariants for such CP maps.

Keywords. Completely positive map; II_1 factor; irrational rotation C^* algebras.

1. Introduction

Quantum dynamical semigroups (QDS), i.e. C_0-semigroups of completely positive (CP) contractive maps on C^* or von Neumann algebras (assumed to be normal in the von Neumann algebra case), are important objects both from physical and mathematical viewpoint. It is thus quite natural to look for a nice classification scheme for them. As in every branch of mathematics, one would like to find one or more computable (preferably numerical) invariants for QDS with respect to some suitable equivalence relation, which is by now accepted as the so-called cocycle congugacy introduced and studied in a series of papers by Arveson [1–3], Powers [11], Bhat [5, 6] and others. There has already been a considerable amount of literature on this problem as well as the related (and in some sense equivalent) problem of classifying product system of Hilbert modules, thanks to the intensive works by a number of mathematicians (see, e.g. [1, 4, 14] and references therein). However, while much is known about QDS on $\mathcal{B}(\mathcal{H})$ (where $\mathcal{H}$ is a separable Hilbert space), and a nice numerical invariant ('Arveson index', see, e.g. [2]) is available in this case, QDS on other types of von Neumann algebra are not so well understood, in the sense that there is not yet any satisfactory numerical (or easily computable) invariant for QDS (or, equivalently, product systems of Hilbert modules) on general von Neumann algebras. In order to construct such invariants, it is reasonable to first restrict attention to just a single CP map instead of semigroup of CP maps. This is what we attempt to do in the present article for the hyperfinite type II_1 factor. While we are not yet able to come up with a satisfactory numerical invariant for an arbitrary normal CP map on the hyperfinite II_1 factor, we do get a nice invariant for an interesting class of CP maps, namely the 'pure' ones. This invariant, though not numerical, is given by a quadruplet (m, ν, b_1, b_2) where m is a nonnegative integer (so it can be thought of as an analogue of 'Arveson index' in this case), ν is a measure (with suitable property) on the two-torus $\mathbb{T}^2$, and $b_1, b_2 \colon \mathbb{T}^2 \to M_m(\mathbb{C})$ are unitary valued ν-measurable maps. Thus, the invariant is in some sense not a very abstract object, and easily computable. It

can be described at least by a sequence of complex numbers, if we describe the measure ν as well as the bounded functions $z \mapsto b_i(k,l) := (k,l)$-th entry of $b_i(z)$, $i = 1, 2$, in terms of the Fourier coefficients. Apart from the pure CP maps, we also study in some detail another class which we call 'extendible'. We give necessary and sufficient conditions for extendibility, and propose some invariants for such CP maps, including numerical invariants.

2. Preliminaries

2.1 *Intertwiner module*

Let $\mathbf{h}$ be a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(\mathbf{h})$ be a unital von Neumann algebra. Given a normal unital CP map $T: \mathcal{A} \to \mathcal{A}$, by Stinespring's and KSGNS theorems (see [10]) we can obtain a Hilbert von Neumann $\mathcal{A}-\mathcal{A}$ bimodule (to be called KSGNS bimodule) $\mathcal{E}_T \subseteq \mathcal{B}(\mathbf{h}, \mathcal{K}_T)$ for some Hilbert space $\mathcal{K}_T$, a normal representation $\pi_T: \mathcal{A} \to \mathcal{L}(\mathcal{E}_T) \subseteq \mathcal{B}(\mathcal{K})$ and $\xi_T \in \mathcal{E}_T$ such that $T(a) = \langle \xi_T, \pi_T(a)\xi_T \rangle, \forall a \in \mathcal{A}$. Here $\langle \xi, \eta \rangle := \xi^* \eta$ is the $\mathcal{A}$-valued inner product on $\mathcal{E}_T$ and $\pi_T(\mathcal{A})\xi_T\mathcal{A}$ is total in $\mathcal{E}_T$ which implies that $\pi_T(\mathcal{A})\xi_T\mathbf{h}$ is total in $\mathcal{K}_T$. Setting $\rho_T: \mathcal{A}' \to \mathcal{B}(\mathcal{K}_T)$ by

$$\rho_T(a')\pi_T(a)\xi h = \pi_T(a)\xi_T a' h,$$

we get a normal representation ρ_T of $\mathcal{A}'$ such that $\pi_T(\mathcal{A})$ and $\rho_T(\mathcal{A}')$ commute and the Hilbert (right) module $\mathcal{E}_T$ coincides with the intertwiner module $I(\rho_T, \mathcal{A}')$ (see [9, 13]) of $(\rho_T, \mathcal{A}')$, which is given by

$$I(\rho_T, \mathcal{A}') = \{R \in \mathcal{B}(\mathbf{h}, \mathcal{K}_T) : Ra' = \rho_T(a')R, \forall a' \in \mathcal{A}'\}.$$

Lemma 2.1. *Let T and $\mathcal{E}_T$ be discussed as above. Then*

$$\mathcal{L}(\mathcal{E}_T) = (\rho_T(\mathcal{A}'))'.$$

Proof. Let $B \in \mathcal{L}(\mathcal{E}_T)$. For $a \in \mathcal{A}$, $a' \in \mathcal{A}'$, $h \in \mathbf{h}$, we have

$$B\rho_T(a')\pi_T(a)\xi_T h = B\pi_T(a)\,\xi_T a' h = \rho_T(a')(B\pi_T(a)\xi_T)h.$$

Since $\pi(\mathcal{A})\xi_T\mathbf{h}$ is total in $\mathcal{K}_T$, $B \in (\rho_T(\mathcal{A}'))'$.

For the converse, let $B \in (\rho_T(\mathcal{A}'))'$. For $a' \in \mathcal{A}'$ and $\xi \in \mathcal{E}_T$ we have

$$B\xi a' = B\rho_T(a')\xi = \rho_T(a')B\xi.$$

This implies that $B\xi$ is in the intertwiner module $\mathcal{E}_T$.

Let us now introduce various equivalence relations on the set CP($\mathcal{A}$) of normal unital CP maps. From now on, unless there is a specific need to mention, we shall omit the adjectives 'normal CP unital'.

The following equivalence relation is motivated by the definition of cocycle conjugacy for CP maps.

DEFINITION 2.2

We say that two elements T, S of CP($\mathcal{A}$) are equivalent, denoted as $T \approx S$, if there exist unitaries u, v in $\mathcal{A}$, such that for all $a \in \mathcal{A}$,

$$S(a) = v^* T(u^* a u) v.$$

We also introduce another (stronger) equivalence relation $\sim$.

DEFINITION 2.3

$T \sim S$ if there exists unitary $u \in \mathcal{A}$ such that $S(a) = T(u^*au)$ for all $a \in \mathcal{A}$.

If $S \approx T$ as above, we have

$$\begin{aligned} S(a) &= v^* T(u^*au)v = v^*\langle \xi_T, \pi_T(u^*au)\xi_T\rangle v \\ &= v^*\xi_T^*\pi_T(u)^*\pi_T(a)\pi_T(u)\xi_T v = \langle \xi_S, \pi_T(a)\xi_S\rangle, \end{aligned}$$

where $\xi_S = \pi_T(u)\xi_T v$. So the corresponding KSGNS bimodules can be chosen to be the same, say $\mathcal{E}$, and $\xi_S = W\xi_T$ for some unitary $W \in \mathcal{L}(\mathcal{E})$.

It is not unreasonable to identify two CP maps if the KSGNS bimodules are isomorphic, i.e. can be chosen to be the same. This leads to the third equivalence relation.

DEFINITION 2.4

We say that T and S in $\mathrm{CP}(\mathcal{A})$ are KSGNS-equivalent, to be denoted by $T \overset{'}{\sim} S$, if the bimodules $\mathcal{E}_T$ and $\mathcal{E}_S$ are isomorphic.

It is clear from our discussion that $T \sim S \Rightarrow T \approx S \Rightarrow T \overset{'}{\sim} S$. However, the converse implications can be shown to be false in general.

2.2 *Second order irrational rotation algebra and its representations*

Fix an irrational number θ, and consider the irrational rotation algebra $\mathcal{A}_\theta$, (see [8]), which is the universal C^*-algebra generated by two unitaries U and V satisfying the Weyl commutation relation $UV = \lambda VU$, where $\lambda = \mathrm{e}^{2\pi i\theta}$. Let τ be the unique faithful normalized trace on $\mathcal{A}_\theta$ and $(\mathcal{H}_\tau \equiv L^2(\mathcal{A}_\theta, \tau), \pi_\tau, 1)$, be the associated GNS triple. From now, we fix the von Neumann algebra $\mathcal{A}$ as the (unique up to isomorphism) hyperfinite type II_1 factor, identified with the weak closure of $\mathcal{A}_\theta$ in $\mathcal{B}(\mathcal{H}_\tau)$. It is well-known that $\mathcal{A}_\theta$ is isomorphic with the crossed product $\mathcal{C}(\mathbb{T}) \rtimes \mathbb{Z}$, with respect to the action described in [8]. Define the unitary operators U_l, V_l, U_r and V_r in $\mathcal{B}(\mathcal{H}_\tau)$ by $U_l a = Ua$, $V_l a = Va$, $U_r a = aU$ and $V_r a = aV$ for $a \in \mathcal{A}_\theta \subseteq \mathcal{H}_\tau$. It is clear that the C^*-algebras $C^*(U_l, V_l)$ and $C^*(U_r, V_r)$ are weakly dense $*$-sub-algebras of $\mathcal{A}$ and $\mathcal{A}'$ respectively. The vector 1 is cyclic and separating both for $\mathcal{A}$ and $\mathcal{A}'$.

Consider the $*$-sub-algebras $\mathcal{A}^{l,\mathrm{fin}}$ and $\mathcal{A}^{r,\mathrm{fin}}$ of $\mathcal{A}$ and $\mathcal{A}'$ respectively, where $\mathcal{A}^{l,\mathrm{fin}}$ is the unital $*$-algebra generated by all polynomials in the unitaries U_l and V_l, and $\mathcal{A}^{r,\mathrm{fin}}$ is the similar algebra obtained by replacing U_l, V_l by U_r, V_r respectively. Clearly, $\mathcal{A}^{l,\mathrm{fin}}$ is norm-dense in $\mathcal{A}_\theta$. Moreover, let $\mathcal{A}^{\mathrm{fin}}$ denote the $*$-algebra of $\mathcal{A}^{l,\mathrm{fin}} \otimes_{\mathrm{alg}} \mathcal{A}^{r,\mathrm{fin}}$ generated by polynomials in all the four unitaries U_l, V_l, U_r and V_r and $\mathcal{B}$ be the C^* algebra obtained by completing $\mathcal{A}^{\mathrm{fin}}$ in the norm-topology of $\mathcal{B}(\mathcal{H}_\tau \otimes \mathcal{H}_\tau)$.

Lemma 2.5. *The C^*-algebra $\mathcal{B}$ is isomorphic (as C^* algebra) with the crossed product C^*-algebra $\mathcal{C}(\mathbb{T}^2) \rtimes \mathbb{Z}^2$ with respect to the action α given by $\alpha(k_1, k_2)(f)(z_1, z_2) = f(\lambda^{-k_1}z_1, \lambda^{-k_2}z_2)$.*

Proof. By definition, the crossed product C^*-algebra $\mathcal{C}(\mathbb{T}^2) \rtimes \mathbb{Z}^2$ with respect to the action α is the universal C^* algebra generated by four unitaries W_1, W_2, W_3, W_4 satisfying

$$W_1W_2 = W_2W_1,\ W_3W_4 = W_4W_3,\ W_1W_3 = \lambda W_3W_1,\ W_2W_4 = \lambda W_4W_2.$$

It can be verified by a straightforward calculation that the linear map $\eta: \mathcal{C}(\mathbb{T}^2) \rtimes \mathbb{Z}^2 \to \mathcal{A}^{\text{fin}}$ defined by

$$\eta(W_1) = U_l \otimes 1,\ \eta(W_2) = 1 \otimes V_r,\ \eta(W_3) = V_l^{-1} \otimes V_r,$$
$$\eta(W_4) = U_l^{-1} \otimes U_r,$$

gives the required isomorphism. □

From the general theory of C^* crossed product, it is easy to show that $\mathcal{B} \cong \mathcal{C}(\mathbb{T}^2) \rtimes \mathbb{Z}^2$ is simple, and thus any nonzero representation is a C^* isomorphism. Thus, the C^* algebras $\mathcal{B}, \mathcal{C}(\mathbb{T}^2) \rtimes \mathbb{Z}^2$ and $C^*(U_l, V_l, U_r, V_r)(\subseteq \mathcal{B}(\mathcal{H}_\tau))$ are all canonically isomorphic, and we shall identify them whenever there is no chance of confusion, and shall call this C^* algebra the 'second order irrational rotation algebra', and denote it by $\mathcal{A}_\theta^{(2)}$. It is actually the 4-dimensional noncommutative torus in the sense of Rieffel and Schwarz [12], corresponding to the skew-symmetric 4×4 matrix $A = ((a_{ij}))$, with $a_{12} = a_{34} = \theta, a_{21} = a_{43} = -\theta$, and all other entries are zero. Moreover, from the proof of the above Lemma, it is clear that the map η gives an algebraic isomorphism between $\mathcal{A}^{\text{fin}}$ and the $*$-algebra generated by $W_i, i = 1, \ldots, 4$.

The weak closure of the C^*-algebra $\mathcal{A}_\theta^{(2)}$ in the weak topology inherited from $\mathcal{B}(\mathcal{H}_\tau \otimes \mathcal{H}_\tau)$ is $\mathcal{A} \otimes \mathcal{A}' \cong W^*(U_l, V_r) \rtimes \mathbb{Z}^2 \cong L^\infty(\mathbb{T}^2) \rtimes \mathbb{Z}^2$, which is again a concrete realization of the (unique up to isomorphism) II_1 hyperfinite factor, with the faithful normal trace $\tau^{(2)}$ given by the vector state $\langle 1 \otimes 1, \ \cdot\ 1 \otimes 1\rangle$. We introduce here some notation for future use. We denote by ξ_0 the vector $1 \otimes 1$, and by $c^{(r)}$ (where $c \in \mathcal{A} \otimes \mathcal{A}'$) the operator of right multiplication by c on $\mathcal{H}_\tau \otimes \mathcal{H}_\tau$, i.e. $c^{(r)}b := bc$, for $b \in \mathcal{A} \otimes \mathcal{A}'$, viewed as an L^2-element. The commutant of $\mathcal{A} \otimes \mathcal{A}'$, i.e. $\mathcal{A}' \otimes \mathcal{A}$, coincides with the set $\{c^{(r)}: c \in \mathcal{A} \otimes \mathcal{A}'\}$. We shall use the notation $\tau^{(2)}(X)$ for $\langle \xi_0, X\xi_0\rangle$ whenever it makes sense, even when the operator X is unbounded but contains ξ_0 in its domain.

Now we make an important observation regarding extension of representations.

PROPOSITION 2.6

Any representation π of the $*$-algebra $\mathcal{A}^{I,\text{fin}}$ ($\mathcal{A}^{\text{fin}}$) extends to the C^*-algebra $\mathcal{A}_\theta$ ($\mathcal{A}_\theta^{(2)}$).

Proof. Let us give a sketch of the proof for $\mathcal{A}^{I,\text{fin}}$ only. The proof for $\mathcal{A}^{\text{fin}}$ is similar. Let $\pi: \mathcal{A}^{I,\text{fin}} \to \mathcal{B}(\mathcal{H}')$ be a representation. Thus, we have two unitaries $\pi(U_l)$ and $\pi(V_l)$ acting on $\mathcal{B}(\mathcal{H}')$, satisfying the commutation relation $\pi(U_l)\pi(V_l) = \lambda\pi(V_l)\pi(U_l)$. Since the commutative C^*-algebra $\mathcal{C}(\mathbb{T})$ is the universal C^*-algebra generated by a unitary, we can get a representation π_0, say, of $\mathcal{C}(\mathbb{T})$ which maps $f \in \mathcal{C}(\mathbb{T})$ to $f(\pi(U_l))$. Clearly, $(\pi_0, \pi(V_l))$ is a covariant representation for the action $(m \cdot f)(z) := f(\lambda^m z)$, and thus, there exists a representation $\tilde{\pi}: \mathcal{C}(\mathbb{T}) \rtimes \mathbb{Z}\ (\equiv \mathcal{A}_\theta)$ into $\mathcal{B}(\mathcal{H}')$ satisfying $\tilde{\pi}(U_l) = \pi(U_l), \tilde{\pi}(V_l) = \pi(V_l)$.

2.3 *Irreducible representations of $\mathcal{A}_\theta^{(2)}$*

Following [7], we say that a separable representation π of the irrational rotation algebra $\mathcal{A}_\theta$ has the uniform multiplicity m if the restriction of π to $\mathcal{C}(\mathbb{T}^2)$ has uniform multiplicity $m \in \{1, 2, \ldots, \infty\}$. The factor representations (in particular, the irreducible representations) are in the above class. In [7], the author gave a nice invariant of uniform multiplicity

representation on the irrational rotation algebra $\mathcal{A}_\theta$. These representations are classified by their multiplicities m, a quasi-invariant Borel measure ν (where quasi-invariance is with respect to rotation by the angle $2\pi\theta$) and a unitary one-cocycle b. It is not difficult to verify that the arguments of [7] can easily be extended to the second order irrational rotation algebra $\mathcal{A}_\theta^{(2)}$. We summarize this fact as a theorem below, the proof of which is omitted since it is very similar to the proofs of analogous results in [7].

Theorem 2.7.

(i) *Any irreducible representation of* $\mathcal{A}_\theta^{(2)} = C(\mathbb{T}^2) \rtimes \mathbb{Z}^2 = C^*(W_1, W_2, W_3, W_4)$ *is unitarily equivalent to a representation of the form* $\pi_{(m,\nu,b_1,b_2)}$ *described below, where* $m \in \mathbb{N} \cup \{0\}$, ν *is an ergodic regular Borel measure on* $\mathbb{T}^2$ *which is also quasi-invariant with respect to the action* α *of* $\mathbb{Z}^2$ *given by* $\alpha_{(n_1,n_2)}z = (\lambda^{n_1}z_1, \lambda^{n_2}z_2)$, *and two* ν*-measurable unitary valued functions* $b_1, b_2\colon \mathbb{T}^2 \to \mathcal{M}_m(\mathbb{C})$ *such that* $\pi_{(m,\nu,b_1,b_2)}\colon \mathcal{A}_\theta^{(2)} \to \mathcal{B}(L^2(\mathbb{T}^2, \nu) \otimes \mathbb{C}^m)$ *is given by*

$$\pi(W_1)f(z) = z_1 f(z),$$

$$\pi(W_2)f(z) = z_2 f(z),$$

$$\pi(W_3)f(z) = b_1(z)\sqrt{\frac{d\nu_{1,0}}{d\nu}}(z)f(\lambda z_1, z_2),$$

$$\pi(W_4)f(z) = b_2(z)\sqrt{\frac{d\nu_{0,1}}{d\nu}}(z)f(z_1, \lambda z_2), \tag{2.1}$$

where $f\colon \mathbb{T}^2 \to M_m(\mathbb{C})$, *and* $\frac{d\nu_{n_1,n_2}}{d\nu}$ *is the Radon–Nikodym derivative of the measure* ν_{n_1,n_2} *given by*

$$\nu_{n_1,n_2}(E) = \nu\{\alpha_{n_1n_2}z\colon z \in E\}$$

with respect to ν. *Note that* ν_{n_1,n_2} *is equivalent to* ν *as* ν *is quasi-invariant.*

(ii) *Two irreducible representations* $\pi_{(m,\nu,b_1,b_2)}$ *and* $\pi_{(\tilde{m},\tilde{\nu},\tilde{b_1},\tilde{b_2})}$ *are unitarily equivalent if and only if*

1. $m = \tilde{m}$ *and* $\nu = \tilde{\nu}$ *can be chosen;*
2. *there exists a unitary-valued,* ν*-measurable map* $W\colon \mathbb{T}^2 \to \mathcal{M}_m(\mathbb{C})$ *satisfying*

$$W(z)b_1(z) = \tilde{b_1}(z)W(\lambda z_1, z_2), \quad W(z)b_2(z) = \tilde{b_2}(z)W(z_1, \lambda z_2).$$

3. Invariants for CP maps on type II_1 factor

3.1 *State associated with CP map*

Given a normal CP unital map T on the type II_1 factor $\mathcal{A}$, consider the associated KSGNS bimodule $\mathcal{E}_T$, cyclic element $\xi_T \in \mathcal{E}_T$, and the representations π_T, ρ_T discussed in §2. Now we define a representation $\tilde{\pi}_T\colon \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}' \to \mathcal{B}(\mathcal{K}_T)$ by

$$\tilde{\pi}_T(a \otimes a') = \pi_T(a)\rho_T(a').$$

This is a homomorphism since $\pi_T(a)$ and $\rho_T(a')$ commute. By Proposition 2.6, the restriction of $\tilde{\pi}$ to the weakly dense subalgebra $\mathcal{A}^{\text{fin}}$ extends to the C^*-algebra $\mathcal{A}_\theta^{(2)}$, and we denote this extension again by the same notation. Since $\pi_T(\mathcal{A})\xi_T\mathcal{H}_\tau$ is dense in $\mathcal{K}_T$, and 1 is cyclic for $\mathcal{A}'$, it is clear that $\xi_T 1$ is cyclic for $(\tilde{\pi}_T, \mathcal{A} \otimes_{\text{alg}} \mathcal{A}', \mathcal{K}_T)$. We define a state ψ_T on $\mathcal{A}_\theta^{(2)}$ by setting

$$\psi_T(\cdot) = \langle \xi_T 1, \tilde{\pi}_T(\cdot)\xi_{1_1} \rangle. \tag{3.1}$$

Since $\tilde{\pi}_T$ is defined also on $\mathcal{A} \otimes_{\text{alg}} \mathcal{A}'$, we shall define ψ_T on $\mathcal{A} \otimes_{\text{alg}} \mathcal{A}'$ too, given by the same expression as in (3.1). Note that we have, $\psi_T(a \otimes a') = \langle 1, T(a)a' 1 \rangle$. It is clear from the construction of ψ_T that $(\mathcal{K}_T, \tilde{\pi}_T, \xi_T 1)$ is a choice of GNS triple for ψ_T. Thus, for two CP maps $T, S \in \text{CP}(\mathcal{A})$, the corresponding state ψ_T and ψ_S have unitarily equivalent GNS representations if and only if $\tilde{\pi}_T$ and $\tilde{\pi}_S$ are unitarily equivalent. This leads to the following observation.

PROPOSITION 3.1

$T \overset{'}{\sim} S$ if and only if ψ_T and ψ_S have unitarily equivalent GNS representations, i.e. $\tilde{\pi}_T$ and $\tilde{\pi}_S$ are unitarily equivalent.

Proof. Suppose first that $T \overset{'}{\sim} S$, and $\mathcal{U}$ is an $\mathcal{A}$-linear unitary from $\mathcal{E}_T$ to $\mathcal{E}_S$ satisfying $\mathcal{U}\pi_T(a)\mathcal{U}^* = \pi_S(a)$ for all $a \in \mathcal{A}$. Using the fact that $\{ev\colon\ e \in \mathcal{E}_T,\ v \in \mathcal{H}_\tau\}$ and $\{ev\colon\ e \in \mathcal{E}_S,\ v \in \mathcal{H}_\tau\}$ are total in $\mathcal{K}_T$ and $\mathcal{K}_S$ respectively, it is easy to verify that the map $U\colon \mathcal{K}_T \to \mathcal{K}_S$ defined by

$$U(ev) := \mathcal{U}(e)v, \quad e \in \mathcal{E}_T,\ v \in \mathcal{H}_\tau,$$

extends to a unitary in the Hilbert space sense, and $U\pi_T(a)U^* = \pi_S(a)$, $U\rho_T(a')U^* = \rho_S(a')$ for all $a \in \mathcal{A}$, $a' \in \mathcal{A}'$, which means that $U\tilde{\pi}_T(\cdot)U^* = \tilde{\pi}_S(\cdot)$.

Conversely, given a unitary $U\colon \mathcal{K}_T \to \mathcal{K}_S$ such that $U\tilde{\pi}_T(\cdot)U^* = \tilde{\pi}_S(\cdot)$, we can define $\mathcal{U}\colon \mathcal{E}_T \to \mathcal{E}_S$ by

$$\mathcal{U}(e) = Ue,$$

where $e \in \mathcal{E}_T$ is viewed as an element of $\mathcal{B}(\mathcal{H}_\tau, \mathcal{K}_T)$. This is easy to prove that $\mathcal{U}$ is an $\mathcal{A}$-linear unitary map, which also intertwins the left actions on the two modules, and thus $\mathcal{E}_T$ and $\mathcal{E}_S$ are isomorphic as bimodules.

We now want to characterize the states on $\mathcal{A}_\theta^{(2)}$ which are of the form ψ_T for some $T \in \text{CP}(\mathcal{A})$. Let us denote by $\mathcal{S}_\tau$ the set of all states ψ on $\mathcal{A}_\theta^{(2)}$ such that

(i) $\psi(1 \otimes a') = \langle 1, a' 1 \rangle \equiv \tau(a')$ for all $a' \in \mathcal{A}^{r,\text{fin}}$; and
(ii) the restriction of the state ψ on $\mathcal{A}^{l,\text{fin}} \otimes 1$ admits a normal extension to $\mathcal{A}$.

PROPOSITION 3.2

The map $T \mapsto \psi_T$ is a bijection from $CP(\mathcal{A})$ to $\mathcal{S}_\tau$.

Proof. It is clear that for $T \in \text{CP}(\mathcal{A})$, ψ_T is an element of $\mathcal{S}_\tau$. Let us first show that it is one-to-one. Let T and T' be two CP maps such that $\psi_T = \psi_{T'}$. For any $a \in \mathcal{A}^{l,\text{fin}}$ and $a', b' \in \mathcal{A}^{r,\text{fin}}$, we have

$$\psi_T(a \otimes a'b') = \langle 1, T(a)(a')^*b' 1 \rangle = \langle 1, T'(a)(a')^*b' 1 \rangle.$$

This gives

$$\langle a'1, T(a)b'1\rangle = \langle a'1, T'(a)b'1\rangle.$$

Since $\{a'1 : a' \in \mathcal{A}^{r,\text{fin}}\}$ is total in $\mathcal{H}_\tau$ it follows that $T(a) = T'(a),\ \forall a \in \mathcal{A}^{l,\text{fin}}$, hence for all $a \in \mathcal{A}$.

Now we show that the map $T \mapsto \psi_T$ is onto. Let $\psi \in \mathcal{S}_\tau$. We define a sesquilinear form $T(a)$ for a in the completion of $\mathcal{A}^{l,\text{fin}}$, i.e. $\mathcal{A}_\theta$ by the following, where $a_1', a_2' \in \mathcal{A}^{r,\text{fin}}$:

$$\langle a_1'\, 1, T(a)a_2'\, 1\rangle = \psi(a \otimes (a_1')^* a_2).$$

For $a_i, i = 1, 2 \ldots n$ in $\mathcal{A}^{r,\text{fin}}$ and $b_i, i = 1, 2 \ldots n$ in $\mathcal{A}^{l,\text{fin}}$, we have

$$\sum_{i,j} \langle a_i, T(b_i^* b_j)a_j\rangle = \sum_{i,j} \psi(b_i^* b_j \otimes a_i^* a_j). \tag{3.2}$$

Since $((b_i^* b_j))$ and $((a_i^* a_j))$ are positive in $M_n(\mathcal{A})$ and $M_n(\mathcal{A}')$ respectively, and each a_i commutes with each b_j, it follows by standard arguments, similar to those used for proving the positivity of Schur product of two positive (numerical) matrices, that the right-hand side of (3.2) is positive. By the condition $\psi|_{1\otimes\mathcal{A}^{r,\text{fin}}} = \tau$, $T(1) = 1$. This, combined with the positivity of the right-hand side of (3.2), suffices to show that $T(a)$ extends to a bounded map from $\mathcal{A}_\theta$ to $\mathcal{B}(\mathcal{H}_\tau)$, to be denoted by the same symbol again. It is also easy to see that T is CP and $T(\mathcal{A}_\theta) \subseteq (\mathcal{A}_\theta)'' = \mathcal{A}$. Finally, as the resriction of the state ψ on $\mathcal{A} \otimes 1$ is normal it follows that the map T extends as a normal CP unital map on $\mathcal{A}$. □

3.2 *Invariant for a pure CP map*

DEFINITION 3.3

An element $T \in \text{CP}(\mathcal{A})$ is said to be a pure CP map (see [**?**16]) if there does not exist any CP (not necessarily unital) normal map S on $\mathcal{A}$, other than scalar multiples of T, such that $T - S$ is CP.

PROPOSITION 3.4

A CP map $T: \mathcal{A} \to \mathcal{A}$ is pure if and only if the state ψ_T is a pure state on the second order irrational rotation C^*-algebra $\mathcal{A}_\theta^{(2)}$.

Proof. First we note that the state ψ_T is a pure state if and only if the GNS representation $\tilde{\pi}_T$ is irreducible. Since $\tilde{\pi}_T$ is constructed out of two commuting representations π_T and ρ_T, the irreducibility of $\tilde{\pi}_T$ is translated into the fact that $\pi_T(\mathcal{A})' \bigcap (\rho_T(\mathcal{A}'))' = \pi_T(\mathcal{A})' \bigcap \mathcal{L}(\mathcal{E}_T)$ is trivial.

Now, it suffices to prove that T is pure if and only if $\pi_T(\mathcal{A})' \bigcap \mathcal{L}(\mathcal{E}_T)$ is trivial. This fact can be deduced from Corollary 3.7 of [16] by some straightforward arguments, or by a direct argument along the lines of the proof of a similar fact for states as in [15]. □

Thus, given any pure CP map T on $\mathcal{A}$, the state ψ_T is a pure state on the second order irrational rotation C^*-algebra $\mathcal{A}_\theta^{(2)}$, i.e. its GNS representation $\tilde{\pi}_T$ is irreducible. Theorem 2.7 provides a quadruplet (m, v, b_1, b_2), which is an invariant for the CP map T under the KSGNS equivalence $\overset{\prime}{\sim}$. Moreover, it is a complete invariant for $\overset{\prime}{\sim}$ by Proposition 3.1. It is also easy to see that if T is pure and S is KSGNS equivalent to T, then S must be pure.

For a measure μ on $\mathbb{T}^2$, let us denote by μ_1 and μ_2 the marginals given by

$$\mu_1(\Delta) := \mu(\Delta \times \mathbb{T}), \quad \mu_2(\Delta) := \mu(\mathbb{T} \times \Delta),$$

where Δ is a Borel subset of $\mathbb{T}$. The restrictions of the representations $\tilde{\pi}$ to the C^*-algebra $\mathcal{A}_\theta = C^*(U_l, V_l)$ and $C^*(U_r, V_r)$ extend to normal representations of $\mathcal{A}$ and $\mathcal{A}'$ (respectively) by our assumption, so it is clear that the marginals ν_1 and ν_2 of the quasi invariant Borel measure ν on $\mathbb{T}^2$ obtained above are absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$. In fact, as ψ_T coincides with the canonical trace τ on $\mathcal{A}'$, ν_2 should be equivalent to the Lebesgue measure on the one-dimensional torus.

We summarise the above discussion in the form of a theorem below.

Theorem 3.5. *A complete invariant for the set of pure CP maps on $\mathcal{A}$ under the KSGNS equivalence $\overset{'}{\sim}$ is given by the quadruplet (m, ν, b_1, b_2) as described in Theorem 2.7, with the additional condition that ν_1 is absolutely continuous with respect to the Lebesgue measure and ν_2 is equivalent to the Lebesgue measure.*

Remark 3.6. It should be emphasized to avoid any confusion that the invariant discussed above is well-defined only after fixing the isomorphism between the weak closure of $\mathcal{A}_\theta$ and the hyperfinite II_1 factor. This is quite natural from the viewpoint of bimodule theory over factors, but this is different from the case of classification theory of automorphisms of a II_1 factor.

3.3 *Extendible CP maps*

It should be noted that the state ψ_T associated to a CP map T may not extend to the von Neumann algebra $\mathcal{A} \otimes \mathcal{A}'$, since given a pair of normal representations π, ρ of $\mathcal{A}$ and $\mathcal{A}'$ respectively (acting on the same Hilbert space) such that $\pi(a)$ and $\rho(a')$ commute for all a, a', it is not in general possible to get a normal representation ϕ of $\mathcal{A} \otimes \mathcal{A}'$ such that the restrictions of ϕ to $\mathcal{A}$ and $\mathcal{A}'$ are respectively π and ρ. In the appendix, we have given a counterexample which justifies this remark. Now, we are going to investigate when ψ_T extends to the type II_1 factor $\mathcal{A} \otimes \mathcal{A}'$ as a normal state. Let us call the map T *extendible* if ψ_T extends to a normal state on $\mathcal{A} \otimes \mathcal{A}'$, or equivalently, $\tilde{\pi}_T$ extends to a normal representation of $\mathcal{A} \otimes \mathcal{A}'$. To give a necessary and sufficient criterion for extendibility, we need the following general result.

PROPOSITION 3.7

Let $\mathcal{C} \subseteq \mathcal{B}(\mathcal{H}_0)$ be a C^* algebra, and $\tilde{\mathcal{C}}$ be its weak closure. Let a discrete group Γ admit a strongly continuous unitary representation on $\mathcal{H}_0$ given by, say, $\Gamma \ni \gamma \mapsto u_\gamma$. Assume furthermore that $u_\gamma \mathcal{C} u_\gamma^* = \mathcal{C}$, so that β_γ defined by $\beta_\gamma(\cdot) = u_\gamma \cdot u_\gamma^*$ defines a Γ-action on $\mathcal{C}$ as well as on $\tilde{\mathcal{C}}$. Consider the C^* crossed product $\mathcal{C} \rtimes \Gamma$ and the von Neumann crossed product $\tilde{\mathcal{C}} \rtimes \Gamma$. Let ϕ: $\mathcal{C} \rtimes \Gamma \to \mathbb{C}$ be a state. Then, ϕ extends to a normal state on $\tilde{\mathcal{C}} \rtimes \Gamma$ if and only if for every $\gamma \in \Gamma$, the bounded linear functional ϕ_γ defined by $\phi_\gamma(c) = \phi(c\delta_\gamma)$ is weakly continuous (where $c\delta_\gamma$ denotes the $\mathcal{C}$-valued function given by $c\delta_\gamma(\gamma') = 0$ if $\gamma' \neq \gamma$, and is c if $\gamma' = \gamma$).

Proof. Let us prove only the nontrivial part, i.e, the 'if' part. Consider the GNS triple $(\mathcal{K}, \pi, \xi)$ for the state ϕ, and denote by π_0 and U_γ the restriction of π to $\mathcal{C}$ and $\pi(\delta_\gamma)$

respectively. It is enough to prove that π_0 is normal. Consider a directed family $0 \leq c_\alpha \uparrow c$, where $c_\alpha, c \in \mathcal{C}$. We have to show that

$$\langle w, \pi_0(c_\alpha)w\rangle \uparrow \langle w, \pi_0(c)\, w\rangle \quad \forall w \in \mathcal{K}. \tag{3.3}$$

Since the family $\pi_0(c_\alpha)$ is uniformly bounded (in operator norm) by $\|\pi_0(c)\|$ (as $0 \leq \pi_0(c_\alpha) \leq \pi_0(c)$), it is enough to verify (3.3) for w belonging to the dense set $\mathcal{D}$ spanned by vectors of the form $\pi_0(c)U_\gamma\xi$, with $c \in \mathcal{C}$ and $\gamma \in \Gamma$. Note that the set $\mathcal{D}$ is dense because ξ is cyclic for $\pi(\mathcal{C} \rtimes \Gamma)$.

Now, it can be verified by a simple calculation that

$$\begin{aligned}
&\langle \pi_0(c_1)U_{\gamma_1}\xi, \pi_0(c_\alpha)\pi_0(c_2)U_{\gamma_2}\xi\rangle \\
&\quad = \phi_{\gamma_1^{-1}\gamma_2}(\beta_{\gamma_1^{-1}}(c_1^* c_\alpha c_2)) \\
&\quad \to \phi_{\gamma_1^{-1}\gamma_2}(\beta_{\gamma_1^{-1}}(c_1^* c c_2)) \\
&\quad = \langle \pi_0(c_1)U_{\gamma_1}\xi, \pi_0(c)\pi_0(c_2)U_{\gamma_2}\xi\rangle,
\end{aligned}$$

by the assumption of weak continuity of ϕ_γ for every γ. This completes the proof. □

For $k \equiv (k_1, k_2) \in \mathbb{Z}^2$, let ψ_T^k be the bounded linear functional on the commutative C^*-algebra $C^*(U_l, V_r) \equiv \mathcal{C}(\mathbb{T}^2)$ defined by

$$\psi_T^k(U_l^m V_r^n) = \psi_T((U_l^m \otimes V_r^n)(V_l^{-1} \otimes V_r)^{k_1}(U_l^{-1} \otimes U_r)^{k_2}), \ \forall m, n \in \mathbb{Z}.$$

Let us denote by μ_T^k the complex measure on $\mathbb{T}^2$ corresponding to ψ_T^k, i.e. $\psi_T^k(U_l^m V_r^n) = \int z_1^m z_2^n \mathrm{d}\mu_T^k(z)$. From Proposition 3.7, we conclude the following.

Theorem 3.8. *The CP map T is extendible if and only if for each $k \in \mathbb{Z}^2$ the measure μ_T^k is absolutly continuous with respect to Lebesgue measure on $\mathbb{T}^2$.*

We end this subsection by mentioning a few examples of extendible maps. The verification of extendibility, using Theorem 3.8, is quite straightforward, and we omit these calculations.

Example 3.9. $T(a) = R^* aR$, with $R \in \mathcal{A}$, and is given by $R = \sum_{i,j\in\mathbb{Z}} c_{i,j} U_l^i V_l^j$ such that $(c_{i,j}) \in l^2(\mathbb{Z}^2)$. Then T is extendible.

Example 3.10. Suppose that for each $k \in \mathbb{Z}$, there exists some positive integer N_k such that $T(f(U_l))V_l^k \in \mathrm{Span}[U_l^m V_l^n : m \in \mathbb{Z}, \ -N_k \leq n \leq N_k]$, $\forall f \in \mathcal{C}(\mathbb{T})$. Then T is extendible.

3.4 *Invariants for extendible CP maps*

Suppose that T is an extendible CP map, and let us denote the normal extension of ψ_T again by the same symbol. So, ψ_T is a normal state on $\mathcal{A}\otimes\mathcal{A}'$, which is also a type II_1 hyperfinite factor, with the canonical trace $\tau^{(2)}(\cdot) = \langle \xi_0, \cdot \xi_0\rangle$ (where $\xi_0 = 1 \otimes 1 \in \mathcal{H}_\tau \otimes \mathcal{H}_\tau$). Recall also the notation $c^{(r)}$ for $c \in \mathcal{A} \otimes \mathcal{A}'$, introduced earlier by us.

Since $\tau^{(2)}$ is faithful normal on $\mathcal{A} \otimes \mathcal{A}'$, any normal state is absolutely continuous with respect to $\tau^{(2)}$. Thus, we can obtain a 'Radon–Nikodym derivative', which is a positive

(possibly unbounded) operator D_T affiliated to the commutant of $\mathcal{A} \otimes \mathcal{A}'$, i.e. $\mathcal{A}' \otimes \mathcal{A}$, such that $\xi_0 \in \mathrm{Dom}(D_T^{\frac{1}{2}})$, and

$$\psi_T(b) = \tau^{(2)}(D_T b) \equiv \langle D_T^{\frac{1}{2}}\xi_0, bD_T^{\frac{1}{2}}\xi_0\rangle, \quad \forall b \in \mathcal{A} \otimes \mathcal{A}'.$$

Note that D_T is actually an element in $L^1(\mathcal{A}' \otimes \mathcal{A}, \tau^{(2)})$.

We now explain how D_T can be used to get numerical invariants for the equivalence $\sim$. Suppose that $S \sim T$, and let $u \in \mathcal{A}$ be such that $S(a) = T(u^*au)$ for all $a \in \mathcal{A}$. By an easy calculation we see that

$$\psi_S(b) = \psi_T(\tilde{u}^* b \tilde{u}), \quad b \in \mathcal{A} \otimes \mathcal{A}',$$

where $\tilde{u} := u \otimes 1$. Using the facts that $D_T^{\frac{1}{2}}$ commutes with $\tilde{u}$, and that $\tilde{u}\xi_0 = \tilde{u} = \tilde{u}^{(r)}\xi_0$, (where $\tilde{u}^{(r)} \in \mathcal{A}' \otimes \mathcal{A}$ denotes right multiplication by $\tilde{u}$), we get the following:

$$\begin{aligned}
\psi_S(b) &= \psi_T(\tilde{u}^* b \tilde{u}) \\
&= \langle \tilde{u} D_T^{\frac{1}{2}}\xi_0, b\tilde{u} D_T^{\frac{1}{2}}\xi_0\rangle \\
&= \langle D_T^{\frac{1}{2}}\tilde{u}^{(r)}\xi_0, b D_T^{\frac{1}{2}}\tilde{u}^{(r)}\xi_0\rangle \\
&= \langle (\tilde{u}^{(r)})^* D_T^{\frac{1}{2}}\tilde{u}^{(r)}\xi_0, b(\tilde{u}^{(r)})^* D_T^{\frac{1}{2}}\tilde{u}^{(r)}\xi_0\rangle.
\end{aligned}$$

From this, we conclude (using the uniqueness of the Radon–Nikodym derivative) that

$$D_S = (\tilde{u}^{(r)})^* D_T \tilde{u}^{(r)}.$$

So, in particular, $\tau^{(2)}(\mathrm{e}^{-tD_S}) = \tau^{(2)}(\mathrm{e}^{-tD_T})$ for all $t \geq 0$. These numbers can be used as invariants for $\sim$. Another possibility is to use $\tau^{(2)}(P_T)$, where P_T denotes the projection (in $\mathcal{A}' \otimes \mathcal{A}$) onto the range of D_T.

Remark 3.11. Since D_T is affiliated to $\mathcal{A}' \otimes \mathcal{A}$, which can be identified with the von Neumann crossed product $L^\infty(\mathbb{T}^2) \rtimes \mathbb{Z}^2$, we can describe D_T by a formal series (which can be made sense of in a suitable L^1-topology) of the form $\sum_{k\in\mathbb{Z}^2} D_T^{(k)}\delta_k$, where $D_T^{(k)}$ is a measurable complex-valued function on $\mathbb{T}^2$. Let us give a formula for the functions $D_T^{(k)}$ for a class of CP maps. Consider $T(a) = R^*aR$ with $R = \sum_{(m_1,m_2)} c_{m_1,m_2} U_l^{m_1} V_l^{m_2}$, where the summation is over a finite set, i.e. $c_m \equiv c_{(m_1,m_2)}$ is zero for all but a finitely many m's. Then T is extendible and $D_T^{(k)}$'s are given by

$$D_T^{(k)}(z) = \sum_{n,m\in\mathbb{Z}^2} c_n \bar{c}_{m_1+n_1,m_2+n_2+k_1-k_2} \lambda^{N(k,n,m)} z_1^{m_1} z_2^{m_2},$$

with $N(k,n,m) = 2k_2m_2 + k_1n_1 + k_2n_2 + k_1k_2 - k_2^2$. This can be verified by a simple calculation.

In particular, for $R = U$, $D_T^{(k)}(z) = \lambda^{k_2^2-k_1k_2} z_2^{k_2-k_1}$, and for $R = U + V$, $D_T^{(k)}(z) = \lambda^{k_2^2-k_1k_2} z_2^{k_2-k_1}\{\lambda^{k_1} + \lambda^{k_2} + \lambda^{2k_2+k_1} z_1^{-1}z_2 + \lambda^{k_2} z_1^{-1}z_2^{-1}\}$.

Let us conclude with a nice computable formula for $\tau^{(2)}(P_T)$ when D_T is L^2.

PROPOSITION 3.12.

If D_T belongs to $L^2(\mathcal{A}' \otimes \mathcal{A}, \tau^{(2)})$, then we have

$$\tau^{(2)}(P_T) = 1 - \inf_{b \in \mathcal{A}^{\text{fin}}} \sup_{c \in \mathcal{A}^{\text{fin}}:\ \tau^{(2)}(c^*c)=1} |\tau^{(2)}(c^*) - \psi_T(c^*b)|^2.$$

Proof. Note that $D_T \in L^2$ implies that $\xi_0 \in \text{Dom}(D_T)$, and thus, $\tau^{(2)}(b) = \langle \xi_0, bD_T\xi_0 \rangle$ for all $b \in \mathcal{A} \otimes \mathcal{A}'$. Moreover, as D_T is affiliated to $(\mathcal{A} \otimes \mathcal{A}')'$, e^{-tD} leaves $\mathcal{A}^{\text{fin}}\xi_0 \subseteq \mathcal{H}_\tau \otimes \mathcal{H}_\tau$ invariant. Hence $\mathcal{A}^{\text{fin}}\xi_0$ is a core for D_T, and hence $\{D_T b\xi_0 : b \in \mathcal{A}^{\text{fin}}\}$ is dense in Ran D. Thus, we have

$$\begin{aligned}
\tau^{(2)}(P_T) &= \|P_T\xi_0\|^2 \\
&= 1 - \text{dist}(\xi_0, \text{Ran } D_T)^2 \\
&= 1 - \inf_{b \in \mathcal{A}^{\text{fin}}} \|\xi_0 - D_T\, b\xi_0\|^2 \\
&= 1 - \inf_{b \in \mathcal{A}^{\text{fin}}} \sup_{c \in \mathcal{A}^{\text{fin}}:\ \tau^{(2)}(c^*c)=1} |\langle c, \xi_0 - D_T b\xi_0 \rangle|^2 \\
&= 1 - \inf_{b \in \mathcal{A}^{\text{fin}}} \sup_{c \in \mathcal{A}^{\text{fin}}:\ \tau^{(2)}(c^*c)=1} |\tau^{(2)}(c^*) - \tau^{(2)}(c^* D_T b)|^2 \\
&= 1 - \inf_{b \in \mathcal{A}^{\text{fin}}} \sup_{c \in \mathcal{A}^{\text{fin}}:\ \tau^{(2)}(c^*c)=1} |\tau^{(2)}(c^*) - \psi_T(c^*b)|^2.
\end{aligned}$$

□

However, in this paper, we do not study these numerical invariants in more detail, and do not investigate whether one or more of such invariants characterise the equivalence $\sim$. We leave these topics for future research.

Appendix

Here we give an example of a commuting pair of normal representations π and ρ of a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_0)$ and its commutant respectively, such that the representation Φ of $\mathcal{A} \otimes_{\text{alg}} \mathcal{A}'$ defined by $\Phi(a \otimes a') = \pi(a)\rho(a')$ does not have a normal extension to the von Neumann algebra $\mathcal{A} \otimes \mathcal{A}'$. To this end, let us take $\mathcal{H}_0 = L^2(\mathbb{T})$ (with Lebesgue measure), and $\mathcal{A} = \mathcal{A}' = L^\infty(\mathbb{T})$. Let $\mathcal{H} = L^2(\mathbb{R}^2, \mu)$, where

$$\mu(E) := l(E \cap D),$$

$D := \cup_{n \in \mathbb{Z}}\{(t, t+n) : t \in \mathbb{R}\}$, and l denotes the Lebesgue measure on $\mathbb{R}$. Define π and ρ from $L^\infty(\mathbb{T})$ to $\mathcal{B}(\mathcal{H})$ by setting:

$$(\pi(\phi)f)(s,t) = \phi(\lambda^s) f(s,t), \quad (\rho(\phi)f)(s,t) = \phi(\lambda^t) f(s,t),$$

for $f \in L^2(\mathbb{R}^2, \mu)$. We claim that $\Phi: L^\infty(\mathbb{T}) \otimes_{\text{alg}} L^\infty(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ given by $\Phi(\phi \otimes \psi) := \pi(\phi)\rho(\psi)$, does not admit a normal extension to $L^\infty(\mathbb{T}^2)$. To see this, note that the set $\Gamma := \{(x, (\lambda^n x) \bmod 1) : n \in \mathbb{Z}\}$ has zero Lebesgue measure as a subset of $\mathbb{T}^2$. Now, assuming, if possible, that Φ admits a normal extension on $L^\infty(\mathbb{T}^2)$, choose a function $F \in \mathcal{C}(\mathbb{T}^2)$ and define F_0 in $L^\infty(\mathbb{T}^2)$ by setting $F_0(z) = 0$ for $z \in \Gamma$, and $F_0(z) = F(z)$ for all z in the complement of Γ. Thus, $F = F_0$ a.e. (Lebesgue), hence $\Phi(F) = \Phi(F_0)$. However,

$$\begin{aligned}\|\Phi(F_0)f\|^2 &= \int_{\mathbb{R}^2} |F_0(\lambda^s, \lambda^t) f(s,t)|^2 \mathrm{d}\mu(s,t) \\ &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} |F_0(\lambda^s, \lambda^n\lambda^s) f(s, s+n)|^2 \mathrm{d}s \\ &= 0,\end{aligned}$$

for every $f \in L^2(\mu)$. Thus, $\Phi(F) = 0$ for all $F \in \mathcal{C}(\mathbb{T}^2)$, contradicting the assumption that Φ extends normally.

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