

Study of the “non-Abelian” current algebra of a non-linear σ -model

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Abstract

A particular form of non-linear σ -model, having a global gauge invariance, is studied. The detailed discussion on current algebra structures reveals the non-Abelian nature of the invariance, with *field dependent structure functions*. Reduction of the field theory to a point particle framework yields a non-linear harmonic oscillator, which is a special case of similar models studied before in [J.F. Carinena et al., Nonlinearity 17 (2004) 1941, math-ph/0406002; J.F. Carinena et al., in: Proceedings of 10th International Conference in Modern Group Analysis, Larnaca, Cyprus, 2004, p. 39, math-ph/0505028; J.F. Carinena et al., Rep. Math. Phys. 54 (2004) 285, hep-th/0501106]. The connection with non-commutative geometry is also established.

1. Introduction

In the present Letter we aim to reveal some interesting properties enjoyed by the current algebra of a particular form of non-linear σ -model. Similar type of models became attractive in the late sixties [1] in the description of mesons, as a non-linear realization of chiral groups, (e.g. $SU(2) \times SU(2)$). Later it appeared that if these models are reduced to their corresponding point particle counterpart, (keeping only time derivatives in the field theory Lagrangian from the operational point of view), they gave rise to non-linear harmonic oscillator models with very interesting behavior [2]. These oscillators were further studied in all their generality by Carinena et al. [3].

The σ -model that we consider here is, in fact, the massless limit of the models analysed in [1]. But because of this feature, there are additional global gauge invariances associated with it, leading to interesting current algebra structures. The current algebra in this self-interacting model turns out to be non-Abelian, with the structure functions being *field dependent*. Generic forms of non-Abelian current algebra have been studied in [4] where it was assumed that the σ -model exhibits a global invariance under some internal symmetry group, represented by a (connected) Lie group G with the corresponding Lie

algebra \mathfrak{g} that obeys $[T^a, T^b] = f_c^{ab} T^c$ in some arbitrary basis (T_a) , with f_{ab}^c being the structure *constants*. We emphasize that in the present work, we have been able to recover all the results of [4] with the field dependent structure function $F_c^{ab}(\phi)$ replacing the structure constant f_c^{ab} . Furthermore, our model reduces to a special case of the non-linear oscillator models studied in [3], which is expected. We will make some brief observations about it towards the end of the Letter.

Works of a similar nature have also appeared in [5,6] where particle [5] and field theoretic [6] models have been studied, which are similar to ours. In [5] the particle model is studied from the perspective of a quantum super-integrable model and [6] consider the duality aspect of different σ -models. As we will mention, these models have some non-trivial connection with the non-commutative geometry framework.

2. The σ -model

We start by considering the Lagrangian,

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi^a \partial_\mu \phi^a + \frac{c}{2} \frac{(\phi^a \partial^\mu \phi^a)(\phi^b \partial_\mu \phi^b)}{1 - c\phi^a \phi^a}. \quad (1)$$

This is a particular form of σ -model, in the sense that,

$$\mathcal{L} = \frac{1}{2} G^{ab}(\phi) \partial^\mu \phi^a \partial_\mu \phi^b, \quad G^{ab}(\phi) = \delta^{ab} + c \frac{\phi^a \phi^b}{1 - c\phi^a \phi^a}. \quad (2)$$

Recalling earlier works [1], our model actually relates to the choice of Gasirowicz–Geffen coordinates (i.e. $\Lambda^2(\varphi^2) = c$ with c being a constant in the representation of the meson field S as in Delbourgo et al. in [1]),

$$S = \sigma + i\tau \cdot \varphi \Lambda(\varphi^2).$$

The variational equation of motion is

$$\partial^\mu \partial_\mu \phi^a + 2c\mathcal{L}\phi^a = 0. \quad (3)$$

For a single field, i.e. $a = 1$, exploiting a mode expansion for $\phi^a(x) = \phi^a e^{ikx}$ in the equation of motion (3), we obtain

$$\frac{k^2}{1 - c\phi^2} \phi = 0, \quad (4)$$

indicating that the “particles” are massless, $k^2 = 0$.

Let us now discuss the symmetries of the model. Besides the obvious spacetime symmetries, the model enjoys a global gauge invariance, with the symmetry transformation,

$$\delta\phi^a = \sqrt{1 - c\phi^2} \epsilon^a, \quad (5)$$

with ϵ^a being an infinitesimal rigid parameter. The corresponding conserved Noether current is given by,

$$\begin{aligned} J_\mu^a &= \sqrt{1 - c\phi^2} \left[\partial_\mu \phi^a + c \frac{(\phi \partial_\mu \phi)}{1 - c\phi^2} \phi^a \right] \\ &\equiv \sqrt{1 - c\phi^2} G^{ab} \partial_\mu \phi^b. \end{aligned} \quad (6)$$

However, generalizing this global gauge invariance to a local invariance is tricky. For a local gauge transformation (5), where $\epsilon^a(x)$ is not constant, the Lagrangian transforms nicely,

$$\delta\mathcal{L} = J_\mu^a \partial^\mu \epsilon^a(x). \quad (7)$$

However, the current transforms in a covariant way,

$$\begin{aligned} \delta J_\mu^a &= c(\phi^a \partial_\mu \phi^b - \phi^b \partial_\mu \phi^a) \epsilon^b = \frac{c}{\sqrt{1 - c\phi^2}} (\phi^a J_\mu^b - \phi^b J_\mu^a) \epsilon^b \\ &= \frac{c}{\sqrt{1 - c\phi^2}} \epsilon^{abc} \epsilon^{cde} \epsilon^b \phi^d J_\mu^e \equiv F^{abc}(\phi) \epsilon^b J_\mu^c. \end{aligned} \quad (8)$$

We have identified the structure function to be,

$$F^{abc} = \frac{c}{\sqrt{1 - c\phi^2}} \epsilon^{abd} \epsilon^{dgc} \phi^g = \frac{c}{\sqrt{1 - c\phi^2}} (\phi^a \delta^{bc} - \phi^b \delta^{ac}), \quad (9)$$

with $F^{abc} = -F^{bac}$.

Hence we can consider an interacting Lagrangian \mathcal{L}_I of the form,

$$\mathcal{L}_I = \mathcal{L} + J_\mu^a A^{a\mu}. \quad (10)$$

In order to achieve local “gauge” symmetry, A_μ^a will have to transform as,

$$\begin{aligned} \delta A_\mu^a &= -\partial_\mu \epsilon^a - \frac{c}{\sqrt{1 - c\phi^2}} \epsilon^{abc} \epsilon^{cde} \epsilon^b \phi^d A_\mu^e \\ &= -(\partial_\mu \epsilon^a + F^{abc} \epsilon^b A_\mu^c) \\ &\equiv -(D_\mu \epsilon)^a. \end{aligned} \quad (11)$$

It will be interesting to see if construction of a locally gauge invariant kinetic term for A_μ^a is possible, which might require a generalization of the real fields ϕ^a complex fields.

3. Current algebra for the σ -model

The canonical definition for the energy–momentum tensor is,

$$\theta^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu \phi^a)} \partial^\nu \phi^a - \eta^{\mu\nu} \mathcal{L}. \quad (12)$$

In the present case, this leads to

$$\theta^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi + c \frac{(\phi \partial^\mu \phi)(\phi \partial^\nu \phi)}{1 - c\phi^2} - \eta^{\mu\nu} \mathcal{L}, \quad (13)$$

which is conserved, symmetric and traceless for $(1+1)$ dimensions.

We now revert to a Hamiltonian framework, which is suitable for studying the current algebra. The conjugate momentum,

$$\begin{aligned} \pi^a &= \frac{\delta\mathcal{L}}{\delta\dot{\phi}^a}, \quad \dot{\phi}^a = \pi^a - c(\phi \cdot \pi)\phi^a, \\ \pi^a &= \dot{\phi}^a + c \frac{(\phi \cdot \dot{\phi})}{1 - c\phi^2} \phi^a, \end{aligned} \quad (14)$$

yields the Hamiltonian and total momentum densities,

$$\begin{aligned} \theta^{00} &\equiv \mathcal{H} \\ &= \frac{1}{2} \left[\pi^a \pi^a + \partial_i \phi^a \partial_i \phi^a - c(\phi \cdot \pi)^2 + c \frac{(\phi \partial_i \phi)(\phi \partial_i \phi)}{1 - c\phi^2} \right], \end{aligned} \quad (15)$$

$$\theta^{0i} \equiv \mathcal{P}^i = \pi^a \partial^i \phi^a, \quad (16)$$

which are the generators of time and spatial translations, respectively.

The gauge current is also expressed in terms of phase space variables,

$$\begin{aligned} J_0^a &= \sqrt{1 - c\phi^2} \pi^a, \\ J_i^a &= \sqrt{1 - c\phi^2} \left(\partial_i \phi^a + c \frac{(\phi \partial_i \phi)}{1 - c\phi^2} \phi^a \right). \end{aligned} \quad (17)$$

We exploit the equal-time canonical Poisson brackets:

$$\begin{aligned} \{\phi^a(x), \pi^b(y)\} &= \delta^{ab} \delta(x - y), \\ \{\phi^a(x), \phi^b(y)\} &= \{\pi^a(x), \pi^b(y)\} = 0. \end{aligned} \quad (18)$$

The internal current algebra is “non-Abelian” in nature:

$$\begin{aligned} \{J_0^a(x), J_0^b(y)\} &= c(\phi^a \pi^b - \phi^b \pi^a) \delta(x - y) \\ &= \epsilon^{abc} \epsilon^{cde} \frac{c}{\sqrt{1 - c\phi^2}} \phi^d J_0^e \delta(x - y) \\ &= F^{abc} J_0^c \delta(x - y), \end{aligned} \quad (19)$$

$$\begin{aligned} \{J_0^a(x), J_i^b(y)\} &= ((1 - c\phi^2(x)) \delta^{ab} + c\phi^a(x) \phi^b(x)) \partial_i \delta(x - y) \\ &\quad + 2c(\phi^a \partial_i \phi^b - (\phi \partial_i \phi) \delta^{ab}) \delta(x - y) \\ &= J^{ab}(y) \partial_i \delta(x - y) + F^{abc} J_i^c \delta(x - y). \end{aligned} \quad (20)$$

The current algebra closes by considering J^{ab} as a composite operator [4],

$$J^{ab} = (1 - c\phi^2) \delta^{ab} + c\phi^a \phi^b, \quad (21)$$

with the algebra,

$$\{J_0^a(x), J^{bc}(y)\} = c\left(\sqrt{1-c\phi^2}\right)(F^{abd}J^{cd} + F^{acd}J^{bd})\delta(x-y). \quad (22)$$

Rest of the commutators are trivial since they do not involve the momenta,

$$\{J_i^a(x), J_j^b(y)\} = \{J_i^a(x), J^{bc}(y)\} = 0. \quad (23)$$

Next we come to the diffeomorphism algebra,

$$\begin{aligned} \{\mathcal{H}(x), \mathcal{H}(y)\} &= (\mathcal{P}_i(x) + \mathcal{P}_i(y))\partial_i\delta(x-y), \\ \{\mathcal{H}(x), \mathcal{P}_i(y)\} &= (\mathcal{H}(x) + \mathcal{H}(y))\partial_i\delta(x-y), \\ \{\mathcal{P}_i(x), \mathcal{P}_j(y)\} &= 0. \end{aligned} \quad (24)$$

Finally we consider the mixed commutators:

$$\begin{aligned} \{\mathcal{H}(x), J_0^b(y)\} &= J_0^b(x)\partial_i\delta(x-y), \\ \{\mathcal{H}(x), J_i^a(y)\} &= J_0^a(x)\partial_i\delta(x-y) - 2c(\partial_0 J_i^a - \partial_i J_0^a)\delta(x-y). \end{aligned} \quad (25)$$

In (25) we have used the relation,

$$\dot{\phi}^a \equiv \{\phi^a(x), H\} = \pi^a - c(\phi \cdot \pi)\phi^a. \quad (26)$$

All the brackets with \mathcal{P}_i are conventional in nature since \mathcal{P}_i has a canonical structure and so they are not shown explicitly. Hence the general current algebra structures discussed in [4] are exactly reproduced in our model with a field dependent non-Abelian structure function.

4. Non-linear harmonic oscillator and relation with non-commutative space

To recover the oscillator model, we formally replace the fields $\phi^a(x)$ in (1) by $X_i(t)$ variables and interpret the latter as particle coordinates. This gives us the Lagrangian,

$$L = \frac{1}{2}\dot{X}^2 + \frac{c(X \cdot \dot{X})^2}{2(1-cX^2)}, \quad (27)$$

where X_i are spatial coordinates, $X^2 = X_i X_i$, $X \cdot \dot{X} = X_i \dot{X}_i$, and the model is in arbitrary space dimension. For convenience, we have taken the mass to be unity and c is a parameter. As mentioned before, this model is a special case of systems discussed in [3] ($\alpha = 0$ in the notation of [3]). In fact this model can be studied as a variable mass problem [7] as well with interpreting the total Lagrangian (27) as the kinetic term. The equation of motion,

$$\ddot{X}_i = -2c\left[\frac{1}{2}\dot{X}^2 + \frac{c(X \cdot \dot{X})^2}{2(1-cX^2)}\right]X_i \equiv -(2cL)X_i, \quad (28)$$

can be reproduced in the Hamiltonian framework with,

$$P_i \equiv \frac{\partial L}{\partial \dot{X}_i} = \dot{X}_i + c\frac{(X \cdot \dot{X})}{1-cX^2}X_i, \quad (29)$$

$$H = \dot{X}_i P_i - L = \frac{1}{2}[P^2 - c(X \cdot P)^2]. \quad (30)$$

Exploiting the canonical Poisson brackets,

$$\{X_i, P_j\} = \delta_{ij}, \quad \{X_i, X_j\} = \{P_i, P_j\} = 0 \quad (31)$$

and the definition $\dot{A} = \{A, H\}$ for any dynamical variable A , we obtain

$$\ddot{X}_i = -[2cH]X_i. \quad (32)$$

Notice that in the present case $L = H$ which justifies the particle to be considered as “free”.

To focus on the striking feature of the non-linear oscillator, we restrict to one-dimensional motion and consider bounded and periodic solution of the form,

$$X_e = \frac{1}{\sqrt{c}}\sin(\sqrt{2cE}t), \quad (33)$$

where the subscript X_e is put in to remind us of its exotic nature. This solution corresponds to the energy E ,

$$E = \frac{1}{2}[P^2 - c(X \cdot P)^2] = \frac{1}{2}A^2\omega^2, \quad \omega = \sqrt{2cE}. \quad (34)$$

For comparison, we write down the behavior of a *normal* harmonic oscillator,

$$H = \frac{1}{2}P^2 + \frac{1}{2}cX^2, \quad (35)$$

that has solution,

$$X = \sqrt{\frac{2E}{c}}\sin(\sqrt{c}t). \quad (36)$$

Notice that for X_e the *amplitude is fixed*, depending upon c whereas the *frequency is energy dependent*, oscillations becoming more rapid with larger energy. This is qualitatively different from the behavior of a normal oscillator and does not reduce to it in any limit.

In order to see the connection with non-commutative geometry, we now discuss the conserved quantities. Obviously angular momentum L_{ij} is conserved:

$$L_{ij} = X_i P_j - X_j P_i, \quad \dot{L}_{ij} = 0. \quad (37)$$

But there are other conserved quantities p_i as well:

$$p_i = \sqrt{1-cX^2}P_i, \quad \dot{p}_i = 0. \quad (38)$$

Now we can forge a connection with a particular form of non-commutative space. Notice that p_i are non-commuting,

$$\{p_i, p_j\} = c(x_i p_j - x_j p_i), \quad (39)$$

and together with the identification,

$$x_i \equiv \frac{X_i}{\sqrt{1-cX^2}}, \quad (40)$$

it is easy to derive,

$$\{x_i, x_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij} + cx_i x_j. \quad (41)$$

Thus, (39) and (41) generate a particular form of non-commutative phase space in (x_i, p_j) that is quite well known in high energy physics [8,9]. This is actually a complimentary form of

the Snyder algebra [9] (see Ghosh in [8]). For $c = 0$ we recover the normal free particle.

The Hamiltonian (30) in (x_i, p_j) -spacetime turns out to be,

$$H = \frac{1}{2}[(1 + cx^2)p^2 - c(x \cdot p)^2], \quad (42)$$

with the subsequent equations of motion in Snyder spacetime,

$$\dot{x}_i = \{x_i, H\} = (1 + cx^2)p_i, \quad \dot{p}_i = \{p_i, H\} = 0. \quad (43)$$

This might be interpreted as a “free” particle in the sense that there is no *external* force but it is moving in a constant curvature Snyder space. One might interpret the non-canonical term in \dot{x}_i in (43) as an anomalous velocity term or equivalently consider a variable mass particle.

5. Conclusion

We have studied a particular non-linear σ -model, that corresponds to the massless limit of derivative coupling models studied earlier [1], in meson phenomenology. Apart from the Poincare invariance, the model enjoys a non-Abelian global gauge invariance, with *field dependent structure functions*.

The field theory studied here is worthwhile for the following reason. In [3] it was pointed out that the non-linear oscillator model is integrable [10], in the sense that it contains conserved quantities in involution, that are same in number as the number of degrees of freedom. It will be interesting to see whether similar conclusions can be drawn for the field theory studied here. From the analysis done so far in this paper, it is not clear whether this analogy can be extended to the level of integrability for the field theory.

In the point particle reduction, our model reduces to a particular form non-linear oscillator, studied earlier in [3]. The

connection of the present oscillator model with a specific form of non-commutative space have also been revealed.

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