Cylindrical and spherical quantum ion acoustic waves

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Cylindrical and spherical deformed Korteweg–de Vries (dKdV) equations are derived for quantum ion acoustic waves in an unmagnetized two species quantum plasma system, comprised of electrons and ions, by the reductive perturbation technique in the weakly nonlinear limit. The properties of quantum ion acoustic solitary waves are studied taking into account the quantum mechanical effects in a nonplanar cylindrical or spherical geometry, which differs from one-dimensional planar geometry. Both analytical and numerical solutions of the dKdV equations are discussed in some detail. There exists a critical value of quantum parameters beyond which the quantum ion acoustic soliton collapses. It is also found that for the critical values of H, viz., $H=2$, some nontrivial analytical solution exists for both cylindrical and spherical dKdV equations. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2409527]

I. INTRODUCTION

Over the recent years quantum effects in plasma[1] have received a lot of attention because of their application in many aspects of plasma like quantum plasma echo,[2] dense plasma particularly in astrophysical and cosmological studies,[3–6] quantum plasma instabilities in Fermi gases,[7] quantum Landau damping,[8] and other important plasma researches. One of the popular models to study quantum effects in plasma is the quantum hydrodynamical model[9–14] (QHD) which has been studied by several authors. Essentially the QHD model is an extension of the usual fluid model in plasma. This model is comprised of a set of equations describing transportation of charge momentum and energy. The deviation from the classical fluid model lies in the fact that a so-called Bohm potential[10] is introduced in the equation of motion for the charged particles. Another important quantum plasma theory is the Wigner Poisson system[15] which involves the integrodifferential system. Haas *et al.*[16] used the QHD model to study quantum ion acoustic waves in the weakly nonlinearized theory and obtained a deformed Korteweg–de Vries (dKdV) equation which depends on the quantum parameter H in a nontrivial way. In the case of quantum ion acoustic waves several characteristic features of pure quantum origin were observed for the linear, weakly nonlinear and fully nonlinear waves. The linear quantum ion acoustic waves are described by a dispersion relation that tends to classical dispersion relation as quantum effects tend to zero in accordance with the correspondence principle, whereas, the weakly nonlinear quantum ion acoustic waves are described through a modified KdV equation depending on $\hbar$. Later Haas[17] used a magnetohydrodynamical quantum model to extend their study in magnetized plasma. Earlier Opher *et al.*[4] studied the effects of highly damped modes in the energy and reaction rates in plasma and discussed the implication of introducing highly damped modes (with $\hbar \neq 0$) in the nuclear reaction rates in a plasma. Garcia *et al.*[18] used the hydrodynamical model to study the modified Zakharov equation in a plasma with a quantum correction. Most of the studies mentioned above were in either one dimension or planar geometry except the one by Haas[17] who used cylindrical geometry to study the magnetostatic equilibrium. However no study is made for nonlinear waves in the framework of nonplanar geometry. In this paper we have investigated the cylindrical and spherical deformed Korteweg–de Vries (dKdV) equation for quantum ion acoustic waves in an unmagnetized two species quantum plasma system, comprised of electrons and ions in nonplanar geometry. The paper is organized as follows: In Sec. II we derived the cylindrical and spherical deformed KdV equations. In Sec. III we discuss an analytical solution for cylindrical and spherical KdV equations. In Sec. IV we discuss the numerical solutions of cylindrical and spherical dKdV equations, while Sec. V is kept for conclusions.

II. DERIVATION OF CYLINDRICAL AND SPHERICAL DEFORMED KDV EQUATIONS

We consider a two species quantum plasma system comprised of electrons and ions in a nonplanar cylindrical or spherical geometry and study the nonlinear propagation ion acoustic solitary waves. The one-dimensional quantum hydrodynamic mode consists of the continuity and momentum balance equations for both electron ions together with the Poisson's equation for the self-consistent potential.[16] The nonlinear dynamics of the ion acoustic waves in quantum plasma system in nonplanar cylindrical and spherical geometries is governed by

$$\frac{\partial n_e}{\partial t} + \frac{1}{r^\nu}\frac{\partial}{\partial r}(r^\nu n_e u_e) = 0, \tag{1}$$

$$\frac{\partial n_i}{\partial t} + \frac{1}{r^\nu}\frac{\partial}{\partial r}(r^\nu n_i u_i) = 0, \tag{2}$$

$$\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial r}=\frac{e}{m_e}\frac{\partial \phi}{\partial r}-\frac{1}{m_e n_e}\frac{\partial p_e}{\partial r}+\frac{\hbar^2}{2m_e^2}\frac{\partial}{\partial r}\left\{\frac{\frac{1}{r^\nu}\frac{\partial}{\partial r}\left(r^\nu\frac{\partial\sqrt{n_e}}{\partial r}\right)}{\sqrt{n_e}}\right\}, \tag{3}$$

$$\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial r}=-\frac{e}{m_i}\frac{\partial \phi}{\partial r}+\frac{\hbar^2}{2m_i^2}\frac{\partial}{\partial r}\left\{\frac{\frac{1}{r^\nu}\frac{\partial}{\partial r}\left(r^\nu\frac{\partial\sqrt{n_i}}{\partial r}\right)}{\sqrt{n_i}}\right\}, \tag{4}$$

$$\frac{1}{r^\nu}\frac{\partial}{\partial r}\left(r^\nu\frac{\partial \phi}{\partial r}\right)=\frac{e}{\varepsilon_0}(n_e-n_i), \tag{5}$$

where $\nu=0$, for one-dimensional geometry and $\nu=1$, 2 for cylindrical and spherical geometries, respectively. n_e, u_e, m_e, $-e$ (n_i, u_i, m_i, e) are the electron (ion) density field, velocity field, mass, and charge, respectively, and ε_0 and $\hbar$ are the dielectric and Planck constant divided by 2π. ϕ is the electrostatic wave potential, p_e, the pressure effects for electrons. Pressure effects for ions are neglected for simplicity. We assume that the electrons obey the equation of state pertaining to a one-dimensional zero temperature Fermi gas,[7]

$$p_e=\frac{m_e v_{Fe}^2}{3n_0^2}n_e^3, \tag{6}$$

where n_0 is the equilibrium density for both electrons and ions, and v_{Fe} is the electronic Fermi velocity connected to the Fermi temperature T_{Fe} by $m_e v_{\mathrm{Fe}}^2/2=k_B T_{\mathrm{Fe}}$, k_B is the Boltzmann's constant. Now we introduce the following normalization:

$$\bar{r}=\omega_{pi}r/t,\quad \bar{t}=\omega_{pi}t,\quad \bar{n}_e=n_e/n_0,\quad \bar{n}_i=n_i/n_0,$$
$$\bar{u}_e=u_e/c_s,\quad \bar{u}_i=u_i/c_s,\quad \bar{\phi}=e\phi/(2k_B T_{\mathrm{Fe}}), \tag{7}$$

where ω_{pe} and ω_{pi} are the corresponding electron and ion plasma frequencies,

$$\omega_{pe}=\left(\frac{n_0e^2}{m_e\varepsilon_0}\right)^{1/2},\quad \omega_{pi}=\left(\frac{n_0e^2}{m_i\varepsilon_0}\right)^{1/2},$$

c_s is the quantum ion acoustic velocity given by

$$c_s=\left(\frac{2k_B T_{Fe}}{m_i}\right)^{1/2}.$$

We have denoted the nondimensional quantum parameter

$$H=\frac{\hbar\,\omega_{pe}}{2k_B T_{Fe}}(>0).$$

Using the above normalization we obtain from Eqs. (3) and (4) (dropping bars)

$$\frac{m_e}{m_i}\left(\frac{\partial u_e}{\partial t}+u_e\frac{\partial u_e}{\partial r}\right)=\frac{\partial \phi}{\partial r}-n_e\frac{\partial n_e}{\partial r}+\frac{H^2}{2}\frac{\partial}{\partial r}\left\{\frac{\frac{1}{r^\nu}\frac{\partial}{\partial r}\left(r^\nu\frac{\partial\sqrt{n_e}}{\partial r}\right)}{\sqrt{n_e}}\right\}, \tag{8}$$

$$\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial r}=-\frac{\partial \phi}{\partial r}+\frac{m_e}{m_i}\frac{H^2}{2}\frac{\partial}{\partial r}\left\{\frac{\frac{1}{r^\nu}\frac{\partial}{\partial r}\left(r^\nu\frac{\partial\sqrt{n_i}}{\partial r}\right)}{\sqrt{n_i}}\right\}. \tag{9}$$

As $m_e/m_i \ll 1$, after integrating Eq. (8) once and assuming the boundary conditions $n_e=1$, $\phi=0$ at infinity, we get

$$\phi=-\frac{1}{2}+\frac{n_e^2}{2}-\frac{H^2}{2\sqrt{n_e}}\frac{1}{r^\nu}\frac{\partial}{\partial r}\left(r^\nu\frac{\partial\sqrt{n_e}}{\partial r}\right). \tag{10}$$

This equation gives the electrostatic potential in terms of electron density and its derivatives. In the momentum equation (9), the quantum diffraction term may be neglected due to $m_e/m_i \ll 1$.

Now the continuity equation (2), momentum equation (9), and Poisson's equations become

$$\frac{\partial n_i}{\partial t}+\frac{1}{r^\nu}\frac{\partial}{\partial r}(r^\nu n_i u_i)=0, \tag{11}$$

$$\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial r}=-\frac{\partial \phi}{\partial r}, \tag{12}$$

$$\frac{1}{r^\nu}\frac{\partial}{\partial r}\left(r^\nu\frac{\partial \phi}{\partial r}\right)=n_e-n_i. \tag{13}$$

Equations (11)–(13) and Eq. (10) are the four basic equations with four unknown quantities n_i, u_i, n_e, and ϕ. The only remaining free parameter is H, which measures the effect of quantum diffraction. Physically H is the ratio between the electron plasmon energy and the electron Fermi energy. The electron fluid velocity can be found from the continuity equation (11) n_e, replacing n_i. We now introduce the stretched coordinates

$$\xi=\epsilon^{1/2}(r-t),\quad \tau=\epsilon^{3/2}t \tag{14}$$

and expand n_i, u_i, and n_e in a power series of ϵ as

$$n_i=1+\epsilon n_i^{(1)}+\epsilon^2 n_i^{(2)}+\cdots, \tag{15}$$

$$u_i=\epsilon u_i^{(1)}+\epsilon^2 u_i^{(2)}+\cdots, \tag{16}$$

$$n_e=1+\epsilon n_e^{(1)}+\epsilon^2 n_e^{(2)}+\cdots. \tag{17}$$

Due to the above expansion of n_e, the expansion for ϕ [Eq. (10)] becomes

$$\phi = \epsilon\left(n_e^{(1)} - \frac{H^2}{4}\frac{\partial^2 n_e^{(1)}}{\partial r^2} - \frac{H^2}{4}\frac{\nu}{r}\frac{\partial n_e^{(1)}}{\partial r}\right) + \frac{\epsilon^2}{2}\left[n_e^{(1)2} + 2n_e^{(2)} + \frac{H^2}{2}\left\{n_e^{(1)}\left(\frac{\partial^2 n_e^{(1)}}{\partial r^2} + \frac{\nu}{r}\frac{\partial n_e^{(1)}}{\partial r}\right) + \frac{1}{2}\left(\frac{\partial n_e^{(1)}}{\partial r}\right)^2 - \frac{\partial^2 n_e^{(2)2}}{\partial r^2} - \frac{\nu}{r}\frac{\partial n_e^{(2)}}{\partial r}\right\}\right]. \tag{18}$$

Now we develop Eqs. (11)–(13) in the form of a power series of ϵ. Then the system of equations can be written as with the help of Eq. (18)

$$\frac{\partial}{\partial \xi}(u_i^{(1)} - n_i^{(1)}) + \epsilon\left\{\frac{\partial n_i^{(1)}}{\partial \tau} + \frac{\partial u_i^{(2)}}{\partial \xi} - \frac{\partial n_i^{(2)}}{\partial \xi} + \frac{\partial}{\partial \xi}(n_i^{(1)}u_i^{(1)}) + \frac{\nu}{\tau}u_i^{(1)}\right\} = O(\epsilon^2), \tag{19}$$

$$\frac{\partial}{\partial \xi}(n_e^{(1)} - u_i^{(1)}) + \epsilon\left\{\frac{\partial u_i^{(1)}}{\partial \tau} - \frac{\partial u_i^{(2)}}{\partial \xi} + u_i^{(1)}\frac{\partial u_i^{(1)}}{\partial \xi} - \frac{H^2}{4}\frac{\partial^3 n_e^{(1)}}{\partial \xi^3} + \frac{1}{2}\frac{\partial}{\partial \xi}(n_e^{(1)2} + 2n_e^{(2)})\right\} = O(\epsilon^2), \tag{20}$$

$$n_i^{(1)} - n_e^{(1)} + \epsilon\left\{n_i^{(2)} - n_e^{(2)} + \frac{\partial^2 n_e^{(1)}}{\partial \xi^2}\right\} = O(\epsilon^2). \tag{21}$$

The zeroth order terms of the above equations together with the assumption the $u_i^{(1)}$ and $n_i^{(1)}$ vanish as $\xi \to 0$ yields

$$n_e^{(1)} = n_i^{(1)} = u_i^{(1)} \equiv U(\xi, \tau) \tag{22}$$

defining a new function $U(\xi, \tau)$.

From (19)–(21), considering the first order terms using Eq. (22) we have

$$\frac{\partial U}{\partial \tau} + \frac{\partial}{\partial \xi}(u_i^{(2)} - n_i^{(2)} + U^2) + \frac{\nu}{\tau}U = 0, \tag{23}$$

$$\frac{\partial U}{\partial \tau} + \frac{\partial}{\partial \xi}\left(n_e^{(2)} - u_i^{(2)} + U^2 - \frac{H^2}{4}\frac{\partial^2 U}{\partial \xi^2}\right) = 0, \tag{24}$$

$$\frac{\partial^2 U}{\partial \xi^2} = n_e^{(2)} - n_i^{(2)}. \tag{25}$$

Combining Eqs. (23)–(25), we deduce a modified deformed KdV equation for quantum ion acoustic waves

$$\frac{\partial U}{\partial \tau} + \frac{\nu}{2\tau}U + 2U\frac{\partial U}{\partial \xi} + \frac{1}{2}\left(1 - \frac{H^2}{4}\right)\frac{\partial^3 U}{\partial \xi^3} = 0. \tag{26}$$

III. AN ANALYTICAL SOLUTION OF CYLINDRICAL AND SPHERICAL KDV EQUATIONS

It can be shown[19] that a suitable coordinate transformation reduces the above cylindrical deformed KdV equation [Eq. (26), for $\nu=1$] into the ordinary KdV equation which can be solved analytically. In this way the exact solution of Eq. (26) for $\nu=1$ can be written as[20]

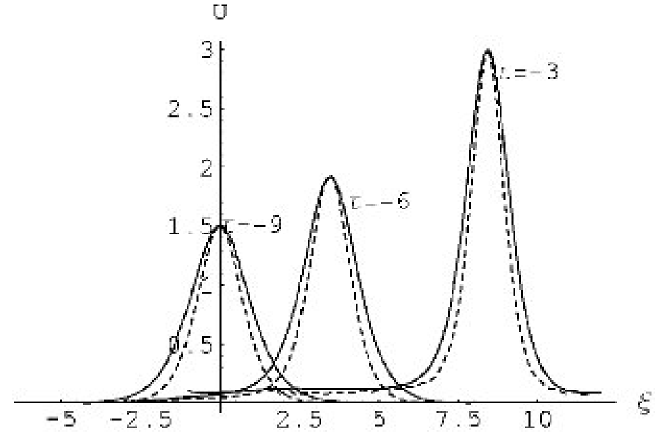

FIG. 1. Numerical solution for Eq. (26), for different values of τ, and for several values of H (solid line for $H=1.0$ and dashed line for $H=1.5$), where $\nu=1$.

$$U = \frac{1}{\tau}\left[\frac{\xi}{4} + \frac{3V}{2}\operatorname{sech}^2\left(\sqrt{\frac{V}{2\left(1 - \frac{H^2}{4}\right)\tau}}(\xi + 2V)\right)\right], \tag{27}$$

where V is the solitary wave velocity and the solution is valid for $(1-H^2/4)\tau > 0$ or $\tau > 0$, $H < 2$ or $\tau < 0$, $H > 2$. It is not valid for $H=2$.

Another analytical solution of the cylindrical and spherical KdV equations can be found by the group analysis method.[20] By this method the solution of cylindrical dKdV equation is given by

$$U = \frac{\xi}{4\tau} + \frac{1}{\tau}(a_0 + a_2 \tanh^2(\xi\tau^{-1/2})), \tag{28}$$

where

$$a_0 = 2\left(1 - \frac{H^2}{4}\right) \quad \text{and} \quad a_2 = -3\left(1 - \frac{H^2}{4}\right). \tag{29}$$

The solution (28) reduces to the solution $U = \xi/4\tau$, when $H=2$. When $H=2$ the dKdV equation reduces to the first order partial differential equation which can be solved exactly by the method of characteristics. In the case of cylindrical geometry one gets the solution

$$U = \frac{1}{\sqrt{\tau}}F(\xi - 4U\tau) \tag{30}$$

and for spherical geometry the solution is

$$U = \frac{\xi}{2\tau \ln \tau} + \frac{F(U\tau)}{\tau \ln \tau} \tag{31}$$

for a constant F the solution reduces to

$$U = \frac{\xi}{2\tau \ln \tau} + \frac{c}{\tau \ln \tau} \tag{32}$$

which is valid for all values of H.

IV. NUMERICAL SOLUTIONS

Equation (26) has the following solitary wave solution for $\nu=0$

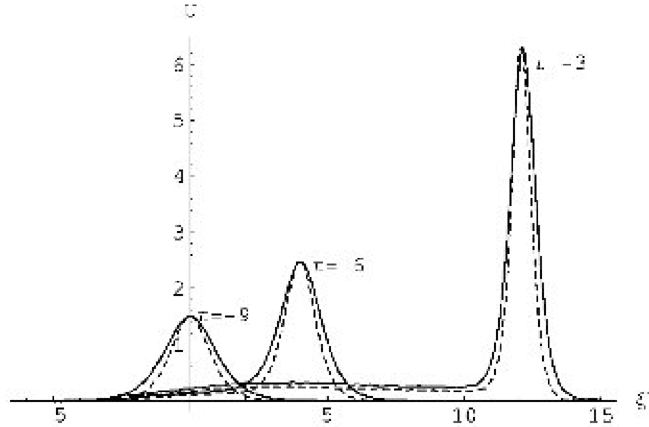

FIG. 2. Numerical solution for Eq. (26), for different values of τ and for several values of H (solid line for $H=1.0$ and dashed line for $H=1.5$), where $\nu=2$.

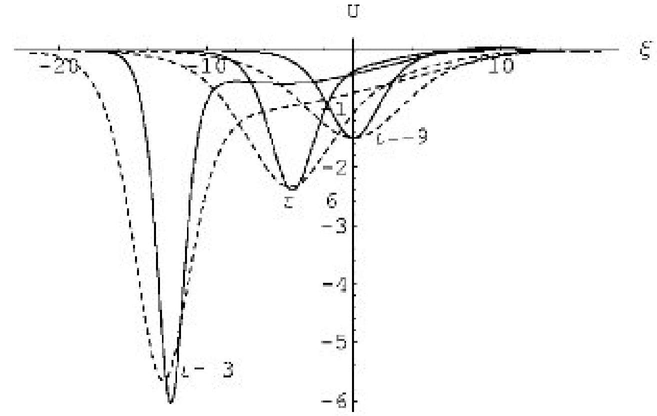

FIG. 4. Numerical solution for Eq. (26), for different values of τ and for several values of H (solid line for $H=4.0$ and dashed line for $H=7.0$), where $\nu=2$.

$$U(\xi,\tau)=\frac{3V}{2}\,\mathrm{sech}^2\left(\sqrt{\frac{V}{2\left(1-\frac{H^2}{4}\right)}}(\xi-V\tau)\right), \tag{33}$$

where V is the solitary wave velocity. With this initial profile at $\tau=-9$ we solve the cylindrical and spherical deformed KdV equation. In Fig. 1 and Fig. 2 we plot the solutions of (26) for several values of τ ranges from $\tau=-3$ to $\tau=-9$ in cylindrical ($\nu=1$) and spherical ($\nu=2$) geometry, respectively, and for different values of H (quantum parameter) for $H<2$. It is seen that as the magnitude of τ increases the solutions look like those for one-dimensional KdV solitons. This is because the extra term $(\nu/2\tau)U$ becomes small for large values of τ and we get back the known KdV solution. Here we have seen that the amplitude of the solitary waves do not change as H increases, but the width of the solitary waves is reduced as H increases for $H<2$. For $H>2$, the soliton velocity V would be negative, otherwise the soliton would cease to exist. For $H>2$, the solutions of (26) are plotted for different values of τ in Fig. 3 and in Fig. 4 for $\nu=1$ and $\nu=2$, respectively, and for several values of H, the quantum parameter. It is seen that for $H>2$, the solitary waves propagate in the negative direction with a decrease of τ. It is also seen that the amplitude of the spherical solitary wave is greater than that of the cylindrical solitary wave. Also the amplitude of the solitary waves remains unchanged, but the width increases as H increases for $H>2$. So for both cases, viz., $H<2$ and $H>2$ the amplitude of the solitary waves remains unaffected, but the width of the solitary waves decreases as H increases for $H<2$, whereas the width of the solitary waves expands as H increases for $H>2$. In Figs. 5 and 6 the exact solution [Eq. (27)] of the cylindrical deformed KdV equation is plotted for $H<2$ ($\tau=1$) and $H>2$ ($\tau=-1$), respectively.

V. CONCLUSION

We have derived cylindrical and spherical dKdV equations for quantum ion acoustic waves in an unmagnetized two species quantum plasma system, comprised of electrons and ions. The standard reductive perturbation technique is employed to derive dKdV equations in the weakly nonlinear limit. It is found that the propagation of quantum ion acoustic waves in nonplanar geometry differs from that in one-dimensional planar geometry. It is also seen that for large values of τ the solution is similar to the one-dimensional

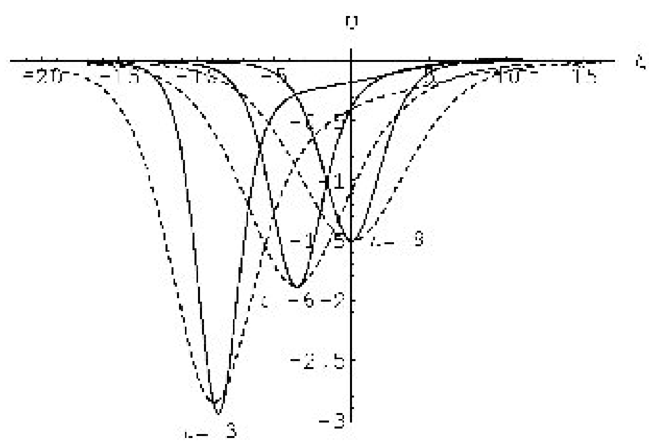

FIG. 3. Numerical solution for Eq. (26), for different values of τ and for several values of H (solid line for $H=4.0$ and dashed line for $H=7.0$), where $\nu=1$.

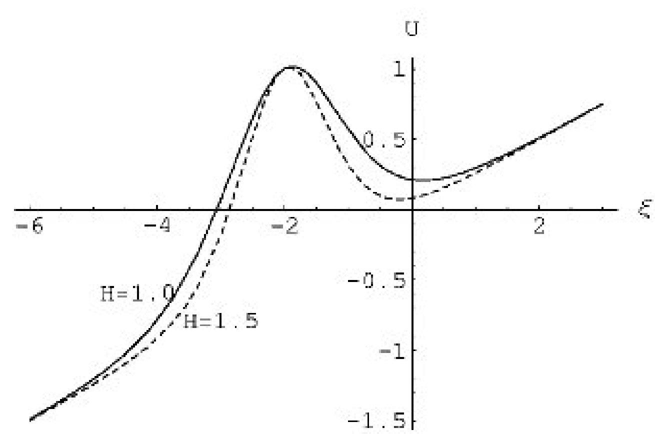

FIG. 5. Plot of the analytical solution for Eq. (27), for different values of H (solid line for $H=1.0$ and dashed line for $H=1.5$), where $\nu=1$ and $\tau=1$.

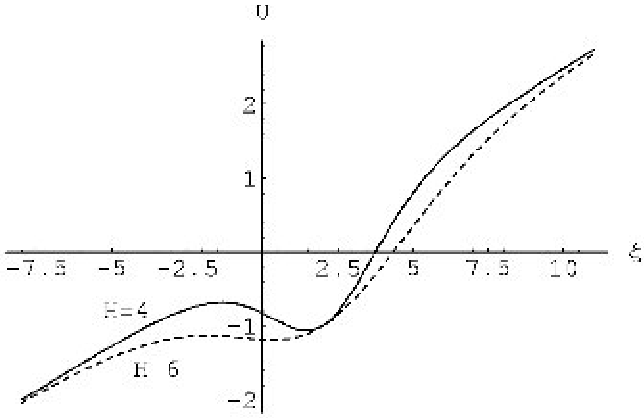

FIG. 6. Plot of the analytical solution for Eq. (27), for different values of H (solid line for H=4.0 and dashed line for H=7.0), where ν=1 and τ=−1.

dKdV soliton, but for small values of τ the soliton solution differs from the one-dimensional soliton. This is because of the extra term $(\nu/2\tau)U$ which becomes small for large values of τ. It is also found that for $H<2$ one gets positive (bright) soliton solution whereas for $H>2$ one gets negative (dark) soliton solution. It is also seen that there exists a critical value of the quantum parameter $(H=2)$ for which the quantum ion acoustic soliton collapses. We have also obtained exact analytical solutions for $H\neq 2$ and an exact solution of the deformed spherical KdV equation for $H=2$. These results may be used for the description of ultracold neutral atom gases.

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[1]Yu L. Klimontovich and V. P. Slin, Z. Phys. **23**, 151 (1952); J. E. Drummond, *Plasma Physics* (McGraw-Hill, New York, 1961); N. Maafa, Phys. Scr. **48**, 351 (1993).
[2]G. Manfredi and M. R. Feix, J. Plasma Phys. **53**, 6460 (1996).
[3]G. Chabrier, F. Douchin, and A. Y. Potekhin, J. Phys.: Condens. Matter **14**, 9133 (2002).
[4]M. Opher, L. O. Silva, D. E. Dauger, V. K. Decyk, and J. M. Dawson, Phys. Plasmas **8**, 2454 (2001).
[5]Y. D. Jung, Phys. Plasmas **8**, 3842 (2001).
[6]D. Kremp, Th. Bornath, M. Bonitz, and M. Schlanges, Phys. Rev. E **60**, 4725 (1999).
[7]G. Manfredi and F. Hass, Phys. Rev. B **64**, 075316 (2001).
[8]N. Suh, M. R. Feix, and P. Bertrand, J. Comput. Phys. **94**, 403 (1991).
[9]E. Madelung, Z. Phys. **40**, 332 (1926).
[10]C. Gardner, SIAM (Soc. Ind. Appl. Math.) J. Numer. Anal. **54**, 409 (1994).
[11]M. G. Acona and G. J. Iafrate, Phys. Rev. B **39**, 9536 (1989); M. V. Kuzelev and A. A. Rukhadze, Phys. Usp. **42**, 687 (1999).
[12]C. Gardner and C. Ringhofer, VLSI Des. **10**, 415 (2000).
[13]I. Gasser and P. A. Markowich, Asymptotic Anal. **14**, 97 (1997).
[14]I. Gasser, C. K. Lin, and P. Markowich, Taiwan. J. Math. **4**, 501 (2000).
[15]P. A. Markowich, C. A. Ringhofer, and C. Schmeiser, *Semiconductor Equations* (Springer, Vienna, 1990).
[16]F. Haas, L. G. Garcia, J. Goedert, and G. Manfredi, Phys. Plasmas **10**, 3858 (2003).
[17]F. Haas, Phys. Plasmas **12**, 062117 (2005).
[18]L. G. Garcia, F. Haas, L. P. L. de Oliveira, and J. Goedert, Phys. Plasmas **12**, 012302 (2005).
[19]R. Hirota, Phys. Lett. **71A**, 393 (1979).
[20]B. Sahu and R. Roychoudhury, Phys. Plasmas **10**, 4162 (2003).