ON TIPPETT'S "RANDOM SAMPLING NUMBERS."

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INTRODUCTION.

In a foreword to Tippett's "Random Sampling Numbers" (*Tracts for Computers No. XV, Camb. Univ. Press, 1927*), Karl Pearson laid emphasis on the importance of testing statistical theories by sampling experiments. How Tippett's numbers could be put to use for this purpose was illustrated by means of examples.

Till the advent of these numbers the work of sampling was being carried out by drawing of balls or tickets from a bag. Karl Pearson drew pointed attention to the fact that the latter method, besides being laborious, failed to yield *perfectly random samples* when the number of samples required was large.

These numbers have been largely used with remarkable success, in the Department of Applied Statistics, University College, London and elsewhere for conducting sampling experiments, but whether they are perfectly random has not yet been established. In fact Karl Pearson concluded the foreword with a warning to users of Tippett's numbers against pronouncing premature judgment on their randomness. It will be pertinent in this connexion to quote a remark of Karl Pearson[2] on his experience with Tippett's numbers:

"This caution is not given wholly inadvisedly. I have not myself made much use of Tippett's numbers, but recently I obtained in 100 trials three such unusual samples that only one should have occurred in 1,000,000 trials."

P. C. Mahalanobis[3] had also noted that certain improbable results were obtained in using Tippett's numbers for verifying sampling moments of the D^2-statistic. He had suggested however that such discrepancies were probably due to using the same random deviates in different combinations, and not to a lack of randomness on the part of Tippett's numbers.

Probability integral tables have been published for most of Pearson's curves and so using Tippett's numbers large scale sampling can be done for populations conforming to these curves of distribution. Individual workers in the past have prepared for their own use, samples of various sizes from many of these populations, but have not published them in a form that might prove beneficial to future workers. P. C. Mahalanobis, with the co-operation of three others, published an exhaustive table of 'random' samples from the Normal population for the 10,400 Tippett's numbers read horizontally only[4]. After choosing a convenient size for class intervals, all the 10,400 values were thrown into a single frequency distribution and were likewise formed 26 and 104 frequency distributions of sizes 400 and 100 respectively. These 131 frequency distributions were tested for 'goodness of fit' with the theoretical normal curve. They obtained satisfactory values for $P(\chi^2)$ thereby "confirming the random character of Tippett's numbers." For the single sample of 10,400 the first six moments were calculated and it was found they were very close to the expected values.

The $P(\chi^2)$ test adopted by them becomes inappropriate when we wish to test for randomness, samples smaller than size 100. It is the case of small samples that has been examined in this paper. The appropriate test used is described later.

NATURE OF TIPPETT'S NUMBERS.

Tippett formed his numbers by taking 41,600 digits at random from Census Reports and combining them by fours to give 10,400 numbers.

By means of Tippett's numbers it is possible to get samples from populations with known probability integrals. If Tippett's 10,400 numbers are each preceded by a decimal point, they can be looked upon as forming a system of probability integrals; and, with the help of a table of probability integrals of the population from which the samples are required to be drawn, we can find the corresponding values of 10,400 individual variates belonging to the sampled population expressed in appropriate units. If it is samples of five that we want, we have to group the 10,400 values by fives, or if it is samples of ten that are required, we have to group them by tens, and so on. It must be noted, however, that as the size of the samples increases, the number of samples available decreases. Thus only 104 independent samples of size 100, and only 26 independent samples of size 400, can be obtained from a system of 10,400 values. To remove this limitation, Karl Pearson suggested that "we may read our numbers backwards or take the last two figures of one column with the first two of the next, or read the four figures diagonally, etc., etc."

SAMPLING FROM CONTINUOUS POPULATION.

If the sampled population is actually represented by a continuous frequency curve its probability integral will take all values between 0 and 1. So, while sampling from such a population all values of the probability integral between 0 and 1 must be given equal representation. In using Tippett's numbers it, however, becomes necessary in practice to limit values of the probability integral to four decimal places. Theoretically a truly random sample from the population will in general contain many individuals represented in the table of probability integrals by figures which run to 5 or more decimal places. All such individuals are excluded when we make use of Tippett's numbers.

This limitation of Tippett's numbers is not perhaps sufficiently recognised while taking large samples. For, here each sample is given as a frequency distribution within a limited number of class intervals, and we are not concerned with the exact value of every individual variate in each sample but only with the group to which it belongs. But this simplification is not possible in the case of small samples, where grouping is not considered desirable. The present position of sampling technique can be well described in the words of Holzinger and Church[2]:—

"At present no really satisfactory method has been devised of sampling from a population which was actually represented by a smooth frequency curve, and not by a series of discrete frequency groups . . . . The necessity of subdividing a sampled population into a limited number of frequency groups before actual sampling can be performed, is one of the greatest difficulties that arise in sampling."

Owing to the rapid development of the "theory of small samples" in recent years, much use is however being made of Tippett's numbers either for verification of distribution laws derived analytically, or for discovering new laws through empirical means based on the results of sampling. It is the object of this paper to describe the method used and results obtained in an investigation on the reliability of Tippett's numbers when used in the customary way for sampling from continuous populations.

THE TEST FOR RANDOMNESS.

In Biometrika Vol. XXV, pp. 379 *et seq.* Karl Pearson had developed a new statistical test which he calls the $P(\lambda_n)$ test "for determining whether a sample of size n, supposed to have been drawn from a parent population having a known probability integral has probably been drawn at random."

The $P(\lambda_n)$ test supersedes the $P(\chi^2)$ test in that the former is applicable to small samples where grouping is not possible. It is also free from certain assumptions which are inevitable in the latter test. Karl Pearson had later shown[3] that the $P(\lambda_n)$ test could also be used as an alternative test of 'goodness of fit' for large 'grouped' samples and was inclined to the view that it was probably a more stringent test than the ordinary $P(\chi^2)$ test, since the former avoided the clubbing together of terminal frequencies.

The $P(\lambda_n)$ test was evolved by Pearson from the following considerations:—

We have a sample of n variates $x_1, x_2, \ldots\ldots x_n$ taken from a population following a given or supposed law of distribution of which we know the probability integral. Let the values of this for our n sample variates $x_1, x_2, \ldots\ldots x_n$ be respectively $p_1, p_2, \ldots\ldots p_n$. These probabilities follow a rectangular distribution. If we take $\lambda_n = p_1 \times p_2 \times \ldots\ldots \times p_n$ (the probability of the occurrence of the particular independent set of probabilities $p_1, p_2, \ldots\ldots p_n$) then the frequency distribution of λ_n is given by[7]

$$df = \frac{1}{(n-1)\,!}\,(-\log_e \lambda_n)^{n-1}\,d\lambda_n$$

and the probability $P(\lambda_n)$ of a combination occurring with a probability value as great as, or greater than λ_n is given by

$$P(\lambda_n) = \frac{1}{(n-1)\,!}\int_{\lambda_n}^{1}(-\log_e \lambda_n)^{n-1}d\lambda_n$$

$$= I\left(n-1,\ -\frac{\log_{10}\lambda_n}{\sqrt{n}\,\log_{10}e}\right)$$

(or if $Q(\lambda_n)$ be the probability of a lower combined probability occurring $Q(\lambda_n) = 1 - P(\lambda_n)$) where $I(p, u)$ is the function tabled in the *Tables of the incomplete Γ-function*.

Now, the proof of this test depends on p_s's $(s = 1, 2, \ldots n)$ being all measured in the *same* direction and since there is no reason why we should choose one end rather than the other, we must find $p'_s = 1 - p_s$ and calculate $\lambda'_n = (1-p_1)(1-p_2)\ldots(1-p_n)$. We have thus two series of probability integrals associated with a particular sample, giving us two $P(\lambda_n)$'s and two $Q(\lambda_n)$'s. Since we must judge of a sample by the test which is the more stringent, *i.e.*, that in which the probability of a more improbable result occurring would be the smaller, we have to select the smaller $Q(\lambda_n)$ as the criterion for judging the randomness of the sample.

For rapid application of this test, tables of values of $P(\lambda_n)$ and $-\log_{10}\lambda_n$ for $n = 2$ to 30 have been provided by Miss F. N. David.[1]

A very remarkable property of the $P(\lambda_n)$ test is that it is essentially a test of randomness. That is to say, the individuals $x_1, x_2, \ldots\ldots x_n$ constituting the sample need not all be drawn from the same parent population, but may be taken from a number of populations, one or more from each. If they are drawn at random from their respective populations their probability integrals $p_1, p_2 \ldots\ldots p_n$ calculated from the appropriate population curves of frequency, will all be random samples from a rectangular distribution and may be combined to serve as a random sample of n from such a distribution. The $P(\lambda_n)$ test is accordingly a test of randomness.

Application of $P(\lambda_n)$ Test to Tippett's Numbers.

It has been pointed out already that Tippett's numbers, each preceded by a decimal point, may be regarded as a system of probability integrals belonging to that population from which we wish to take random samples. Tippett's numbers accordingly supply the p's with which the x's are found from probability integral tables. If Tippett's numbers were really a random collection these integrals formed out of them must behave as a random selection of individuals following the rectangular distribution. The x's will be random only if the p's are random.

TABLE I. FREQUENCY DISTRIBUTION OF $\Gamma(\lambda_s)$.

Plate:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	Total
·00 — ·05	4	2	4	2	5	7	4	1	2	2	5	1	3	5	5	7	3	3	3	5	3	3	0	2	3	6	90
·05 — ·10	3	6	2	5	0	2	1	3	1	7	0	2	6	2	5	0	3	4	4	6	7	7	5	2	5	3	91
·10 — ·15	5	3	2	9	6	1	2	3	7	3	2	5	7	5	5	6	5	3	9	2	3	2	4	3	4	2	108
·15 — ·20	5	3	6	7	2	6	6	4	4	4	7	3	4	2	2	3	5	2	5	3	5	6	2	3	2	3	104
·20 — ·25	1	9	2	0	3	5	5	9	6	3	6	3	1	4	0	3	4	2	4	2	4	3	4	8	7	2	100
·25 — ·30	2	5	3	4	3	5	3	5	4	5	7	2	2	4	10	5	5	2	5	2	3	5	5	3	3	2	104
·30 — ·35	7	2	1	1	2	2	1	2	4	2	4	10	3	5	5	4	2	4	3	5	6	6	3	4	5	1	94
·35 — ·40	5	4	2	2	5	2	5	2	7	2	6	3	4	4	2	5	1	7	6	2	3	3	4	4	2	3	95
·40 — ·45	6	2	1	5	5	3	8	5	4	2	3	7	5	3	10	5	1	2	0	3	4	2	6	6	6	3	107
·45 — ·50	2	3	5	5	4	7	7	5	0	5	6	6	7	5	2	2	4	3	4	4	2	5	3	5	5	4	110
·50 — ·55	4	3	6	6	7	4	3	2	2	4	4	2	5	5	6	3	8	5	2	5	4	4	5	4	2	3	108
·55 — ·60	4	4	4	4	4	3	3	7	7	6	5	3	3	3	3	4	8	3	1	5	6	3	3	4	6	5	111
·60 — ·65	5	2	4	2	4	5	5	5	6	6	3	4	1	3	2	5	6	6	3	2	6	5	2	4	4	3	103
·65 — ·70	2	4	6	3	7	8	9	6	2	7	3	6	3	2	3	2	2	4	3	6	3	6	5	5	4	5	116
·70 — ·75	2	3	6	4	7	4	6	4	1	4	4	3	6	4	3	5	1	3	6	3	4	4	6	4	4	6	107
·75 — ·80	4	5	5	5	3	2	3	1	2	2	5	6	4	3	2	2	4	6	3	8	3	1	7	2	1	1	90
·80 — ·85	4	6	3	2	1	4	3	4	6	4	1	3	2	2	3	4	2	2	2	4	2	3	4	3	4	3	81
·85 — ·90	6	7	2	3	4	5	2	0	7	2	1	3	3	5	5	4	5	10	8	5	3	4	4	4	5	9	116
·90 — ·95	6	5	9	8	4	4	0	5	3	3	4	2	5	10	2	7	6	4	6	4	3	4	2	3	3	10	122
·95 — 1·00	3	2	7	3	4	1	4	7	5	7	4	6	6	4	5	4	5	5	3	4	6	4	6	7	5	6	123
Total ...	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	80	2080

TABLE 2. FREQUENCY DISTRIBUTION OF $P(\lambda_{10})$.

Plate:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	Total
·00 — ·05	2	4	0	5	1	2	1	2	2	4	1	0	2	1	1	2	2	1	1	2	2	2	0	2	1	1	44
·05 — ·10	1	0	4	1	1	1	4	0	1	0	2	1	3	4	2	3	2	1	2	1	2	4	2	2	1	0	45
·10 — ·15	4	4	2	2	4	3	1	1	1	1	3	2	4	0	3	4	2	2	3	1	4	2	1	2	3	1	60
·15 — ·20	1	1	1	2	3	3	2	3	4	3	4	2	2	0	2	4	2	3	3	1	1	1	0	2	3	0	53
·20 — ·25	2	1	2	2	0	2	1	2	1	2	2	2	2	3	4	1	1	0	3	2	2	0	2	2	3	0	44
·25 — ·30	0	0	1	1	2	2	2	2	5	1	2	1	0	4	2	3	2	2	0	1	3	2	2	0	1	1	42
·30 — ·35	4	2	3	2	1	3	2	4	2	0	3	4	4	1	1	0	0	1	2	1	1	2	3	1	3	6	56
·35 — ·40	2	3	2	2	0	2	3	2	2	0	2	4	4	2	4	1	1	0	2	2	1	2	0	3	4	0	50
·40 — ·45	4	1	0	2	3	2	2	0	1	5	3	3	0	0	3	0	2	2	3	5	2	5	2	3	1	0	54
·45 — ·50	3	2	1	1	1	3	2	5	1	2	2	3	0	2	3	2	4	2	2	0	2	3	4	0	2	1	53
·50 — ·55	2	2	2	0	3	1	5	1	0	2	0	1	2	1	1	1	0	0	1	2	4	2	1	3	3	4	44
·55 — ·60	0	1	1	2	3	0	1	0	0	2	2	2	0	3	3	2	3	3	2	2	2	2	1	0	0	5	42
·60 — ·65	2	1	0	2	2	1	1	1	5	2	1	0	4	1	1	2	3	5	2	4	2	1	3	1	1	0	48
·65 — ·70	3	3	1	1	1	1	2	3	2	2	2	0	0	3	1	3	0	1	3	3	3	2	3	2	2	5	52
·70 — ·75	1	1	2	1	5	1	4	2	2	2	3	4	0	5	1	0	4	3	2	1	1	2	5	2	4	1	59
·75 — ·80	1	5	4	3	3	6	2	0	1	0	0	3	1	2	1	2	2	3	1	5	1	2	0	6	1	2	57
·80 — ·85	1	3	0	3	0	4	2	3	1	5	3	0	2	2	3	1	3	2	2	0	1	0	5	3	1	0	50
·85 — ·90	2	1	6	3	4	1	0	2	4	2	2	2	4	1	1	2	2	3	1	2	0	2	4	1	1	5	58
·90 — ·95	0	5	5	5	3	1	2	6	2	2	2	2	1	1	1	3	2	4	4	4	4	3	2	4	4	6	78
·95 —1·00	5	0	3	0	0	1	1	1	3	3	1	4	5	4	2	4	3	2	1	1	2	1	0	1	1	2	51
Total ...	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	40	1040

TABLE 3. FREQUENCY DISTRIBUTION OF $P(\lambda_{21})$.

Plate:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	Total
.00 — ·05	1	0	0	2	1	1	1	1	1	0	0	0	1	0	2	1	1	1	1	1	1	2	0	1	1	0	21
·05 — ·10	2	0	0	1	0	0	0	0	0	1	2	1	2	0	0	0	0	0	1	0	1	1	1	0	1	0	14
·10 — ·15	0	0	1	1	1	0	1	1	0	2	3	1	0	1	1	0	0	1	1	0	0	0	0	0	1	1	17
·15 — ·20	2	1	2	1	0	1	2	0	1	1	1	0	1	1	1	1	0	0	0	0	3	2	0	1	1	2	25
·20 — ·25	0	1	1	1	1	2	0	0	2	0	1	1	1	1	1	0	1	0	0	0	2	1	0	0	1	0	18
·25 — ·30	1	3	0	0	0	2	1	1	0	0	1	1	2	1	0	2	0	0	0	1	1	0	1	1	0	0	19
·30 — ·35	1	1	1	0	2	0	2	3	1	1	0	2	0	0	3	0	2	1	2	0	0	0	0	2	0	1	25
·35 — ·40	1	2	0	0	0	2	1	0	1	0	0	0	0	0	1	2	2	1	2	1	0	2	1	0	1	0	20
·40 — ·45	1	0	0	1	1	0	1	0	0	0	0	0	0	0	0	3	1	1	2	1	1	1	2	0	1	0	17
·45 — ·50	0	1	0	0	0	1	2	1	1	1	0	3	0	0	0	1	1	0	0	0	0	0	1	0	1	0	14
·50 — ·55	0	1	0	0	2	1	0	1	0	0	1	0	0	2	1	0	2	1	0	2	1	0	2	1	0	0	18
·55 — ·60	1	1	0	2	0	0	0	1	3	0	0	1	1	1	1	1	0	0	1	1	1	0	0	0	1	0	17
·60 — .65	2	1	0	0	2	3	0	1	1	2	1	0	0	0	0	0	0	0	2	4	0	2	2	2	1	2	28
·65 — ·70	1	0	2	2	1	0	0	0	0	2	1	0	1	4	1	0	0	2	0	2	0	1	1	3	2	1	27
·70 — ·75	0	0	1	0	2	0	0	0	0	0	1	0	2	0	0	1	0	1	0	1	1	0	0	1	0	1	12
·75 — ·80	0	1	0	1	0	1	1	1	0	1	1	1	1	1	1	0	1	0	0	0	0	2	2	1	1	0	18
·80 — ·85	0	0	2	0	2	1	2	2	1	2	2	1	1	0	0	1	1	3	1	0	1	1	0	1	0	2	27
·85 — ·90	1	1	2	2	0	0	0	1	2	2	1	1	0	1	2	2	1	1	1	1	1	0	0	1	1	·2	27
·90 — ·95	0	2	1	0	0	1	2	1	1	0	0	1	2	3	0	0	2	2	2	0	1	0	1	1	1	1	25
·95 — 1·00	2	0	3	2	1	0	0	1	1	1	0	2	1	0	1	1	1	1	0	1	1	1	2	0	1	3	27
Total ...	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	416

It follows therefore that the randomness of Tippett's numbers in connexion with its use in sampling from continuous frequency curves can be judged by means of the $P(\lambda_n)$ test.

The *modus operandi* is now quite clear. Since the quantities entered in Miss David's tables are $-\log_{10}\lambda_n$ and $P(\lambda_n)$ for $n=2$ to 30, we have first of all to prepare tables of $-\log_{10}p$ corresponding to the 10,400 numbers of Tippett. Since each number consisted only of four digits the ordinary four-figure logarithm tables could conveniently be used. By adding values of $-\log_{10}p$ by n's we can then get $-\log_{10}\lambda_n$. Since we are concerned here not with the testing of an isolated sample but of a large number of samples it was felt unnecessary to calculate both λ_n and λ'_n.

The present investigation was confined to samples of sizes 5, 10 and 25. The 2080 values of $-\log_{10}\lambda_5$, the 1040 values of $-\log_{10}\lambda_{10}$ and the 416 values of $-\log_{10}\lambda_{25}$ were obtained directly. By means of Miss David's tables corresponding interpolated values of $P(\lambda_5)$, $P(\lambda_{10})$ and $P(\lambda_{25})$ were then calculated, but have not been reproduced here owing to lack of space.

$P(\lambda_n)$ being in itself a probability integral, it must follow the rectangular law of distribution. The values of $P(\lambda_5)$, $P(\lambda_{10})$ and $P(\lambda_{25})$ observed on each of the 26 plates of Tippett's Numbers were first thrown into twenty frequency groups (class interval=0·05). These are shown in Tables 1, 2, and 3. The "goodness of fit" test of the right hand marginal totals with expectations of 104, 52 and 20·8 respectively in each class gave the following values for χ^2 and $P(\chi^2)$ with 19 degrees of freedom :—

	$P(\lambda_5)$	$P(\lambda_{10})$	$P(\lambda_{25})$
χ^2	28·04	25·78	24·00
$P(\chi^2)$	0·236	0·138	0·196

Judging from the five per cent. level of significance, the observed distributions of the three $P(\lambda)$'s may be considered probable samples from a rectangular population.

A "goodness of fit" test of the observed frequency distributions of the three $P(\lambda)$'s on each of the 26 plates should be more illuminating. To ensure the validity of the $P(\chi^2)$ test, the class intervals had to be widened to 0·20 in the case of $P(\lambda_5)$, to 0·25 in the case of $P(\lambda_{10})$ and to 0·50 in the case of $P(\lambda_{25})$. The 26 values of χ^2 obtained for each of them and the appropriate degrees of freedom are entered in Table 4.

TABLE 4. OBSERVED VALUES OF χ^2.

	$P(\lambda_5)$	$P(\lambda_{10})$	$P(\lambda_{25})$		$P(\lambda_5)$	$P(\lambda_{10})$	$P(\lambda_{25})$
D. F.	4	3	1	D. F.	4	3	1
Plate: 1	1·25	1·40	0·25	Plate: 15	4·00	2.60	0·25
2	3·50	2·40	0·25	16	1·13	4·00	1·00
3	7·38	9·00	2·25	17	3·38	0.60	0·00
4	9·38	4·00	0·25	18	3·75	3·80	2·25
5	4·25	2·60	1·00	19	7·50	0·60	0·25
6	1·13	5·00	0·25	20	2·25	1·80	4·00
7	8·50	2·00	2·25	21	0·50	1·00	0·25
8	2·38	2·60	0·25	22	0·51	2·20	0·25
9	5·50	0·40	0·25	23	2·51	3·60	1·00
10	1·51	0·80	1·00	24	3·50	3·80	2·25
11	5·88	1·60	0·00	25	1·50	0·60	0·00
12	2·88	4·40	0·25	26	13·38	11·80	4·00
13	4·50	3·80	0·25				
14	2·88	1·40	4·00	Total χ^2 ...	104·83	77.80	28·00

It will be seen that Plate 26 gives the highest value for χ^2 in all the three cases. In the case of $P(\lambda_{25})$ the biggest value of χ^2 *viz.* 4·00 occurs thrice in 26 cases whereas by chance such a value should occur in less than five per cent. of cases only. Plate 26 can be considered on the whole unsatisfactory.

Because of the additive property of χ^2 the total χ^2 given at the bottom of Table 4 should be distributed with 104, 78 and 26 degrees of freedom respectively. None of them is significantly large.

The present investigation shows that for samples of size 5, 10 and 25, we may consider to be random, Tippett's numbers as they have been actually used for sampling experiments, within the limitations imposed by the use of four decimal figures in the probability integrals. Greater precision is, of course, to be expected if we use 6 or 8 figure numbers formed from Tippett's numbers, but there are other considerations which will not always permit of such refinements in practice.

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DISCUSSION ON K. R. NAIR'S PAPER.

PROFESSOR R. A. FISHER remarked that the $P(\lambda_n)$ test did not necessarily give a more stringent test than the $P(\chi^2)$ test, for judging the goodness of fit. For example, it is conceivable that for a given set of observations ($x_1, x_2, \ldots\ldots x_n$) a number of parent curves could be found yielding the same λ_n. It will not be possible by means of the $P(\lambda_n)$ test to choose the best fitting among these curves.

PROF. P. C. MAHALANOBIS thought the $P(\lambda_n)$ test had one advantage, namely, that it could be used in the case of small samples which was not possible with the $P(\chi^2)$ test. For judging the goodness of fit of large samples he supposed there was not much to choose between the two tests as pointed out by Prof. Fisher. But in the present paper the $P(\lambda_n)$ test had been used not to judge the goodness of any fitted curve, but merely to test the randomness of a sample on the assumption that the population form was known. In fact as the 'random numbers' had entered in the calculations merely as values of a probability integral, the work was independent of the form of the parent population.