Determinant of the distance matrix of a tree with matrix weights

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Abstract

Let T be a tree with n vertices and let D be the distance matrix of T. According to a classical result due to Graham and Pollack, the determinant of D is a function of n, but does not depend on T. We allow the edges of T to carry weights, which are square matrices of a fixed order. The distance matrix D of T is then defined in a natural way. We obtain a formula for the determinant of D, which involves only the determinants of the sum and the product of the weight matrices.

Key words and phrases: tree, distance matrix, Laplacian matrix, matrix weights, determinant

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1 Introduction

We consider simple graphs, that is, graphs which have no loops or parallel edges. Thus a *graph* $G = (V(G), E(G))$ consists of a finite set of *vertices,* $V(G)$, and a set of *edges,* $E(G)$, each of whose elements is a pair of distinct vertices. We generally take $V(G) = \{1, 2, \dots, n\}$ and $E(G) = \{e_1, \dots, e_m\}$, unless stated otherwise. We will assume familiarity with basic graph-theoretic notions, see, for example, [2, 3].

Let G be a connected graph. The *distance* between vertices i, j of G, denoted by d_{ij}, is defined to be the length (i.e., the number of edges) in a shortest path from i to j in the graph. The *distance matrix* of G, denoted by $D(G)$, or simply by D, is the $n \times n$ matrix with its (i, j)-entry equal to $d_{ij}; i, j = 1, 2, \dots, n$. Note that $d_{ii} = 0, i = 1, 2, \dots, n$.

If T is a tree with n vertices, then according to a well-known result of Graham and Pollack [4], the determinant of D is $(-1)^{n-1}(n-1)2^{n-2}$. Thus the determinant of D is a function of n but does not depend on the tree itself. An extension of this result to weighted trees, the weights being scalars, was obtained in [1].

In this paper we consider a tree with each of its edges bearing a square matrix as weight. All the weight matrices will be of a fixed order, to be generally denoted by s. If i and j are vertices of T, then there is a unique path from i to j, and the distance between i and j is defined to be the sum of the matrices associated as weights to the edges of the path. The distance matrix D of T is then a block matrix, of order $ns \times ns$, with its (i, j)-block d_{ij} equal to the distance between i and j, if $i \neq j$ and is the $s \times s$ null matrix if $i = j$. We obtain a formula for the determinant of D which contains the classical formula due to Graham and Pollack [4] as a special case.

We introduce some more notation. The $n \times 1$ vector of all ones and the identity matrix of order n will be denoted by $\mathbf{1}_n$ and I_n respectively. Let δ_i denote the degree of the vertex i, let $\tau_i = 2 - \delta_i, i = 1, 2, \dots, n$, and let $\tau = [\delta_1, \dots, \delta_n]^T$. Note that

$$\sum_{i=1}^{n} \tau_i = \sum_{i=1}^{n} (2 - \delta_i) = 2n - 2(n-1) = 2. \tag{1}$$

The Kronecker product of matrices will be denoted by $\otimes$.

2 The Main Result

We first prove a preliminary result.

Lemma 1 *Let T be a tree with n vertices, let W_i be the $s \times s$ edge weight matrix associated with the edge $e_i, i = 1, 2, \ldots, n-1$, let τ be the vector with $\tau_i = 2 - \delta_i, i = 1, 2, \ldots, n$, and let D be the distance matrix of T. Then*

$$D(\tau \otimes I_s) = \mathbf{1}_n \otimes (\sum_{i=1}^{n-1} W_i).$$

Proof: Recall that D is a block matrix, of order $ns \times ns$, with its (i, j)-block equal to d_{ij}. Let i be fixed, $1 \le i \le n$. Then we must prove that

$$\sum_{r=1}^{n} \tau_r d_{ir} = \sum_{j=1}^{n-1} W_j. \tag{2}$$

For $1 \le j \le n-1, 1 \le k \le n$, let $p_{kj} = 1$ if the (unique) path from i to k in T passes through e_j and let $p_{kj} = 0$ otherwise. Then

$$\sum_{r=1}^{n} \tau_r d_{ir} = \sum_{j=1}^{n-1} (\sum_{k=1}^{n} p_{kj} \tau_k) W_j. \tag{3}$$

For $j, 1 \le j \le n-1$, let T_j be the component of $T \setminus e_j$ that does not contain i and let $V(T_j)$ be the vertex set of T_j. Let $u \in V(T_j)$ be an end-vertex of e_j. Note that for $k \in V(T_j)$, the degree of k in T and in T_j coincide if $k \neq u$, while the degree of u in T exceeds the degree of u in T_j by 1. This observation and (1) imply that

$$\begin{aligned}
\sum_{k=1}^{n} p_{kj} \tau_k &= \sum_{k \in V(T_j)} \tau_k \\
&= \sum_{k \in V(T_j)} (2 - \delta_k) \\
&= \sum_{k \in V(T_j), k \neq u} (2 - \delta_k) + (2 - \delta_u + 1) - 1 \\
&= 2 - 1 = 1.
\end{aligned}$$

Substituting the above expression in (3) we see that (2) is proved. ■

Theorem 2 *Let T be a tree with n vertices, let W_i be the $s \times s$ edge weight matrix associated with the edge $e_i, i = 1, 2, \ldots, n-1$, and let D be the distance matrix of T. Then*

$$\det D = (-1)^{(n-1)s} 2^{(n-2)s} \det(\prod_{i=1}^{n-1} W_i) \det(\sum_{i=1}^{n-1} W_i).$$

Proof: If $n = 2$, then $D = \begin{bmatrix} 0 & W_1 \\ W_1 & 0 \end{bmatrix}$. It is easily verified that

$$\det D = (-1)^s (\det W_1)^2,$$

and the proof is complete in this case. Let $n \geq 3$, assume the result to be true for a tree with $n-1$ vertices, and proceed by induction.

Now, as in the hypothesis, let T be a tree with n vertices, $n \geq 3$. We assume, without loss of generality, that vertex n is a pendant vertex and that it is adjacent to vertex $n-1$. We also assume that the edge with end-vertices n and $n-1$ is e_{n-1}. Let T_1 be the subtree of T obtained by deleting vertex n and let D_1 be the distance matrix of T_1.

We think of D as a block matrix with each block being an $s \times s$ matrix. The blocks are indexed by $(i, j), i, j = 1, 2, \ldots, n$. In D, subtract block $(n-1, i)$ from block $(n, i), i = 1, 2, \ldots, n$ and then subtract block $(i, n-1)$ from block $(i, n), i = 1, 2, \ldots, n$. The resulting matrix, denoted $\tilde{D}$, is given by

$$\tilde{D} = \left[\begin{array}{ccc|c} & & & W_{n-1} \\ & D_1 & & \vdots \\ & & & W_{n-1} \\ \hline W_{n-1} & \cdots & W_{n-1} & -2W_{n-1} \end{array}\right].$$

Since the theorem is assumed to hold for trees with $n-1$ vertices, then

$$\det D_1 = (-1)^{(n-2)s} 2^{(n-3)s} \det(\prod_{i=1}^{n-2} W_i) \det(\sum_{i=1}^{n-2} W_i). \quad (4)$$

We first assume that $\prod_{i=1}^{n-2} W_i$ and $\sum_{i=1}^{n-2} W_i$ are nonsingular, so that D_1 is nonsingular as well. The general case then follows by a continuity argument.

By the well-known formula for the determinant of a partitioned matrix,

$$\begin{aligned}\det D &= \det \tilde{D} \\ &= (\det D_1)\det(-2W_{n-1} - [W_{n-1},\cdots,W_{n-1}]D_1^{-1}\begin{bmatrix} W_{n-1} \\ \vdots \\ W_{n-1}\end{bmatrix}).\end{aligned} \tag{5}$$

Note that

$$\begin{aligned}D_1^{-1}(\mathbf{1}_{n-1}\otimes W_{n-1}) &= D_1^{-1}(\mathbf{1}_{n-1}\otimes(\sum_{i=1}^{n-2}W_i)(\sum_{i=1}^{n-2}W_i)^{-1}W_{n-1}) \\ &= D_1^{-1}(\mathbf{1}_{n-1}\otimes(\sum_{i=1}^{n-2}W_i))(\sum_{i=1}^{n-2}W_i)^{-1}W_{n-1}.\end{aligned} \tag{6}$$

The degree of vertex $n-1$ in T_1 is $\delta_{n-1}-1$. Therefore an application of Lemma 1 gives

$$D_1\left(\begin{bmatrix}\tau_1 \\ \vdots \\ \tau_{n-1}+1\end{bmatrix}\otimes I_s\right) = \mathbf{1}_{n-1}\otimes(\sum_{i=1}^{n-2}W_i),$$

and hence

$$D_1^{-1}(\mathbf{1}_{n-1}\otimes(\sum_{i=1}^{n-2}W_i)) = \begin{bmatrix}\tau_1 \\ \vdots \\ \tau_{n-1}+1\end{bmatrix}\otimes I_s. \tag{7}$$

It follows from (6) and (7) that

$$[W_{n-1},\cdots,W_{n-1}]D_1^{-1}\begin{bmatrix} W_{n-1} \\ \vdots \\ W_{n-1}\end{bmatrix} = (\tau_1+\cdots+\tau_{n-1}+1)W_{n-1}(\sum_{i=1}^{n-2}W_i)^{-1}W_{n-1}. \tag{8}$$

Since $\tau_n = 1$, by (1) we have $\tau_1+\cdots+\tau_{n-1}+1 = 2$ and hence (8) implies that

$$[W_{n-1},\cdots,W_{n-1}]D_1^{-1}\begin{bmatrix} W_{n-1} \\ \vdots \\ W_{n-1}\end{bmatrix} = 2W_{n-1}(\sum_{i=1}^{n-2}W_i)^{-1}W_{n-1}. \tag{9}$$

In view of (4),(5) and (9),

$$\begin{aligned}
\det D &= (\det D_1)\det(-2W_{n-1} - 2W_{n-1}(\sum_{i=1}^{n-2} W_i)^{-1}W_{n-1}) \\
&= (\det D_1)(\det W_{n-1})\det(-2I - 2(\sum_{i=1}^{n-2} W_i)^{-1}W_{n-1}) \\
&= (-1)^{(n-2)s}2^{(n-3)s}\det(\prod_{i=1}^{n-2} W_i)\det(\sum_{i=1}^{n-2} W_i) \\
&\quad \times \det(W_{n-1})(-2)^s\det(I + (\sum_{i=1}^{n-2} W_i)^{-1}W_{n-1}) \\
&= (-1)^{(n-2)s}2^{(n-3)s}(-2)^s\det(\prod_{i=1}^{n-2} W_i) \\
&\quad \times(\det W_{n-1})\det(\sum_{i=1}^{n-2} W_i)\det(I + (\sum_{i=1}^{n-2} W_i)^{-1}W_{n-1}) \\
&= (-1)^{(n-1)s}2^{(n-2)s}\det(\prod_{i=1}^{n-1} W_i)\det(\sum_{i=1}^{n-1} W_i)
\end{aligned}$$

and the proof is complete. ■

As an application, if A, B and C are $s \times s$ matrices, then by using Theorem 2 we get the following determinantal identity. (Here the tree is taken to be the path on four vertices.)

$$\det \begin{bmatrix} 0 & A & A+B & A+B+C \\ A & 0 & B & B+C \\ A+B & B & 0 & C \\ A+B+C & B+C & C & 0 \end{bmatrix} = (-1)^s 2^{2s}\det(ABC)\det(A+B+C).$$

It is known that the distance matrix of an unweighted tree or a tree with positive numbers as edge weights has exactly one positive eigenvalue (see, for example, [1]). An analogous property in the case of positive definite matrix weights is proved in the next result.

Theorem 3 *Let T be a tree with n vertices, let W_i be a positive definite $s \times s$ edge weight matrix associated with the edge $e_i, i = 1, 2, \ldots, n-1$, and let D be the distance matrix of T. Then D has s positive and $(n-1)s$ negative eigenvalues.*

Proof: First suppose that each weight matrix is the $s \times s$ identity matrix, and let D_1 be the corresponding distance matrix. Also, let D_2 be the $n \times n$ distance matrix of the tree T where each edge is assigned the weight 1. As remarked earlier, D_2 has 1 positive and $n-1$ negative eigenvalues. Then, since $D_1 = D_2 \otimes I_s$, it follows that D_1 has s positive and $(n-1)s$ negative eigenvalues.

For $0 \leq \alpha \leq 1$, let the edge weights of T be $(1-\alpha)W_i + \alpha I_s, i = 1, 2, \dots, n-1$, and let D_α be the corresponding distance matrix. Since each D_α is nonsingular by Theorem 2, D_0 and D_1 have the same inertia. Thus $D = D_0$ has s positive and $(n-1)s$ negative eigenvalues. ■

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