LEVEL-CROSSING PROBABILITIES AND FIRST-PASSAGE TIMES FOR LINEAR PROCESSES

GOPAL K. BASAK,* *University of Bristol*

KWOK-WAH REMUS HO,** *Hong Kong University of Science and Technology*

Abstract

Discrete time-series models are commonly used to represent economic and physical data. In decision making and system control, the first-passage time and level-crossing probabilities of these processes against certain threshold levels are important quantities. In this paper, we apply an integral-equation approach together with the state-space representations of time-series models to evaluate level-crossing probabilities for the AR(p) and ARMA(1, 1) models and the mean first passage time for AR(p) processes. We also extend Novikov's martingale approach to ARMA(p, q) processes. Numerical schemes are used to solve the integral equations for specific examples.

Keywords: First passage time; level crossing probability; integral equation; martingale; AR process; ARMA process

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1. Introduction

Let ξ_t, $t \in \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ be a sequence of independent and identically distributed (i.i.d.) random variables on a probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$. A general linear process $\{X_t\}$ can be defined as $X_t = \sum_{i=-\infty}^{\infty} a_i \xi_{t-i}$, where $\{a_i\}$ is a sequence of real numbers. The autoregressive AR(p), moving average MA(q), autoregressive moving average ARMA(p, q) and autoregressive integrated moving average ARIMA(p, d, q) models are all special cases of the linear process. We define a stationary ARMA(p, q) process to be a process satisfying the following equation:

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \theta_1 \xi_{t-1} + \cdots + \theta_q \xi_{t-q} + \xi_t, \tag{1.1}$$

where $t \in \mathbb{Z}$, $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$ are constants, and where $\phi_1, \dots, \phi_p$ are such that all the roots of the characteristic polynomial of the AR(p) part (that is, $\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$) are outside the unit disk. When $\theta_1, \dots, \theta_q$ are all zero, we say that the above process is a stationary AR(p) process. In this case,

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \xi_t. \tag{1.2}$$

In this paper we study the level-crossing probabilities for ARMA(p, q) processes using two techniques: (i) the martingale approach used by Novikov [7] and (ii) an integral-equation

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* Postal address: Department of Mathematics, University of Bristol, Bristol BS8 1TW, UK.
Email address: magkb@bris.ac.uk

** Postal address: Department of Information and Systems Management, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. Email address: imremus@ust.hk

approach. These processes are commonly used in econometrics to model various data. In making decisions, we may want to know how likely it is that the process will attain a certain high level before it drops back to or even below the present level, or we may want to know the expected time for the process to reach a certain level. The results of this paper answer the above questions for some of the above-mentioned linear processes by evaluating the following quantities: (i) the probability of crossing a level b before a level a and (ii) the expectation of the first-passage times $\mathcal{T}(b)$ and $\mathcal{T}(a, b)$ defined as

$$\begin{aligned} \mathcal{T}(b) &= \inf\{t : Y_t > b\}, \qquad b > x_0, x_{-1}, \dots, x_{-p+1}, \\ \mathcal{T}(a, b) &= \inf\{t : Y_t > b \text{ or } Y_t < a\}, \qquad b > x_0, x_{-1}, \dots, x_{-p+1} > a. \end{aligned}$$

The paper is organized as follows. We present background material in Section 2. Section 3 deals with the representation of mean first-passage times for ARMA(p, q) processes using the martingale approach of Novikov [7]. We extend his work from AR(1) to ARMA(p, q) processes. In Section 4, level-crossing probabilities and mean first-passage times for AR(p) processes are discussed with an example. Section 5 deals with ARMA(1, 1) processes with an example while Section 6 gives conclusions and discussions. Some of the detailed calculations are given in the appendices.

2. Background

In this section we discuss some background material. Section 2.1 deals with the state-space representations of AR(p) and ARMA(p, q) processes used in Sections 3–5. In Section 2.2, some of the theory of Fredholm integral equations of the second kind is described. These results on Fredholm-integral equations are used throughout Sections 3–5. Section 2.3 describes the collocation method, which is used in Section 5.

2.1. State-space representation and stationarity

For the stationary time series satisfying (1.2), we have the following state-space representation:

$$\begin{aligned} Y_t &= G\vec{X}_t, \qquad t \in \mathbb{Z}, \\ \vec{X}_{t+1} &= F\vec{X}_t + H\xi_t, \qquad t \in \mathbb{Z}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \vec{X}_t &= (Y_{t-p+1}, \dots, Y_{t-1}, Y_t)^\top, \\ H &= (0, \dots, 0, 1)^\top, \\ G &= (0, \dots, 0, 1), \\ F &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \phi_p & \phi_{p-1} & \cdots & \phi_2 & \phi_1 \end{pmatrix}. \end{aligned}$$

The state equation demonstrates the underlying Markov property of the AR(p) model that is crucial in our discussion. ARMA(p, q) processes, defined in (1.1), can also be given a state-space representation:

$$\begin{aligned} Y_t &= G\vec{Z}_t, \qquad t \in \mathbb{Z}, \\ \vec{Z}_{t+1} &= F_0\vec{Z}_t + \vec{\varepsilon}_t, \qquad t \in \mathbb{Z}, \end{aligned} \tag{2.2}$$

where

$\vec{Z}_t = (Y_{t-p+1}, \ldots, Y_{t-1}, Y_t, \xi_{t-q+1}, \ldots, \xi_t)^\top = \begin{pmatrix} \vec{X}_t \\ \vec{\nu}_t \end{pmatrix}$,

$G = (0, \ldots, 0, 1, 0, \ldots, 0)$ (a $(p+q)$-dimensional vector with 1 at the pth place),

$\vec{\varepsilon}_t = (0, \ldots, 0, 1, 0, \ldots, 0, 1)^\top \xi_{t+1}$ (a $(p+q)$-dimensional vector with ξ_{t+1} at the pth and $(p+q)$th places),

$F_0 = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$

with $F_{11} = F$, $F_{21} = \mathbf{0}_{q\times p}$,

$$F_{12} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \theta_q & \theta_{q-1} & \cdots & \theta_1 \end{pmatrix}, \qquad F_{22} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Thus, the Markov property also holds for the state equation of the ARMA(p, q) model.

2.2. Introduction to integral equations

In this section, we will only focus on results that will be useful in later discussions and the theorems stated here will not be proved. Background material in this and the following subsection is mainly taken from [1]. The equations that we are interested in are of Fredholm type of the second kind. The general form is

$$\lambda x(t) - \int_D K(t, s)x(s)\,\mathrm{d}s = y(t), \qquad t \in D, \lambda \neq 0,$$

where D is a closed and bounded set in $\mathbb{R}^m$ for some $m \geq 1$. The function $K(\cdot, \cdot)$ is called the kernel and is assumed to be absolutely integrable. The function $x(\cdot)$ is the unknown to be solved for. Since the probabilities and expectations we consider should be continuous functions of the initial states, we assume that $x(\cdot) \in \mathrm{C}(D)$. Next, define the integral operator $\mathcal{K}$ by

$$\mathcal{K}x(t) = \int_D K(t, s)x(s)\,\mathrm{d}s, \qquad t \in D, x \in \mathrm{C}(D). \tag{2.3}$$

Definition 2.1. Let $\mathcal{B}$ and $\mathcal{C}$ be normed vector spaces and let $\mathcal{K} : \mathcal{B} \to \mathcal{C}$ be linear. Then $\mathcal{K}$ is compact if the set

$$\{\mathcal{K}x \mid \|x\| \leq 1\}$$

has compact closure in $\mathcal{C}$.

Theorem 2.1. *The integral operator $\mathcal{K}$ defined in (2.3) is bounded and compact in* $\mathrm{C}(D)$ *equipped with the supremum norm* $\|\cdot\|_\infty$ *under the following conditions:*

(i) $K(s,t)$ *is Riemann integrable in* s *for all* $t \in D$;

(ii)

$$\lim_{h\to 0}\max_{t,\tau\in D}\max_{|t-\tau|\le h}\int_D |K(t,s)-K(\tau,s)|\,\mathrm{d}s = 0;$$

(iii)

$$\max_{t\in D}\int_D |K(t,s)|\,\mathrm{d}s < \infty.$$

Notice that the above conditions are fulfilled if $K(t,s)$ *is continuous in* s *and* t.

For compact operators, there is a central theorem [1].

Theorem 2.2. (Fredholm alternative.) *Let* $\mathcal{B}$ *be a Banach space and let* $\mathcal{K}:\mathcal{B}\to\mathcal{B}$ *be compact. Then the equation* $(\lambda-\mathcal{K})x = y$, $\lambda\neq 0$, *has a unique solution* $x\in\mathcal{B}$ *if and only if the homogeneous equation* $(\lambda-\mathcal{K})z = 0$ *has only the trivial solution* $z = 0$. *In such a case, the operator* $\lambda-\mathcal{K}:\mathcal{B}\to\mathcal{B}$ *has a bounded inverse* $(\lambda-\mathcal{K})^{-1}$.

Next, we state a version of the useful contraction mapping theorem [1].

Theorem 2.3. *Let* $\mathcal{B}$ *be a Banach space and let* $\mathcal{A}$ *be a bounded operator from* $\mathcal{B}$ *into* $\mathcal{B}$, *with* $\|\mathcal{A}\| < 1$. *Then* $I-\mathcal{A}:\mathcal{B}\to\mathcal{B}$ *is one-to-one and onto and* $(I-\mathcal{A})^{-1}$ *is a bounded linear operator, where* $I:\mathcal{B}\to\mathcal{B}$ *is the identity operator.*

2.3. Collocation method

The collocation method is a convenient method commonly used to solve integral equations. A general introduction can be found in [1] and here we give a description for our case. To evaluate equations like (5.3) in Section 5 numerically in the domain $\mathcal{B} = \mathrm{C}[\alpha,\beta]$, we introduce the Lagrange basis functions for piecewise linear interpolation:

$$l_i(t) = \begin{cases} 1-\dfrac{|t-t_i|}{h}, & t_{i-1}\le t\le t_{i+1},\ i = 0,1,\ldots,n,\\ 0 & \text{otherwise.}\end{cases}$$

Here $t_0 = \alpha$, $t_n = \beta$, $t_i = t_0 + ih$ for $i = 1,\ldots,n-1$, $h = (\beta-\alpha)/n$ and we define the following projection operator on $\mathrm{C}[\alpha,\beta]$: for $f\in\mathrm{C}[\alpha,\beta]$,

$$\mathcal{P}_n f(x) = f_n(x) = \sum_{i=0}^{n} f(x_i)l_i(x), \qquad x_i = \alpha + ih,\ i = 0,1,\ldots,n,\ x\in[\alpha,\beta]. \quad (2.4)$$

It was proved in [1, p. 59] that $\mathcal{P}_n$ is a bounded linear operator and $\mathcal{P}_n f\to f$ as $n\to\infty$ for all $f\in\mathrm{C}[\alpha,\beta]$. The projection operator $\mathcal{P}_n$ maps any $f\in\mathcal{B}$ to a function f_n in $\mathcal{B}_n$, the $(n+1)$-dimensional subspace of $\mathcal{B}$ that contains functions of the form $\sum_{i=0}^{n} f(x_i)l_i(x)$. Thus, if we represent equations like (5.3) in the form of an operator equation (with P in (5.3) replaced by f to avoid confusion and P_1 as defined in Section 4.2), we have

$$(1-\mathcal{K}_1)f = P_1, \qquad f\in\mathcal{B}. \quad (2.5)$$

We intend to approximate the solution of (2.5) by the solution f_n of the following equation:

$$(1 - \mathcal{P}_n \mathcal{K}_1) f_n = P_1, \qquad f_n \in \mathcal{B}_n. \tag{2.6}$$

We state the following theorems for general Banach spaces from [1, pp. 55, 57] which ensure the convergence under certain conditions.

Theorem 2.4. *Let $\mathcal{B}$ be a Banach space and assume that $\mathcal{K} : \mathcal{B} \to \mathcal{B}$ is bounded and that $1 - \mathcal{K} : \mathcal{B} \to \mathcal{B}$ is one-to-one and onto. Furthermore, assume that*

$$\|\mathcal{K} - \mathcal{P}_n \mathcal{K}\| \to 0 \quad as\ n \to \infty,$$

where P_n is as defined in (2.4). Then the solution f_n of (2.6) converges uniformly to the solution f of (2.5).

Theorem 2.5. *Let $\mathcal{B}$ be some Banach space and let $\mathcal{P}_n$ be a family of bounded projections on $\mathcal{B}$ such that*

$$\mathcal{P}_n f \to f \quad as\ n \to \infty\ for\ f \in \mathcal{B}.$$

If $\mathcal{K} : \mathcal{B} \to \mathcal{B}$ is compact, then

$$\|\mathcal{K} - \mathcal{P}_n \mathcal{K}\| \to 0 \quad as\ n \to \infty.$$

3. ARMA(p, q) processes: a martingale approach

The approach we use here comes from Novikov's work [7] on AR(1) processes which we modify for ARMA(p, q) processes. It is clearer if we start with AR(p) processes. We have to impose a mild condition on the stationary AR(p) processes under consideration. For an AR(p) process satisfying the state-space representation (2.1), we assume that $\phi_p > 0$. Under this constraint, the coefficient matrix F is nonsingular. Since the characteristic polynomial of F is $\Phi_F(x) = x^p - \phi_1 x^{p-1} - \phi_2 x^{p-2} - \cdots - \phi_p$, $\Phi_F(0) = -\phi_p < 0$ and $\Phi_F(x) \to \infty$ as $x \to \infty$. This means that the characteristic polynomial of F has a positive root, i.e. F has a positive eigenvalue, say λ. Assume that the corresponding row eigenvector is $\vec{c}$ and notice that $0 < \lambda < 1$ by stationarity. We can then rewrite (2.1) as

$$\vec{c}\vec{X}_{t+1} = \vec{c}F\vec{X}_t + \vec{c}H\xi_{t+1}.$$

By choosing $\vec{c}$ such that $\vec{c}H = 1$ or such that the last element of $\vec{c}$ is 1, we have

$$\vec{c}\vec{X}_{t+1} = \lambda\vec{c}\vec{X}_t + \xi_{t+1}. \tag{3.1}$$

Thus, $W_t := \vec{c}\vec{X}_t$ satisfies an AR(1) equation,

$$W_{t+1} = \lambda W_t + \xi_{t+1},$$

with $0 < \lambda < 1$.

We now develop a similar AR(1) equation for ARMA(p, q) processes. Besides stationarity and invertibility of Y_t, we further assume that the characteristic polynomial for the autoregressive part and the characteristic equation for the moving average part have no common roots. This is a quite natural assumption in order to have a unique causal representation of an ARMA process Y_t. With the same notation as in Section 2.1, we observe that

$$\det(\lambda I_{p+q} - F_0) = \det(\lambda I_p - F)\lambda^q.$$

Therefore, the eigenvalues of F_0 are the eigenvalues of F and q zeros. Hence, for $\phi_p > 0$, we get a positive eigenvalue; this eigenvalue is λ, as above. We take the same eigenvector $\vec{c}$ and observe that $\vec{c}F_{11} + \vec{d}F_{21} = \lambda\vec{c}$ for any q-dimensional vector $\vec{d}$ as $F_{21} = \mathbf{0}$ and $F_{11} = F$. We find $\vec{d}$ by solving the following (equivalent) equations:

$$\vec{c}F_{12} + \vec{d}F_{22} = \lambda\vec{d}, \qquad c_p(\theta_q, \ldots, \theta_1) + \vec{d}F_{22} = \lambda\vec{d}, \qquad \vec{d}(\lambda I - F_{22}) = c_p(\theta_q, \ldots, \theta_1).$$

The solution is unique since, whenever $0 < \lambda < 1$, $\lambda I - F_{22}$ is nonsingular (in fact, $\det(\lambda I - F_{22}) = \lambda^q$). Also, it can be noted from the above equations that

$$\begin{aligned} c_p\theta_q &= \lambda d_1, \\ c_p\theta_{q-i} + d_i &= \lambda d_{i+1} \quad \text{for } i = 1, \ldots, q-1. \end{aligned} \tag{3.2}$$

Hence, $c_p\theta_1 + d_{q-1} = \lambda d_q$. Notice that, if $d_q = -c_p$, then, from (3.1),

$$\begin{aligned} c_p\theta_q &= \lambda d_1 = \lambda(\lambda d_2 - c_p\theta_{q-1}) \\ &= \lambda^2 d_2 - c_p\lambda\theta_{q-1} \\ &= \cdots \\ &= \lambda^q d_q - c_p(\lambda\theta_{q-1} + \lambda^2\theta_{q-2} + \cdots + \lambda^{q-1}\theta_1) \\ &= -c_p(\lambda^q + \lambda\theta_{q-1} + \lambda^2\theta_{q-2} + \cdots + \lambda^{q-1}\theta_1). \end{aligned}$$

This implies that

$$c_p(\theta_q + \lambda\theta_{q-1} + \lambda^2\theta_{q-2} + \cdots + \lambda^{q-1}\theta_1 + \lambda^q) = 0$$

and, since $c_p \neq 0$, this means that λ is a root of the characteristic polynomial of the moving average part as well. But, as $\vec{c}F_{11} = \lambda\vec{c}$, λ is an eigenvalue of F_{11} and λ is also a root of the characteristic polynomial of the autoregressive part. This is not possible under the assumption that the autoregressive and moving average parts have no common root. Hence, $d_q \neq -c_p$. Now take

$$\vec{c}_1 = \frac{1}{c_p + d_q}(\vec{c}, \vec{d}\,) = (\vec{c}_{10}, \vec{d}_{10}),$$

with $\vec{d}$ as above and where $\vec{c}_{10} = (1/(c_p + d_q))\vec{c}$ and $\vec{d}_{10} = (1/(c_p + d_q))\vec{d}$. Then $\vec{c}_1$ is the eigenvector corresponding to the eigenvalue λ. Hence, if we define $V_t = \vec{c}_1\vec{Z}_t$, then

$$V_{t+1} = \vec{c}_1\vec{Z}_{t+1} = \vec{c}_1F_0\vec{Z}_t + \vec{c}_1\vec{\varepsilon}_{t+1} = \lambda\vec{c}_1\vec{Z}_t + \frac{c_p + d_q}{c_p + d_q}\xi_{t+1} = \lambda V_t + \xi_{t+1}. \tag{3.3}$$

Next we assume that ξ_1 has finite expectation and that

$$\mathrm{E}\exp(u\xi_1) = \exp\{\psi(u)\} < \infty \quad \text{for all } u \geq 0.$$

The function $\psi(u) := \ln \mathrm{E}(u\xi_1)$ is convex and is bounded by a linear function for small u. Define

$$\varphi_\lambda(u) = \sum_{k=0}^{\infty} \psi(\lambda^k u), \qquad u \geq 0. \tag{3.4}$$

Then, $\varphi_\lambda(\cdot)$ is bounded on any finite interval and

$$\varphi_\lambda(u) = \varphi_\lambda(\lambda u) + \psi(u).$$

In the case of a normal distribution, say $\xi_1 \sim \mathrm{N}(\mu, \sigma^2)$,

$$\psi(u) = \mu u + \tfrac{1}{2}\sigma^2 u^2,$$
$$\varphi_\lambda(u) = \frac{\mu}{1-\lambda}u + \frac{1}{2}\frac{\sigma^2}{1-\lambda^2}u^2, \qquad |u| < \infty.$$

Now suppose that $\vec{Z}_0 = (y_{-p+1}, \ldots, y_0, \mathbf{0}_{1\times q})^\top$ is the initial vector with $y_0, y_{-1}, \ldots, y_{-p+1} \le b$. Let $\mathcal{F}_t = \sigma(\vec{X}_s)_{s\le t}$. Following Novikov's construction [7], we define a martingale, with respect to $\{\mathcal{F}_t\}$,

$$\begin{aligned} G_t &:= \int_0^\infty u^{-1}[\exp(u\vec{c}_1\vec{Z}_t) - \exp(u\vec{c}_1\vec{Z}_0)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u - t\log\left(\frac{1}{\lambda}\right) \\ &= \int_0^\infty u^{-1}[\exp(uV_t) - \exp(uV_0)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u - t\log\left(\frac{1}{\lambda}\right), \end{aligned}$$

whenever

$$\int_1^\infty u^{-1}\exp(u\vec{c}_{10}\vec{X}_0 - \varphi_\lambda(u))\,\mathrm{d}u < \infty. \tag{3.5}$$

A proof of the martingale property of $G(t)$ is given in Appendix B.

Theorem 3.1. *Assume that*

$$\int_1^\infty u^{-1}\exp(uK - \varphi_\lambda(u))\,\mathrm{d}u < \infty \tag{3.6}$$

for any positive constant K. Then $\mathrm{E}\,\mathcal{T}(a,b) < \infty$ and

$$\mathrm{E}\,\mathcal{T}(a,b) = \frac{1}{\log(1/\lambda)}\,\mathrm{E}\int_0^\infty u^{-1}[\exp(u\vec{c}_1\vec{Z}_{\mathcal{T}(a,b)}) - \exp(u\vec{c}_1\vec{Z}_0)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u. \tag{3.7}$$

This theorem is proved in Appendix B.

The above result is particularly useful when the disturbance follows a normal distribution. In that case, ξ_t satisfies the basic assumptions and the condition (3.6). Thus, for stationary and invertible Gaussian ARMA(p,q) processes with no roots common to the autoregressive and moving average parts, $\mathrm{E}\,\mathcal{T}(a,b)$ is finite for any a, b such that $b > y(-p+1), \ldots, y(0) > a$.

Theorem 3.2. *Assume that the autoregressive coefficients ϕ_j are nonnegative for $j = 1, \ldots, p-1$ and $\phi_p > 0$ and that the moving average coefficients θ_i are nonnegative for $i = 1, \ldots, q$. Furthermore, assume that*

$$\int_1^\infty u^{-1}\exp(uK - \varphi_\lambda(u))\,\mathrm{d}u < \infty \tag{3.8}$$

for any positive constant K. Then $\mathrm{E}\,\mathcal{T}(b) < \infty$ and

$$\mathrm{E}\,\mathcal{T}(b) = \frac{1}{\log(1/\lambda)}\,\mathrm{E}\int_0^\infty u^{-1}[\exp(u\vec{c}_1\vec{Z}_{\mathcal{T}(b)}) - \exp(u\vec{c}_1\vec{Z}_0)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u. \tag{3.9}$$

This result is proved in Appendix B.

Stationary AR(p) processes are important special cases. We can achieve similar results with a slightly milder assumption.

Theorem 3.3. *Assume that*

$$\int_1^\infty u^{-1}\exp(u\,|||\vec{c}\vec{b}_a|||-\varphi_\lambda(u))\,\mathrm{d}u<\infty,$$

where $\vec{b}_a=(b\vee(-a),\dots,b\vee(-a))^\top$ *and* $|||\vec{x}\vec{y}|||=|x_1y_1|+\cdots+|x_py_p|$ *for* $\vec{x},\ \vec{y}\in\mathbb{R}^p$. *Then* $\mathrm{E}\,\mathcal{T}(a,b)<\infty$ *and*

$$\mathrm{E}\,\mathcal{T}(a,b)=\frac{1}{\log(1/\lambda)}\,\mathrm{E}\int_0^\infty u^{-1}[\exp(u\vec{c}\vec{X}_{\mathcal{T}(a,b)})-\exp(u\vec{c}\vec{X}_0)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u.$$

Thus, for stationary Gaussian AR(p) processes with positive ϕ_p, $\mathrm{E}\,\mathcal{T}(a,b)$ is finite for any a,b with $b>y(-p+1),\dots,y(0)>a$.

Theorem 3.4. *Assume that the autoregressive coefficients* ϕ_j *are nonnegative for* $j=1,\dots,p-1$ *and* $\phi_p>0$. *Furthermore, assume that*

$$\int_1^\infty u^{-1}\exp(u\vec{c}_0\vec{b}-\varphi_\lambda(u))\,\mathrm{d}u<\infty, \tag{3.10}$$

where $\vec{b}=(b,\dots,b)^\top$. *Then* $\mathrm{E}\,\mathcal{T}(b)<\infty$ *and*

$$\mathrm{E}\,\mathcal{T}(b)=\frac{1}{\log(1/\lambda)}\,\mathrm{E}\int_0^\infty u^{-1}[\exp(u\vec{c}_0\vec{X}_{\mathcal{T}(b)})-\exp(u\vec{c}_0\vec{X}_0)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u.$$

4. AR(p) processes: an integral-equation approach

The main objective of this section is to derive integral equations for AR(p) processes that lead to the evaluation of (i) the probability of crossing a given level b before another given level a and (ii) the mean first-passage time to attain a level b. In our formulation, we depend heavily on the Markov nature of the state-space representation of the time series. The form of the state vectors of AR(p) processes means that the integral equation is of Fredholm type of the second kind and can be handled through developed numerical schemes.

4.1. Time-homogeneous Markov processes

We define a discrete-time real-valued Markov process $\{X_n\}$ on a probability space $\{\Omega,\mathcal{F},\mathcal{P}\}$ with stationary continuous transition density $f(x,y)$ continuous in both x and y. The term $f(x,y)$ denotes the transition density of reaching y at the next step given that the present state is x. Suppose that $X_0=x_0$ and that we are given levels $b>a$, where $x_0\in[a,b]$. Define

$$\begin{aligned}P_n(x_0)&:=P_n^{a,b}(x_0)\\&:=\mathrm{P}(a\le X_1\le b,\dots,a\le X_{n-1}\le b,X_n>b\mid X_0=x_0).\end{aligned}$$

By looking at the first step and using the Markov property, we have, for $n\ge 2$,

$$P_n(x_0)=\int_a^b P_{n-1}(y)f(x_0,y)\,\mathrm{d}y. \tag{4.1}$$

Summing the terms $P_n(x_0)$ in (4.1) for $n\ge 1$ gives

$$P(x_0)=\int_a^b P(y)f(x_0,y)\,\mathrm{d}y+P_1(x_0), \tag{4.2}$$

where $P(x):=\sum_{n=1}^\infty P_n(x)$. The equation (4.2) is a Fredholm integral equation of the second kind. The first concern is the existence and uniqueness of a solution to (4.2).

Theorem 4.1. *The equation (4.2) exhibits a unique solution in* $\mathrm{C}[a, b]$ *given that*

$$\int_a^b f(x, y)\,\mathrm{d}y < 1 \quad \textit{for all } x \in [a, b]. \tag{4.3}$$

Proof. By virtue of the contraction mapping theorem (Theorem 2.3), it suffices to show that the integral operator $\mathcal{A}(P)(x) := \int_a^b P(y) f(x, y)\,\mathrm{d}y$ has operator norm less than 1. Here we use the supremum norm in $\mathrm{C}[a, b]$. The condition is fulfilled when (4.3) holds.

Thus, solving (4.2) leads to the probability of crossing the level b before the level a. To formulate the result for mean first-passage time to a given level, we first have the following well-known result which we now prove in a new way.

Theorem 4.2. *Under the condition (4.3),* $\mathrm{P}(\mathcal{T}_{a,b}(x_0) < \infty) = 1$. *Here* $\mathcal{T}_{a,b}(x_0)$ *denotes the exit time of the interval* $[a, b]$ *provided that the Markov process defined in this section starts at* $x_0 \in [a, b]$.

Proof. We can formulate an integral equation similar to (4.2) by replacing P by P' and P_1 by P_1', where $P'(x_0) = \sum_{n=1}^{\infty} P_n'(x_0)$ and $P_n'(x_0) = \mathrm{P}(a \le y_1 \le b, \dots, a \le y_{n-1} \le b$ and $y_n > b$ or $y_n < a \mid X_0 = x_0)$. Thus,

$$P'(x_0) = \int_a^b P'(y) f(x_0, y)\,\mathrm{d}y + P_1'(x_0). \tag{4.4}$$

We see that $P'(x_0) = \mathrm{P}(\mathcal{T}_{a,b}(x_0) < \infty)$. It is clear that $P'(\cdot) \equiv 1$ satisfies (4.4). So, by the uniqueness of a continuous solution, the result follows.

Next define $M(x_0, z) := \sum_{n=1}^{\infty} P_n'(x_0) z^n$ where $0 < z \le 1$. Analogous to the formulation of (4.2), we have

$$M(x_0, z) = z \int_a^b M(y, z) f(x_0, y)\,\mathrm{d}y + z P_1'(x_0). \tag{4.5}$$

Differentiating (4.5) with respect to z and evaluating at $z = 1$ give

$$\mathrm{E}(\mathcal{T}_{a,b}(x_0)) = \int_a^b \mathrm{E}(\mathcal{T}_{a,b}(u)) f(x_0, u)\,\mathrm{d}u + \int_a^b M(u, 1) f(x_0, u)\,\mathrm{d}u + P_1'(x_0). \tag{4.6}$$

Since $M(u, 1) = \mathrm{P}(\mathcal{T}_{a,b}(u) < \infty) = 1$ for all $u \in [a, b]$, by Theorem 4.2, the last two terms on the right-hand side of (4.6) sum to 1. Thus,

$$\mathrm{E}(\mathcal{T}_{a,b}(x_0)) = \int_a^b \mathrm{E}(\mathcal{T}_{a,b}(u)) f(x_0, u)\,\mathrm{d}u + 1. \tag{4.7}$$

Although, in general, we can only solve the integral equations numerically, there are some special cases where the integral equations can be solved analytically. One of these cases was addressed by Greenberg [6] who used a Markov-chain approximation to evaluate a kind of mean level passage time. In his paper, the innovations follow a hyperexponential distribution which is useful as an approximation to other positive distributions (see [2]).

4.2. AR(p) processes: level-crossing probability

On a probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$, define an AR($p$) process as in (1.2):

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \xi_t,$$

where ξ_t are i.i.d. random variables.

Hereafter, we will assume that ξ_t has $\mathrm{N}(0, \sigma^2)$ distribution, but the results also hold for most disturbances with a continuous density function satisfying certain conditions. Now, (1.2) has a state-space representation

$$\begin{aligned} Y_t &= G\vec{X}_t, \qquad t \in \mathbb{Z}, \\ \vec{X}_{t+1} &= F\vec{X}_t + H\xi_t, \qquad t \in \mathbb{Z}, \end{aligned}$$

where $\vec{X}_t = (Y_{t-p+1}, \ldots, Y_{t-1} Y_t)^\top$ and F, G, H are as defined in Section 2.1.

Notice that the state vector $\vec{X}_t$ consists of exactly the past p states of the original process. This is crucial for our integral equation to remain of Fredholm type. Analogous to the previous section, we define, for $n \geq 1$,

$$\begin{aligned} P_n(\vec{x}_0) &:= P_n^{a,b}(\vec{x}_0) \\ &:= \mathrm{P}(a \leq y_1 \leq b, \ldots, a \leq y_{n-1} \leq b, y_n > b \mid \vec{X}_0 = \vec{x}_0), \end{aligned}$$

where $\vec{x}_0 = (y_{-p+1}, \ldots, y_{-1}, y_0)^\top$ and $a \leq y_i \leq b$ for $i = 0, -1, \ldots, -p+1$. So,

$$P_{n+1}(\vec{x}_0) = \frac{1}{\sqrt{2\pi}\sigma} \int_a^b P_n(\tilde{x}_1) \exp\biggl(-\frac{(u - \vec{\phi}\vec{x}_0)^2}{2\sigma^2}\biggr) \mathrm{d}u, \tag{4.8}$$

where $\tilde{x}_1 = (y_{-p+2}, \ldots, y_0, u)^\top$ and $\vec{\phi} = (\phi_p, \ldots, \phi_1)^\top$.

Define $P(\vec{x}_0, z) \equiv \sum_{n=1}^\infty P_n(\vec{x}_0) z^n$ for $z \in (0, 1]$. Then, multiplying (4.8) by z^n and summing over all $n \geq 1$, we get

$$P(\vec{x}_0, z) = \frac{z}{\sqrt{2\pi}\sigma} \int_a^b P(\tilde{x}_1, z) \exp\biggl(-\frac{(u - \vec{\phi}\vec{x}_0)^2}{2\sigma^2}\biggr) \mathrm{d}u + zP_1(\vec{x}_0). \tag{4.9}$$

The integral equation of the probability generating function will be useful in determining the mean first-passage time. For calculating the level-crossing probability, set $z = 1$ and rename $P(\vec{x}_0, 1)$ as $P(\vec{x}_0)$. Then iterate (4.9) p times to get an integral equation of order p:

$$\begin{aligned} P(\vec{x}_0) = \frac{1}{(2\pi\sigma^2)^{p/2}} \int_a^b \cdots \int_a^b P(u_1, u_2, \ldots, u_p) \\ \times \exp\biggl(-\frac{\sum_{i=1}^p (e_p \vec{x}_i - \vec{\phi}\vec{x}_{i-1})^2}{2\sigma^2}\biggr) \mathrm{d}u_p \ldots \mathrm{d}u_1 + \sum_{j=1}^p P_j(\vec{x}_0), \end{aligned} \tag{4.10}$$

where $\vec{x}_0 = (y_{-p+1}, \ldots, y_0)^\top$, $\vec{x}_1 = (y_{-p+2}, \ldots, y_0, u_1)^\top$, $\vec{x}_2 = (y_{-p+3}, \ldots, y_0, u_1, u_2)^\top$, etc., and $e_p = (0, \ldots, 0, 1)^\top$.

The equation (4.10) is a standard Fredholm integral equation of the second kind. The existence and uniqueness of a continuous solution is guaranteed by the contraction mapping theorem because the kernel is just a p-dimensional Gaussian kernel and the integral is taken over a compact set.

4.3. AR(p) processes: mean first-passage time

In this subsection, we want to calculate $\mathrm{E}(\mathcal{T}_b(\vec{x}_0))$, that is, the mean first-passage time over a given level b of an AR(p) process starting at an initial state vector $\vec{x}_0$. We proceed by adding a lower boundary a first. Let $\mathcal{T}_{a,b}(\vec{x}_0)$ be the first-passage time of an AR(p) process over the upper level b or the lower level a given the initial state vector $\vec{x}_0$ in $[a, b]^p$, let

$$P'(\vec{x}, z) = \sum_{i=1}^{\infty} P_i'(\vec{x}) z^i$$

and

$$P_i'(\vec{x}) = \mathrm{P}(a \le y_1 \le b, \ldots, a \le y_{i-1} \le b, y_i > b \text{ or } y_i < a \mid \vec{X}_0 = \vec{x}).$$

In line with the argument of Theorem 4.2, differentiating (4.9) with respect to z and evaluating at $z = 1$ give

$$\begin{aligned}\mathrm{E}(\mathcal{T}_{a,b}(\vec{x}_0)) &= \frac{1}{\sqrt{2\pi}\sigma} \int_a^b \mathrm{E}(\mathcal{T}_{a,b}(\vec{x}_1)) \exp\biggl(-\frac{(u - \vec{\phi}\vec{x}_0)^2}{2\sigma^2}\biggr) \mathrm{d}u \\ &\quad + \frac{1}{\sqrt{2\pi}\sigma} \int_a^b P'(\vec{x}_1, 1) \exp\biggl(-\frac{(u - \vec{\phi}\vec{x}_0)^2}{2\sigma^2}\biggr) \mathrm{d}u + P_1'(\vec{x}_0). \qquad (4.11)\end{aligned}$$

From a result analogous to Theorem 4.2, we find that the last two terms on the right-hand side of (4.11) sum to 1. Thus,

$$\mathrm{E}(\mathcal{T}_{a,b}(\vec{x}_0)) = \frac{1}{\sqrt{2\pi}\sigma} \int_a^b \mathrm{E}(\mathcal{T}_{a,b}(\vec{x}_1)) \exp\biggl(-\frac{(u - \vec{\phi}\vec{x}_0)^2}{2\sigma^2}\biggr) \mathrm{d}u + 1. \qquad (4.12)$$

Through iterating (4.11) p times as in the formulation of (4.10), we get a Fredholm integral equation of order p and we can calculate $\mathrm{E}(\mathcal{T}_{a,b}(\vec{x}_0))$ for any given a, b and initial state vector $\vec{x}_0$ in $[a, b]^p$.

Since $\mathrm{E}(\mathcal{T}_{a,b}(\vec{x}_0))$ converges monotonically to $\mathrm{E}(\mathcal{T}_b(\vec{x}_0))$ as $a \to -\infty$, we can get an approximation of $\mathrm{E}(\mathcal{T}_b(\vec{x}_0))$ by evaluating $\mathrm{E}(\mathcal{T}_{a,b}(\vec{x}_0))$ as $a \to -\infty$ instead.

Notice that, under the sufficient condition (3.10) of Section 3, $\mathrm{E}(\mathcal{T}_b(\vec{x}_0)) < \infty$. Clearly, the Gaussian kernel satisfies this condition.

4.4. Numerical example: an AR(2) process

As shown in previous subsections, the determination of $P(\vec{x}_0)$ and $\mathrm{E}(\mathcal{T}_{a,b}(\vec{x}_0))$ through solving (4.11) and (4.12) in general can only be treated numerically. In this subsection, we give a numerical scheme for an AR(2) Gaussian process and some numerical examples. Consider the integral equation

$$f(x, y) = \int_a^b \int_a^b K(x, y, \xi, \eta) f(\xi, \eta) \, \mathrm{d}\xi \, \mathrm{d}\eta + \Psi(x, y), \quad (x, y) \in [a, b] \times [a, b], \quad (4.13)$$

where the function $K : [a, b]^4 \to \mathbb{R}$ is continuous and integrable with respect to all variables and the function $\Psi : [a, b]^2 \to \mathbb{R}$ is continuous in both variables and is not identically 0. We adopt the Nyström method and a particular quadrature rule as discussed in Appendix C to handle (4.13) (see [1] for details).

We study an example of an AR(2) process:

$$y_t = 0.2 y_{t-1} + 0.3 y_{t-2} + \xi_t, \qquad \xi_t \sim \mathrm{N}(0, 1).$$

TABLE 1: Probability of crossing level b before -1 for y_t in (4.14).

n	$b = 1$	$b = 2$	$b = 3$	$b = 4$
3	0.5908	0.2426	0.0343	0.0016
4	0.5911	0.2426	0.0343	0.0016
5	0.5913	0.2426	0.0343	0.0016
6	0.5914	0.2426	0.0343	0.0016
Simulation	0.5912	0.2433	0.0352	0.0015

TABLE 2: Mean first-passage time of level a or level $b = 1$ for y_t in (4.15).

n	$a = -1$	$a = -2$	$a = -3$	$a = -4$	$a = -5$	$a = -6$	$a = -7$	$a = -8$
4	3.0404	5.2297	6.5769	6.8044	6.8281	6.8567	6.9731	7.3217
5	3.0414	5.2304	6.5750	6.7974	6.8141	6.8228	6.8358	6.8690
6	3.0420	5.2309	6.5743	6.7949	6.8086	6.8132	6.8197	6.8284
7	3.0424	5.2312	6.5741	6.7938	6.8063	6.8089	6.8125	6.8176
8	3.0426	5.2315	6.5740	6.7933	6.8052	6.8068	6.8094	6.8120
9	3.0427	5.2316	6.5740	6.7931	6.8046	6.8059	6.8070	6.8089
10	3.0427	5.2317	6.5740	6.7929	6.8043	6.8051	6.8059	6.8071
11	3.0427	5.2317	6.5740	6.7928	6.8041	6.8048	6.8053	6.8061
12	3.0427	5.2317	6.5740	6.7928	6.8040	6.8044	6.8052	6.8054

We further assume that $y_0 = 0.5$ and $y_{-1} = 0.5$. We study the following cases:

(a) Fix $a = -1$. We calculate the probabilities of crossing a level b before a level a by solving the following equation:

$$P(\vec{x}_0) = \frac{1}{2\pi\sigma^2}\int_a^b\int_a^b P(u_1, u_2)\exp\biggl(-\frac{\sum_{i=1}^{2}(e_2\vec{x}_i - \vec{\phi}\vec{x}_{i-1})^2}{2\sigma^2}\biggr)\,\mathrm{d}u_2\,\mathrm{d}u_1 + \sum_{j=1}^{2} P_j(\vec{x}_0). \tag{4.14}$$

We can use the numerical methods discussed here to obtain the results in Table 1.

(b) Next we derive the mean first-passage time for y_t for two specific levels. We solve the following equation, letting $a \to -\infty$:

$$\mathrm{E}\,\mathcal{T}_{a,b}(\vec{x}_0) = \frac{1}{2\pi\sigma^2}\int_a^b\int_a^b \mathrm{E}\,\mathcal{T}_{a,b}(u_1, u_2)\exp\biggl(-\frac{\sum_{i=1}^{2}(e_2\vec{x}_i - \vec{\phi}\vec{x}_{i-1})^2}{2\sigma^2}\biggr)\,\mathrm{d}u_2\,\mathrm{d}u_1 + 1 + \frac{1}{\sqrt{2\pi}\,\sigma}\int_a^b \exp\biggl(-\frac{(u_1 - \vec{\phi}\vec{x}_0)^2}{2\sigma^2}\biggr)\,\mathrm{d}u_1. \tag{4.15}$$

The results for $b = 1$ are given in Table 2.

The mean first-passage time in this case is about 6.805 while the simulation result is around 6.845.

5. ARMA(1, 1) models

In previous sections, we mainly discussed the derivations for AR(p) processes where we naturally used their Markov properties in the state-space representations and formed standard Fredholm integral equations that can be solved. In this section, we will follow the same idea and use the Markov nature of the state-space representations of the ARMA model to form similar integral equations. However, the integral equations formed will not be of standard format and will need further considerations in the numerical procedures. Moreover, we restrict the discussions to the Gaussian ARMA(1, 1) case to simplify the calculations, but the main idea can be applied to processes of other types or with higher dimensions.

We give the formulations of the integral equations in Subsection 5.1 and discuss the solvability in Subsection 5.2. We then use the collocation method introduced in Subsection 2.3 to obtain numerical results for specific examples.

5.1. Formulation of integral equations for level-crossing probabilities

The process $\{y_t,\ t \in \mathbb{Z}\}$ is said to be an ARMA$(1,1)$ process if $\{y_t\}$ is stationary and if, for all t,

$$y_t = \phi y_{t-1} + \theta z_{t-1} + z_t,$$

where $\{z_t\}$ are i.i.d. random variables. Here we assume that $\{z_t\} \sim \mathrm{N}(0, \sigma^2)$. This process has the following state-space representation:

$$y_t = X_t + z_t, \qquad t \in \mathbb{Z}, \tag{5.1}$$

$$X_{t+1} = \phi X_t + \psi z_t, \qquad t \in \mathbb{Z}. \tag{5.2}$$

Here $\psi = \phi + \theta$ and note that $\psi = 0$ implies that $y_t = z_t$, which is an i.i.d. case that we shall not consider; we assume that $\psi > 0$. We use the notation from Section 4.2 where

$$P_n(x_1) := P_n^{a,b}(x_1) := \mathrm{P}(a \le y_1 \le b, \dots, a \le y_{n-1} \le b, y_n > b \mid X_1 = x_1),$$

$$P(x_1) := \sum_{n=1}^{\infty} P_n(x_1).$$

Here the level-crossing probability is a function of the initial state variable x_1 in (5.1) and (5.2) because that variable captures the necessary initial information for the evolution of $y_1, y_2, \dots$. By considering the first step, using (5.1), the Markov property of (5.2) and a change of variables, we get, for $x_1 \in \mathbb{R}$,

$$\begin{aligned} P_n(x_1) &= \mathrm{P}(a \le y_1 \le b, \dots, a \le y_{n-1} \le b, y_n > b \mid X_1 = x_1) \\ &= \frac{1}{\sqrt{2\pi}\sigma\psi} \int_{\psi a + (\phi-\psi)x_1}^{\psi b + (\phi-\psi)x_1} \exp\left(-\frac{(x_2 - \phi x_1)^2}{2\sigma^2\psi^2}\right) \\ &\qquad\qquad \times \mathrm{P}(a \le y_2 \le b, \dots, a \le y_{n-1} \le b, y_n > b \mid X_2 = x_2)\,\mathrm{d}x_2 \\ &= \frac{1}{\sqrt{2\pi}\sigma\psi} \int_{\psi a + (\phi-\psi)x_1}^{\psi b + (\phi-\psi)x_1} P_{n-1}(x_2) \exp\left(-\frac{(x_2 - \phi x_1)^2}{2\sigma^2\psi^2}\right) \mathrm{d}x_2. \end{aligned}$$

Summing for all $n \ge 1$, we have

$$P(x_1) = \frac{1}{\sqrt{2\pi}\sigma\psi} \int_{\psi a + (\phi-\psi)x_1}^{\psi b + (\phi-\psi)x_1} P(x_2) \exp\left(-\frac{(x_2 - \phi x_1)^2}{2\sigma^2\psi^2}\right) \mathrm{d}x_2 + P_1(x_1).$$

TABLE 3: Probabilities of crossing a level b before 0.

n	$b = 1$	$b = 2$	$b = 3$	$b = 4$
2	0.6081	0.3468	0.3099	0.2310
4	0.6111	0.3722	0.2251	0.1721
8	0.6130	0.3361	0.1209	0.0407
16	0.6136	0.3180	0.1048	0.0229
32	0.6135	0.3156	0.0997	0.0187
64	0.6134	0.3150	0.0980	0.0177
128	0.6134	0.3151	0.0977	0.0174
256	0.6134	0.3151	0.0976	0.0174
Simulation	0.6120	0.3148	0.1002	0.0177

When $\psi < 0$, we just interchange the limits of the integral and everything else remains unchanged. Next, letting $y = x_2 - (\phi - \psi)x_1$, we get the following integral equation with constant limits that we are going to solve in later subsections:

$$P(x_1) = \frac{1}{\sqrt{2\pi}\sigma\psi}\int_{\psi a}^{\psi b} P(y + (\phi - \psi)x_1)\exp\biggl(-\frac{(y - \psi x_1)^2}{2\sigma^2\psi^2}\biggr)\mathrm{d}y + P_1(x_1). \tag{5.3}$$

5.2. Existence and uniqueness of a solution

Analogous to the AR(p) case, we prove the uniqueness of the solution for (5.3) in $\mathrm{C}(D)$ where D is compact. Define the operator $\mathcal{K}_1$, by

$$\mathcal{K}_1(f)(x) = \frac{1}{\sqrt{2\pi}\sigma\psi}\int_{\psi a}^{\psi b} f(y + (\phi - \psi)x)\exp\biggl(-\frac{(y - \psi x)^2}{2\sigma^2\psi^2}\biggr)\,\mathrm{d}y. \tag{5.4}$$

Theorem 5.1. *The operator $\mathcal{K}_1$ defined in (5.4) is compact in $\mathrm{C}[\alpha, \beta]$ for some $\beta > \alpha$.*

The proof is given in Appendix B.

5.3. Numerical examples

In this subsection, we use the collocation method described in Subsection 2.3 to deal with two ARMA(1, 1) processes which correspond to the cases where θ is positive and where θ is negative. The range $[\alpha, \beta]$ should be chosen according to the criteria established in Appendix D. In these cases, all conditions in the theorems of Subsection 2.3 are satisfied and so we can use (2.6) to approximate (2.5). In practice, we solve the following system of equations for $i = 0, 1, \ldots, n$ and $x_i = \alpha + ih$:

$$P_n(x_i) = \frac{1}{\sqrt{2\pi}\sigma\psi}\int_{\psi a}^{\psi b}\sum_{j=0}^{n} P_n(x_j)l_j(y + (\phi - \psi)x_i)\exp\biggl(-\frac{(y - \psi x_i)^2}{2\sigma^2\psi^2}\biggr)\mathrm{d}y + P_1(x_i).$$

After solving the values at the collocation nodes, we can approximate the whole solution through (2.4).

Simulation results are also provided for comparison.

Case 1. Here we choose $\phi = 0.5$, $\theta = 0.4$ and $\sigma = 1$ as the process parameters and the lower barrier $a = 0$ and the initial state $x_1 = 1$. The results are given in Table 3.

TABLE 4: Probabilities of crossing a level b before 0.

n	$b = 1$	$b = 2$	$b = 3$	$b = 4$
2	0.5436	0.2239	0.0465	0.0043
4	0.4980	0.1952	0.0347	0.0027
8	0.5333	0.1757	0.0353	0.0020
16	0.5807	0.2417	0.0331	0.0018
32	0.6430	0.2407	0.0326	0.0017
64	0.6431	0.2406	0.0324	0.0017
128	0.6431	0.2406	0.0324	0.0017
256	0.6431	0.2406	0.0325	0.0017
Simulation	0.6445	0.2430	0.0325	0.0016

Case 2. Here we choose $\phi = 0.5, \theta = -0.4$ and $\sigma = 1$ as the process parameters and the lower barrier $a = 0$ and the initial state $x_1 = 1$. The results are given in Table 4.

The result is quite satisfactory and a faster rate can be obtained if higher-order polynomials are used in interpolations.

6. Conclusion and discussion

In this paper, we have proposed an integral-equation approach to evaluating the probability of AR(p) and ARMA$(1, 1)$ processes crossing a level before another level and the mean first-passage time to cross a level for AR(p) processes. We have also extended a martingale approach, developed by Novikov [7] for autoregressive processes, to ARMA(p, q) processes for a representation of the mean first-passage time. While time-series processes are commonly used in modelling data such as exchange rate, GDP and unemployment rate, this type of level-crossing probability is useful in making decisions where we have to weight the gain against the loss with some threshold levels in mind.

Our method relies heavily on the Markov nature of the state-space representations of time-series models. As discussed in [3], any ARMA or ARIMA model can be represented as a finite-dimensional model. Thus, the integral-equation approach can be extended to handle more complex time series. Of course, some conditions on the parameters will be needed as in the ARMA$(1, 1)$ case to make the integral equation solvable.

We further aim to develop a methodology for long-memory, ARFIMA(p, d, q), processes. Since long-memory processes have infinite-dimensional state-space representations (see [4] for details), we believe that, using a truncated state-space representation (as used in [4]), an approximate level-crossing probability can be found and proved to converge as the dimension tends to infinity. We intend to pursue this in future.

Appendix A. Characteristics of $\psi(\cdot)$

We first prove that $\psi(\cdot)$ defined in Section 3 is convex. For all $x, y \geq 0$ and $0 < \lambda < 1$,

$$\begin{aligned}
\lambda\psi(x) + (1-\lambda)\psi(y) &= \lambda \ln(\mathrm{E}\,\mathrm{e}^{x\xi_1}) + (1-\lambda)\ln(\mathrm{E}\,\mathrm{e}^{y\xi_1}) \\
&= \ln(\mathrm{E}\,\mathrm{e}^{x\xi_1})^{\lambda}(\mathrm{E}\,\mathrm{e}^{y\xi_1})^{1-\lambda} \\
&\geq \ln(\mathrm{E}\,\mathrm{e}^{\lambda x\xi_1 + (1-\lambda)y\xi_1}) \quad \text{(by Hölder's inequality)} \\
&= \psi(\lambda x + (1-\lambda)y).
\end{aligned}$$

So $\psi(\cdot)$ is convex. Next we show that $\psi(u)$ is bounded by a linear function for $u < 1$. Consider

$$\begin{aligned}\frac{\psi(u)}{u} &= \frac{\ln(\mathrm{E}\,\mathrm{e}^{u\xi_1})}{u} \\ &= \ln[(\mathrm{E}\,\mathrm{e}^{u\xi_1})^{1/u}] \\ &\le \ln(\mathrm{E}\,\mathrm{e}^{\xi_1}) \quad \text{for } u < 1 \\ &= \psi(1).\end{aligned}$$

So $\psi(u)$ is bounded by a linear function for $u < 1$. The boundedness of $\varphi_\lambda(u)$ comes directly from the definition and the above properties of $\psi(u)$.

Appendix B. Proofs

B.1. Proof of the martingale property of G_t

We show that G_t is a martingale:

$$\begin{aligned}&\mathrm{E}(G_{t+1} \mid \mathcal{F}_t) \\ &\quad = \mathrm{E}\biggl[\int_0^\infty u^{-1}[\exp(u\lambda\vec{c}_1\vec{Z}_t + u\xi_{t+1}) - \exp(u\vec{c}_1\vec{Z}_0)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u \biggm| \mathcal{F}_t\biggr] \\ &\qquad - (t+1)\log\Bigl(\frac{1}{\lambda}\Bigr) \\ &\quad = \int_0^\infty u^{-1}[\exp(u\lambda\vec{c}_1\vec{Z}_t + \psi(u)) - \exp(u\vec{c}_1\vec{Z}_0)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u - (t+1)\log\Bigl(\frac{1}{\lambda}\Bigr) \\ &\quad = \int_0^\infty u^{-1}[\exp(u\lambda\vec{c}_1\vec{Z}_t) - \exp(u\lambda\vec{c}_1\vec{Z}_0)]\exp(-\varphi_\lambda(\lambda u))\,\mathrm{d}u - (t+1)\log\Bigl(\frac{1}{\lambda}\Bigr) \\ &\qquad - \int_0^\infty u^{-1}[\exp(u\vec{c}_1\vec{Z}_0 - \varphi_\lambda(u)) - \exp(u\lambda\vec{c}_1\vec{Z}_0 - \varphi_\lambda(\lambda u))]\,\mathrm{d}u.\end{aligned}$$

The last integral equals $\log\lambda$ by the Frullani identity,

$$\int_0^\infty u^{-1}[f(au) - f(bu)]\,\mathrm{d}u = f(0)\log(b/a), \qquad b > a > 0,$$

which holds whenever the function f is continuous at 0 and $\int_0^\infty u^{-1}f(u)\,\mathrm{d}u$ converges (see [5] for details). Thus the result follows.

B.2. Proof of Theorem 3.1

Note that (3.6) implies (3.5). Hence, by Doob's optional stopping theorem on martingales, (3.7) holds if $\mathcal{T}(a,b)$ is replaced by any bounded stopping time, say by $\mathcal{T}(a,b) \wedge t$. As $\vec{Z}_0^\top = (\vec{X}_0, \vec{0}_q)^\top$,

$$\begin{aligned}&\mathrm{E}(\mathcal{T}(a,b) \wedge t)\log(1/\lambda) \\ &\quad = \mathrm{E}\int_0^\infty u^{-1}[\exp(u\vec{c}_1\vec{Z}_{\mathcal{T}(a,b)\wedge t}) - \exp(u\vec{c}_1\vec{Z}_0)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u \\ &\quad = \mathrm{E}\int_0^\infty u^{-1}[\exp(u\vec{c}_{10}\vec{X}_{\mathcal{T}(a,b)\wedge t} + u\vec{d}_{10}\vec{v}_{\mathcal{T}(a,b)\wedge t}) - \exp(u\vec{c}_{10}\vec{X}_0)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u.\end{aligned} \tag{B.1}$$

We intend to take the limit as $t \to \infty$ and interchange the limit and integral using the dominated convergence theorem. Split the domain of the integral on the right-hand side of (B.1) into the sets $\{t < \mathcal{T}(a,b)\}$, $\{\mathcal{T}(a,b) \le t, \vec{d}_{10}\vec{v}_{\mathcal{T}(a,b)} \le k(u^2+1)^{-1}\}$ and $\{\mathcal{T}(a,b) \le t, \vec{d}_{10}\vec{v}_{\mathcal{T}(a,b)} > k(u^2+1)^{-1}\}$, where $k > 0$. Then the right-hand side of (B.1) does not exceed the following sum:

$$\begin{aligned}
&\int_0^\infty u^{-1}[\exp(u\|\vec{c}_{10}\vec{b}_a\| + uK)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u \\
&\quad + \int_0^\infty u^{-1}\left[\exp\left(u\|\vec{c}_{10}\vec{b}_a\| + \frac{ku}{u^2+1}\right) - \exp(u\vec{c}_{10}\vec{X}_0)\right]^+ \exp(-\varphi_\lambda(u))\mathrm{d}u \\
&\quad + \mathrm{E}\int_0^\infty \mathbf{1}\{\mathcal{T}(a,b) \le t, \vec{d}_{10}\vec{v}_{\mathcal{T}(a,b)} > k(u^2+1)^{-1}\}u^{-1} \\
&\qquad\qquad \times [\exp(u\|\vec{c}_{10}\vec{b}_a\| + u\vec{d}_{10}\vec{v}_{\mathcal{T}(a,b)}) - \exp(u\vec{c}_1\vec{Z}_0)]^+\exp(-\varphi_\lambda(u))\,\mathrm{d}u. \qquad \text{(B.2)}
\end{aligned}$$

Invertibility of the ARMA process implies that

$$\vec{d}_{10}\vec{v}_t \le Cb_a\frac{\sum_{i=1}^q |d_i|}{d_q + c_p} = K \quad \text{say,}$$

where $C = \sum_{j=0}^\infty |\psi_j| < \infty$ and where the ψ_j are such that $\xi_t = \sum_{j=0}^\infty \psi_j Y_{t-j}$.

By (3.6) and the boundedness of $\varphi_\lambda(\cdot)$ (defined in (3.4)) on $[0, 1]$, the first and second integrals in the sum (B.2) are bounded and independent of t. Denote the upper bound by C_1. Now, for a stopping time τ for $\vec{X}$ and for a nonnegative function f, observe that

$$\begin{aligned}
\mathrm{E}(f(\vec{d}_{10}\vec{v}_\tau)\mathbf{1}_{\tau<t}) &= \mathrm{E}\sum_{i=1}^{\tau\wedge t} f(\vec{d}_{10}\vec{v}_i)\mathbf{1}_{\tau=i} \\
&\le \mathrm{E}\sum_{i=1}^{\tau\wedge t} f(\vec{d}_{10}\vec{v}_i).
\end{aligned}$$

The last inequality is valid as $\mathbf{1}_{\tau=i} \le 1$ and f is nonnegative. Now, using the above argument (which is similar to Wald's identity) the third integral in (B.2) can be bounded as follows:

$$\begin{aligned}
&\mathrm{E}\sum_{i=1}^{\mathcal{T}(a,b)\wedge t}\int_0^\infty \mathbf{1}\{\vec{d}_{10}\vec{v}_i > k(u^2+1)^{-1}\}u^{-1}[\mathrm{e}^{u\|\vec{c}_{10}\vec{b}_a\| + u\vec{d}_{10}\vec{v}_i} - \mathrm{e}^{u\vec{c}_{10}\vec{X}_0}]^+\mathrm{e}^{-\varphi_\lambda(u)}\,\mathrm{d}u \\
&\quad \le \mathrm{E}(\mathcal{T}(a,b)\wedge t)\int_0^\infty u^{-1}\,\mathrm{E}\,\mathbf{1}\{\vec{d}_{10}\vec{v}_q > k(u^2+1)^{-1}\} \\
&\qquad\qquad \times [\mathrm{e}^{u\|\vec{c}_{10}\vec{b}_a\| + u\vec{d}_{10}\vec{v}_q} - \mathrm{e}^{u\vec{c}_{10}\vec{X}_0}]^+\mathrm{e}^{-\varphi_\lambda(u)}\,\mathrm{d}u \\
&\quad + \mathrm{E}\sum_{i=1}^{q-1}\int_0^\infty u^{-1}\,\mathrm{E}\,\mathbf{1}\{\vec{d}_{10}\vec{v}_i > k(u^2+1)^{-1}\}[\mathrm{e}^{u\|\vec{c}_{10}\vec{b}_a\| + u\vec{d}_{10}\vec{v}_i} - \mathrm{e}^{u\vec{c}_{10}\vec{X}_0}]^+\mathrm{e}^{-\varphi_\lambda(u)}\,\mathrm{d}u.
\end{aligned} \qquad \text{(B.3)}$$

Letting $k \to \infty$, the above integrals decrease monotonically to zero by the dominated convergence theorem, (3.1) and (3.6). So, by choosing k large enough, the values of the integrals in (B.3) can be made arbitrarily small, say smaller than $\varepsilon\log(1/\lambda)$ with $0 < \varepsilon < 1$ for the first integral while the sum of integrals in the second term can be bounded by a constant C_2. So, from (B.1), (B.2) and the above,

$$\mathrm{E}(\mathcal{T}(a,b)\wedge t)\log(1/\lambda) \leq \frac{C_1+C_2}{1-\varepsilon} < \infty.$$

This implies that $\mathrm{E}\,\mathcal{T}(a,b) < \infty$ simply by Fatou's lemma. Using a similar argument for $\mathcal{T}(a,b)$ as is done above for $\mathcal{T}(a,b)\wedge t$, we obtain that

$$\mathrm{E}\int_0^\infty u^{-1}[\exp(u\vec{c}_1\vec{Z}_{\mathcal{T}(a,b)}) - \exp(u\vec{c}_1\vec{Z}_0)]\exp(-\varphi_\lambda(u)\,\mathrm{d}u < \infty. \tag{B.4}$$

In addition to the integrand of (B.4), which acts as an upper bound, we also need a lower bound to apply the dominated convergence theorem in (B.1). To find this, note that

$$\begin{aligned}
&\int_0^\infty u^{-1}[\exp(u\vec{c}_1\vec{Z}_0) - \exp(u\vec{c}_1\vec{Z}_{\mathcal{T}(a,b)\wedge t})]\exp(-\varphi_\lambda(u))\,\mathrm{d}u\\
&\quad\leq \int_1^\infty u^{-1}\exp[(u\vec{c}_1\vec{Z}_0) - \varphi_\lambda(u)]\,\mathrm{d}u\\
&\qquad + \int_0^1 u^{-1}\exp[(u\vec{c}_1\vec{Z}_0) - \varphi_\lambda(u)](1-\exp[-u\,\|\vec{c}_1(\vec{Z}_{\mathcal{T}(a,b)\wedge t} - \vec{Z}_0)\|])\,\mathrm{d}u\\
&\quad\leq \int_1^\infty u^{-1}\exp[(u\vec{c}_1\vec{Z}_0) - \varphi_\lambda(u)]\,\mathrm{d}u\\
&\qquad + \sup_{0\leq u\leq 1}\exp[(u\vec{c}_1\vec{Z}_0) - \varphi_\lambda(u)]\,\|\vec{c}_1(\vec{Z}_{\mathcal{T}(a,b)\wedge t} - \vec{Z}_0)\|.
\end{aligned}$$

Since $\vec{c}_1\vec{Z}_t = \lambda^t\vec{c}_1\vec{Z}_0 + \sum_{i=1}^t \lambda^{t-i}\xi_i$,

$$\|\vec{c}_1\vec{Z}_{\mathcal{T}(a,b)\wedge t} - \vec{c}_1\vec{Z}_0\| \leq \|\vec{c}_1\vec{Z}_0\| + \sum_{i=0}^{\mathcal{T}(a,b)\wedge t}|\xi_i|.$$

Also, since $\mathrm{E}\,\mathcal{T}(a,b) < \infty$ and by Wald's identity,

$$\mathrm{E}\,\|\vec{c}_1\vec{Z}_{\mathcal{T}(a,b)\wedge t} - \vec{c}_1\vec{Z}_0\| < \infty.$$

With the upper and lower bounds, we have (3.7) by the dominated convergence theorem.

B.3. Proof of Theorem 3.2

For an ARMA(p,q) process satisfying the state-space representation (2.2), if we assume that $\phi_p > 0$ and $\phi_1,\dots,\phi_{p-1} \geq 0$, then the coefficient matrix F is nonnegative and nonsingular. As a result of the Perron–Frobenius theorem, $F^\top$ possesses a largest real positive eigenvalue λ and that corresponds to a real positive eigenvector $\vec{c}$. We further notice that $0 < \lambda < 1$ by stationarity. Again, from (3.2) it is clear that, if $\vec{c}$ and $\vec{\theta}$ are nonnegative and $\lambda > 0$, then $\vec{d}$ is also nonnegative. In fact, if $\theta_q > 0$, then $\vec{d}$ is positive as $c_p > 0$. So we can rewrite (2.2) as:

$$\vec{c}_1\vec{Z}_{t+1} = \vec{c}_1F_0\vec{Z}_t + \vec{c}_1\vec{\varepsilon}_{t+1}.$$

As in (3.3),

$$\vec{c}_1\vec{Z}_{t+1} = \lambda\vec{c}_1\vec{Z}_t + \xi_{t+1}. \tag{B.5}$$

Thus, $V_t := \vec{c}_1\vec{Z}_t$ satisfies an AR(1) equation with $0 < \lambda < 1$.

Now we mimic the the proof of Theorem 3.1, but with a variation, to give the upper and lower bounds for the right-hand side of (B.6) below.

Note that (3.8) implies (3.5). Hence, by Doob's optional stopping theorem on a martingale, (3.9) holds if $\mathcal{T}(b)$ is replaced by any bounded stopping time, say by $\mathcal{T}(b) \wedge t$. So,

$$\mathrm{E}(\mathcal{T}(b) \wedge t) \log(1/\lambda) = \mathrm{E} \int_0^\infty u^{-1}[\exp(u\vec{c}_1 \vec{Z}_{\mathcal{T}(b)\wedge t}) - \exp(u\vec{c}_1 \vec{Z}_0)] \exp(-\varphi_\lambda(u))\, \mathrm{d}u. \quad \text{(B.6)}$$

We intend to take the limit as $t \to \infty$ and interchange the limit and integral by the dominated convergence theorem. Split the domain of the integral on the right-hand side of (B.6) into the sets

$$\{t < \mathcal{T}(b)\},$$
$$\{\mathcal{T}(b) \le t, \vec{d}_{10}\vec{\upsilon}_{\mathcal{T}(b)} \le k(u^2+1)^{-1}\}$$

and

$$\{\mathcal{T}(b) \le t, \vec{d}_{10}\vec{\upsilon}_{\mathcal{T}(b)} > k(u^2+1)^{-1}\},$$

where $k > 0$. Then the right-hand side of (B.6) does not exceed the following sum:

$$\begin{aligned}
&\int_0^\infty u^{-1}[\exp(u\vec{c}_1\vec{b} + uK - \exp(u\vec{c}_1\vec{Z}_0)] \exp(-\varphi_\lambda(u))\, \mathrm{d}u \\
&\quad + \int_0^\infty u^{-1}\left[\exp\left(u\vec{c}_{10}\vec{b} + \frac{ku}{u^2+1}\right) - \exp(u\vec{c}_1\vec{Z}_0)\right]^+ \exp(-\varphi_\lambda(u))\, \mathrm{d}u \\
&\quad + \mathrm{E}\int_0^\infty \mathbf{1}\{\mathcal{T}(b) \le t, \vec{d}_{10}\vec{\upsilon}_{\mathcal{T}(b)} > k(u^2+1)^{-1}\} u^{-1} \\
&\qquad \times [\exp(u\vec{c}_{10}\vec{b} + u\vec{d}_{10}\vec{\upsilon}_{\mathcal{T}(b)}) - \exp(u\vec{c}_1\vec{Z}_0)]^+ \exp(-\varphi_\lambda(u))\, \mathrm{d}u. \qquad \text{(B.7)}
\end{aligned}$$

Here, K is derived as in the proof of Theorem 3.1. By (3.8) and the boundedness of $\varphi_\lambda(\cdot)$ on $[0, 1]$, the first and second integrals of (B.7) are bounded and independent of t. Denote the upper bound by G_1. Now, using Wald's identity, the third integral of (B.7) can be bounded as follows:

$$\begin{aligned}
&\mathrm{E}\sum_{i=1}^{\mathcal{T}(b)\wedge t} \int_0^\infty \mathbf{1}\{\vec{d}_{10}\vec{\upsilon}_i > k(u^2+1)^{-1}\} u^{-1}[\mathrm{e}^{u\vec{c}_1\vec{b}+u\vec{d}_{10}\vec{\upsilon}_i} - \mathrm{e}^{u\vec{c}_1\vec{Z}_0}]^+ \mathrm{e}^{-\varphi_\lambda(u)}\, \mathrm{d}u \\
&\le \mathrm{E}(\mathcal{T}(b) \wedge t) \int_0^\infty u^{-1}\, \mathrm{E}\,\mathbf{1}\{\vec{d}_{10}\vec{\upsilon}_q > k(u^2+1)^{-1}\}[\mathrm{e}^{u\vec{c}_1\vec{b}+u\xi_1} - \mathrm{e}^{u\vec{c}_1\vec{Z}_0}]^+ \mathrm{e}^{-\varphi_\lambda(u)}\, \mathrm{d}u \\
&\quad + \mathrm{E}\sum_{i=1}^{q-1} \int_0^\infty u^{-1}\, \mathrm{E}\,\mathbf{1}\{\vec{d}_{10}\vec{\upsilon}_i > k(u^2+1)^{-1}\}[\mathrm{e}^{u\|\vec{c}_{10}\vec{b}\|+u\vec{d}_{10}\vec{\upsilon}_i} - \mathrm{e}^{u\vec{c}_1\vec{Z}_0}]^+ \mathrm{e}^{-\varphi_\lambda(u)}\, \mathrm{d}u. \quad \text{(B.8)}
\end{aligned}$$

Letting $k \to \infty$, the above integrals decrease monotonically to zero by the dominated convergence theorem, (B.5) and (3.8). So, by choosing k large enough, the values of the integrals on the right-hand side of (B.8) can be made arbitrarily small, say smaller than $\varepsilon \log(1/\lambda)$ with $0 < \varepsilon < 1$ for the first integral while the sum of integrals in the second term can be bounded by a constant G_2. So,

$$\mathrm{E}(\mathcal{T}(b) \wedge t) \log(1/\lambda) \le \frac{G_1 + G_2}{1 - \varepsilon} < \infty.$$

This implies that $\mathrm{E}\,\mathcal{T}(b) < \infty$ simply by Fatou's lemma. Similarly,

$$\mathrm{E}\int_0^\infty u^{-1}[\exp(u\vec{c}_1\vec{Z}_{\mathcal{T}(b)}) - \exp(u\vec{c}_1\vec{Z}_0)]\exp(-\varphi_\lambda(u))\,\mathrm{d}u < \infty. \tag{B.9}$$

In addition to the integrand of (B.9), which acts as an upper bound, we also need a lower bound to apply the dominated convergence theorem in (B.6). To find this, note that

$$\begin{aligned}\int_0^\infty & u^{-1}[\exp(u\vec{c}_1\vec{Z}_0) - \exp(u\vec{c}_1\vec{Z}_{\mathcal{T}(b)\wedge t})]\exp(-\varphi_\lambda(u))\,\mathrm{d}u \\ &\le \int_1^\infty u^{-1}\exp[(u\vec{c}_1\vec{Z}_0) - \varphi_\lambda(u)]\mathrm{d}u \\ &\quad + \sup_{0\le u\le 1}\exp[(u\vec{c}_1\vec{Z}_0) - \varphi_\lambda(u)]\|\|\vec{c}_1(\vec{Z}_{\mathcal{T}(b)\wedge t} - \vec{Z}_0)\|\|.\end{aligned}$$

Since $\vec{c}_1\vec{Z}_t = \lambda^t\vec{c}_1\vec{Z}_0 + \sum_{i=1}^t \lambda^{t-i}\xi_i$,

$$\|\|\vec{c}_1\vec{Z}_{\mathcal{T}(b)\wedge t} - \vec{c}_1\vec{Z}_0\|\| \le \|\|\vec{c}_1\vec{Z}_0\|\| + \sum_{i=0}^{\mathcal{T}(b)\wedge t} |\xi_i|.$$

Since $\mathrm{E}\,\mathcal{T}(b) < \infty$ and by Wald's identity,

$$\mathrm{E}\,\|\|\vec{c}_1\vec{Z}_{\mathcal{T}(b)\wedge t} - \vec{c}_1\vec{Z}_0\|\| < \infty.$$

With these upper and lower bounds, we obtain (3.9) by the dominated convergence theorem.

B.4. Proof of Theorem 5.1

In (5.4), we see that there is a shift in the argument of f in the integrand that may stop us from defining f just inside a compact interval. However, if the domain of f is chosen suitably, this problem can be avoided. For any nonzero values of ψ, we can show that a sufficient condition for considering f just on $[\alpha, \beta]$ is that $-1 < \theta < 1$ and that $[\alpha, \beta]$ be sufficiently large (see Appendix D).

Now, with $-1 < \theta < 1$ and $[\alpha, \beta]$ large enough as mentioned above, we can show that a unique solution of (5.3) exists in $\mathrm{C}[\alpha, \beta]$. First we observe that $\mathcal{K}_1$ in (5.4) is a linear operator on the Banach space $\mathrm{C}[\alpha, \beta]$ equipped with the supremum norm. Next we see that this operator has norm less than 1 once $[a, b]$ is a proper subset of $\mathbb{R}$. So, by the contraction mapping theorem (Theorem 2.3), we know that a unique solution exists for (5.3). Moreover, $\mathcal{K}_1$ is a compact operator on $\mathrm{C}[\alpha, \beta]$. To justify this fact, we know from the Arzelà–Ascoli theorem that any subset $\mathcal{S} \subset \mathrm{C}(D)$ has compact closure if (i) $\mathcal{S}$ is a uniformly bounded set of functions and (ii) $\mathcal{S}$ is an equicontinuous family. Now consider the set

$$\mathcal{S} = \{\mathcal{K}_1 f \mid f \in \mathrm{C}(D), \|f\|_\infty \le 1\}.$$

So, by Definition 2.1, we just have to show that $\mathcal{S}$ satisfies (i) and (ii). Firstly, $\mathcal{S}$ is uniformly bounded as the norm of $\mathcal{K}_1 < 1$. Secondly, let $x, s \in [\alpha, \beta]$ and, for convenience, let $x > s$

and $\phi - \psi > 0$. As $x \to s$,

$$
\begin{aligned}
&|\mathcal{K}_1 f(x) - \mathcal{K}_1 f(s)| \\
&= \left| \frac{1}{\sqrt{2\pi}\sigma\psi} \int_{\psi a}^{\psi b} f(y + (\phi - \psi)x) \exp\left(-\frac{(y - \psi x)^2}{2\sigma^2\psi^2}\right) \mathrm{d}y \right. \\
&\quad \left. - \frac{1}{\sqrt{2\pi}\sigma\psi} \int_{\psi a}^{\psi b} f(y + (\phi - \psi)s) \exp\left(-\frac{(y - \psi s)^2}{2\sigma^2\psi^2}\right) \mathrm{d}y \right| \\
&\leq \frac{1}{\sqrt{2\pi}\sigma\psi} \left| \int_{\psi a + (\phi - \psi)x}^{\psi b + (\phi - \psi)x} f(z) \exp\left(-\frac{(z - \phi x)^2}{2\sigma^2\psi^2}\right) \mathrm{d}z \right. \\
&\qquad \left. - \int_{\psi a + (\phi - \psi)s}^{\psi b + (\phi - \psi)s} f(z) \exp\left(-\frac{(z - \phi s)^2}{2\sigma^2\psi^2}\right) \mathrm{d}z \right| \\
&\leq \frac{1}{\sqrt{2\pi}\sigma\psi} \left| \int_{\psi a + (\phi - \psi)s}^{\psi b + (\phi - \psi)s} f(z) \left[\exp\left(-\frac{(z - \phi x)^2}{2\sigma^2\psi^2}\right) - \exp\left(-\frac{(z - \phi s)^2}{2\sigma^2\psi^2}\right) \right] \mathrm{d}z \right| \\
&\quad + \frac{1}{\sqrt{2\pi}\sigma\psi} \left| \int_{\psi b + (\phi - \psi)s}^{\psi b + (\phi - \psi)x} f(z) \exp\left(-\frac{(z - \phi x)^2}{2\sigma^2\psi^2}\right) \mathrm{d}z \right| \\
&\quad + \frac{1}{\sqrt{2\pi}\sigma\psi} \left| \int_{\psi a + (\phi - \psi)s}^{\psi a + (\phi - \psi)x} f(z) \exp\left(-\frac{(z - \phi x)^2}{2\sigma^2\psi^2}\right) \mathrm{d}z \right| \\
&\leq \frac{1}{\sqrt{2\pi}\sigma\psi} \int_{\psi a + (\phi - \psi)s}^{\psi b + (\phi - \psi)s} \left| \exp\left(-\frac{(z - \phi x)^2}{2\sigma^2\psi^2}\right) - \exp\left(-\frac{(z - \phi s)^2}{2\sigma^2\psi^2}\right) \right| \mathrm{d}z \\
&\quad + \frac{1}{\sqrt{2\pi}\sigma\psi} \int_{\psi b + (\phi - \psi)s}^{\psi b + (\phi - \psi)x} \left| \exp\left(-\frac{(z - \phi x)^2}{2\sigma^2\psi^2}\right) \right| \mathrm{d}z \\
&\quad + \frac{1}{\sqrt{2\pi}\sigma\psi} \int_{\psi a + (\phi - \psi)s}^{\psi a + (\phi - \psi)x} \left| \exp\left(-\frac{(z - \phi x)^2}{2\sigma^2\psi^2}\right) \right| \mathrm{d}z \quad \text{since } \|f\|_\infty \leq 1.
\end{aligned}
$$

Since the kernels are continuous and x, s were chosen from a compact set, this upper bound converges to 0 as x tends to s. As the convergence rate is independent of f and by uniform continuity, we established the equicontinuity of $\mathcal{S}$. Thus, by the Azelà–Ascoli theorem, the operator $\mathcal{K}_1$ is compact. A similar argument holds when $\phi - \psi < 0$.

Appendix C. Numerical integration over triangles

The background material in this appendix is mainly taken from [1]. The first step of the numerical scheme is to divide the domain of integration into small triangles and apply a quadrature rule to perform numerical integration. Since our domain here is simply the square $[a, b] \times [a, b]$, we naturally divide it into $2n^2$ triangles as in Figure 1 (with $n = 2$ as an example).

Suppose that $\triangle$ is one such triangle, with vertices v_1, v_2, v_3. Introduce the unit simplex

$$\sigma = \{(s, t) \mid s, t \geq 0, s + t \leq 1\}.$$

Define a one-to-one and onto mapping $T : \sigma \to \triangle$ by

$$T(s, t) = (1 - s - t)v_1 + tv_2 + sv_3.$$

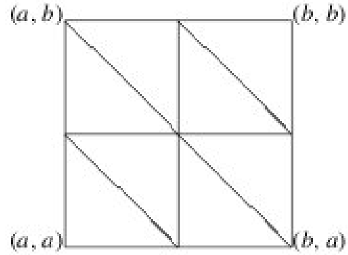

FIGURE 1: The square $[a, b] \times [a, b]$ divided into $8 = 2 \times 2^2$ triangles.

Through the change of variables $(x, y) = T(s, t)$, we have

$$\int_{\triangle} g(x, y)\, \mathrm{d}x\, \mathrm{d}y = 2\mathrm{area}(\triangle) \int_{\sigma} g(T(s, t))\, \mathrm{d}\sigma,$$

where the function $g : [a, b]^2 \to \mathbb{R}$ is continuous and integrable. We can thus use schemes developed on the unit simplex to do the numerical integrations on the triangle $\triangle$. The quadrature rule used here is a seven-point formula provided in [1]:

$$\int_{\sigma} g(s, t)\, \mathrm{d}\sigma \approx \tfrac{9}{40} g(\tfrac{1}{3}, \tfrac{1}{3}) + \tfrac{1}{40}[g(0, 0) + g(0, 1) + g(1, 0)] + \tfrac{1}{15}[g(0, \tfrac{1}{3}) + g(\tfrac{1}{2}, \tfrac{1}{2}) + g(\tfrac{1}{2}, 0)]. \tag{C.1}$$

This rule has degree of precision equal to 3, i.e. there is no error if g is a polynomial of degree no greater than 3, and it can be derived using the method of undetermined coefficients. When it is used in the composite formula, the nodes from adjacent triangles overlap. Thus, the total number of integration nodes will be $6n^2 + 4n + 1$ instead of $14n^2$.

Suppose that $K(x, y, \xi, \eta)$ is continuous over $[a, b] \times [a, b]$ and that we use the composite numerical integration rule (C.1), which can be written as

$$\int_{\sigma} g(s, t)\, \mathrm{d}\sigma \approx \sum_{i=1}^{7} w_i g(\mu_i), \tag{C.2}$$

where $\mu_i \in \{(\frac{1}{3}, \frac{1}{3}), (0, 0), (0, 1), (1, 0), (0, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0)\}$ and $w_i \in \{\frac{9}{40}, \frac{1}{40}, \frac{1}{15}\}$ is the corresponding coefficient. We can now approximate the integral in (4.13) by

$$\begin{aligned}\int_a^b \int_a^b K(x, y, \xi, \eta) f(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta &\approx 2 \sum_{k=1}^{2n^2} \mathrm{area}(\triangle_k) \sum_{i=1}^{7} w_i K(x, y, T_k(\mu_i)) f(T_k(\mu_i)) \\ &= \sum_{j=1}^{m} \omega_j K(x, y, \xi_j, \eta_j) f(\xi_j, \eta_j),\end{aligned}$$

where $T_k = T : \sigma \to \triangle_k$, $m = 6n^2 + 4n + 1$ and ω_j is the weight of the jth node. Thus, the integral equation (4.13) can be approximated by

$$f_n(x, y) - \sum_{j=1}^{m} \omega_j K(x, y, \xi_j, \eta_j) f_n(\xi_j, \eta_j) = \Psi(x, y), \qquad (x, y) \in [a, b] \times [a, b]. \tag{C.3}$$

We can then evaluate the values of f_n on the node points by solving the following system of linear equations:

$$f_n(\xi_i, \eta_i) - \sum_{j=1}^{m} \omega_j K(\xi_i, \eta_i, \xi_j, \eta_j) f_n(\xi_j, \eta_j) = \Psi(\xi_i, \eta_i), \qquad i = 1, \ldots, m.$$

Finally, we can use the Nyström interpolation formula to obtain the remaining values of f_n on $[a, b] \times [a, b]$:

$$f_n(x, y) = \sum_{j=1}^{m} \omega_j K(x, y, \xi_j, \eta_j) f_n(\xi_j, \eta_j) + \Psi(x, y).$$

We state the following theorem (from [1]) to justify the convergence of f_n to f and to calculate the rate of convergence.

Theorem C.1. *Let R be a polygonal region in $\mathbb{R}^2$ and let $\{\mathcal{T}_n\}$ be a sequence of triangulations of R. Let*

$$\delta_n := \max_{k=1,\ldots,n} \operatorname{diameter}(\triangle_k) \to 0$$

and assume that $\delta_n \to 0$ as $n \to \infty$. Assume that the integral equation $(1 - \mathcal{K})f = \Psi$ is uniquely solvable for $\Psi \in \mathrm{C}(\mathbb{R})$, with $\mathcal{K}$ a compact operator on $\mathrm{C}(\mathbb{R})$. Assume that the integration formula (C.2) has degree of precision $d \geq 0$.

(a) *For all sufficiently large n, say $n \geq N$, the approximating equation (C.3) is uniquely solvable, and the inverses $(1 - \mathcal{K}_n)^{-1}$ are uniformly bounded on $\mathrm{C}(\mathbb{R})$. For the error in f_n,*

$$f - f_n = (1 - \mathcal{K})^{-1}(\mathcal{K}f - \mathcal{K}_n f)$$

and $f_n \to f$ as $n \to \infty$.

(b) *If $K(x, y, \cdot, \cdot) \in \mathrm{C}^{d+1}(R)$, for all $(x, y) \in R$, and $f \in \mathrm{C}^{d+1}(R)$, then*

$$\| f - f_n \|_\infty \leq c\delta_n^{d+1}, \qquad n \geq N.$$

The proof is omitted here; the interested reader can consult the reference [1]. Note that the quadrature formula (C.1) we used is of precision $d = 3$, so the rate of convergence is $O(n^{-4})$.

Appendix D. Sufficient condition for (5.3) to have a unique solution

For fixed a, b we want to make sure that (5.3) can be solved uniquely in $\mathrm{C}(D)$, where D is a compact interval.

Case 1. $\psi > 0, \theta < 0$. We want

$$\psi a - \theta\alpha \geq \alpha \quad \text{and} \quad \psi b - \theta\beta \leq \beta,$$

in other words,

$$\psi a \geq (1 + \theta)\alpha \quad \text{and} \quad \psi b \leq (1 + \theta)\beta,$$

which is the case if and only if

$$\alpha \leq \frac{\psi a}{1 + \theta} \quad \text{and} \quad \beta \geq \frac{\psi b}{1 + \theta} \quad \text{if } \theta > -1.$$

Thus, if $0 > \theta > -1$ and α is chosen sufficiently small and β sufficiently large, we can guarantee that the argument inside f will fall in $[\alpha, \beta]$.

Case 2. $\psi > 0, \theta > 0$. We want

$$\psi a - \theta\beta \geq \alpha \quad \text{and} \quad \psi b - \theta\alpha \leq \beta,$$

which is the case if and only if

$$\psi a \geq \alpha + \theta\beta \quad \text{and} \quad \psi b \leq \beta + \theta\alpha.$$

So, if $\theta < 1$, we choose $\alpha = -\beta$ and the above inequalities become

$$\psi a \geq (\theta - 1)\beta \quad \text{and} \quad \psi b \leq (1 - \theta)\beta.$$

Thus, β can be chosen large to satisfy the inequalities.

In the cases where $\psi < 0$, the proof is similar.

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