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The Helgason Fourier Transform for semisimple Lie groups I: the case of $\mathrm{SL}_2(\mathbb{R})$

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THE HELGASON FOURIER TRANSFORM FOR SEMISIMPLE LIE GROUPS I: THE CASE OF $\mathrm{SL}_2(\mathbb{R})$

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Abstract. We consider a Helgason-type Fourier transform on $\mathrm{SL}_2(\mathbb{R})$ and prove various results on L^1-harmonic analysis on the full group analogous to those on symmetric spaces.

1. Introduction

Consider a connected semisimple Lie group G with a fixed maximal compact subgroup K and let $G = KAN$ be an Iwasawa decomposition. Given a *suitably nice* function f on G, one studies the "group-theoretic" Fourier transform $f \mapsto \widehat{f}$, where for an irreducible unitary representation π, $\widehat{f}(\pi) = \pi(f)$ is an operator on H_π (the Hilbert space on which π is realized), defined by $\pi(f) = \int_G f(x)\pi(x^{-1})dx$, the integral being suitably interpreted. However if f is a right K-invariant function, then $\pi(f) = 0$ unless π is a representation of class one. Even when π is of class one, $\pi(f)v = 0$, if $v \in \{v_0\}^\perp$ where v_0 is the essentially unique K-fixed vector. So $\pi(f)$ is completely determined by $\pi(f)v_0$. As is well known, the class one representations are given by the spherical principal series $\{\pi_\lambda\}$, where λ ranges over a suitable subset of $\mathfrak{a}^*_{\mathbb{C}}$ ([11]). All the π_λ are realized on close subspaces of $L^2(K)$ and v_0 then is just the constant function on K. Thus one is led to consider the function of two variables (λ, k), λ as above and $k \in K$, given by $\widetilde{f}(\lambda, k) = (\pi_\lambda(f)v_0)(k)$. This is essentially the Helgason Fourier transform, which can also be expressed as $\int_G e_{\lambda,k}(g)f(g)\,dg$, where $e_{\lambda,k}$ are eigenfunctions of the Laplace–Beltrami operator on the Riemannian symmetric space G/K, which are constant on horocycles ([15]). These eigenfunctions serve as analogues of plane waves in the case of Euclidean space. (For an excellent overview of non-Euclidean analysis, see [13].) In this paper we introduce a Helgason-type Fourier transform for complex valued functions on the *full group* $\mathrm{SL}_2(\mathbb{R})$, in the spirit of Camporesi [5]. Camporesi's definition, applied to $\mathrm{SL}_2(\mathbb{R})$, would amount to considering only those functions of a

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particular right K-type, where $K = \mathrm{SO}(2)$. However, no restrictions are made here on the K-types of the functions involved. Secondly, Camporesi's main interest is in C_c^∞-functions, while we are interested in L^1-functions. For a general function f, the inversion formula (see Sections 3, 4) can be thought of as an eigenfunction expansion involving a sequence of elliptic operators Δ_n, or alternatively an eigenfunction expansion for the Casimir operator Ω. Our formulation is particularly well-suited to stating various theorems of L^p-harmonic analysis on the whole group, such as theorems of the Wiener–Tauberian type and Benedicks' Theorem (see [22, 6] and Sections 5 and 6 of this paper). In Section 7, we describe how $\mathrm{SL}_2(\mathbb{R})$ can be identified with the semisimple symmetric space $\mathrm{SO}_0(2,2)/\mathrm{SO}_0(1,2)$ ([1]) and we translate our results in the language of harmonic analysis on this specific symmetric space. In a subsequent paper we will indicate how the definition of the Helgason Fourier transform and some of the results of this paper can be generalized to a class of semisimple Lie groups. One of the reasons for restricting to the case of $\mathrm{SL}_2(\mathbb{R})$ is that an analogue of the Wiener–Tauberian Theorem is known only in this case and as matters stand the case of general semisimple Lie group is intractable. (However some reasonable analogues have been found for symmetric spaces of rank one ([23, 18]).)

2. Preliminaries

Let G be the group $\mathrm{SL}_2(\mathbb{R})$ and $\mathfrak{g}$ its Lie algebra ($= \mathfrak{sl}(2,\mathbb{R})$). Suppose that

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \text{ and } n_\xi = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}.$$

Then $K = \{k_\theta \mid \theta \in [0, 2\pi)\}$, $A = \{a_t \mid t \in \mathbb{R}\}$ and $N = \{n_\xi \mid \xi \in \mathbb{R}\}$ are three particular subgroups of G of which K is the (maximal) compact subgroup $\mathrm{SO}(2)$ of G. Let $\mathfrak{k}$ be the Lie algebra of K. Let $G = KAN$ be an Iwasawa decomposition and for $x \in G$, let $x = k_\theta a_t n_\xi$ be its corresponding decomposition. Then we will write $H(x)$ for t and $K(x)$ for k_θ. Let M be the subgroup $\{\pm I_2\}$, where I_2 is the 2×2 identity matrix. Let $\widehat{M}$ denote the equivalence classes of irreducible representations of M. Then $\widehat{M} = \{\sigma_0, \sigma_1\}$ where σ_0 is the trivial representation of M and σ_1 is the unique nontrivial irreducible representation of M. For convenience we will denote them simply by 0 and 1 respectively. We define

$$\begin{aligned} & \mathbb{Z}^\sigma = 2\mathbb{Z} \text{ and } \mathbb{Z}^{-\sigma} = 2\mathbb{Z}+1 \quad \text{if } \sigma = 0 \\ \text{and} \quad & \mathbb{Z}^\sigma = 2\mathbb{Z}+1 \text{ and } \mathbb{Z}^{-\sigma} = 2\mathbb{Z} \quad \text{if } \sigma = 1. \end{aligned}$$

We also define for $n \in \mathbb{Z}$, $\sigma(n) = 0$ if n is even and $\sigma(n) = 1$ if n is odd.

For $n \in \mathbb{Z}$, we define e_n by $e_n(k_\theta) = e^{in\theta}$. Then $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(K, d\theta/2\pi)$ and e_0 is the constant function 1 on K.

Let $\mathfrak{a}$ be the Lie algebra of A, $\mathfrak{a}^*$ its dual, and $\mathfrak{a}^*_{\mathbb{C}}$ the complexification of the dual. Then $\mathfrak{a}^*_{\mathbb{C}}$ can be identified with $\mathbb{C}$ via $\lambda \leftrightarrow \lambda\rho$, where ρ is the half sum of the positive roots for the adjoint action of $\mathfrak{a}$.

For $\sigma \in \widehat{M} = \{0, 1\}$ and $\lambda \in \mathbb{C}$ we have the principal series representations $(\pi_{\sigma,\lambda}, H_\sigma)$ of $\mathrm{SL}_2(\mathbb{R})$ where H_σ is the subspace of $L^2(K)$ generated by the orthonormal basis $\{e_n \mid n \in \mathbb{Z}^\sigma\}$. Representations $\{\pi_{\sigma,\lambda} \mid (\sigma, \lambda) \in \widehat{M} \times \mathbb{C}\}$ are parametrized such that $\{\pi_{\sigma,\lambda} \mid \lambda \in i\mathbb{R}\}$ are unitary irreducible representations, except for $\pi_{1,0}$, which is unitary but not irreducible. For a detailed account on the action of the principal series representations and their reducibility we refer to [2, 4.1 and p. 16].

3. The Helgason Fourier transform on G

Let dx be a fixed Haar measure on G and $dk_\theta = d\theta/2\pi$ be a Haar measure on K such that $\int_K dk_\theta = 1$. Let $\mu(\sigma, \lambda)\, d\lambda, \lambda \in i\mathbb{R}$ be the Harish-Chandra's Plancherel measure restricted to the spherical and the nonspherical unitary principal series depending on whether σ is trivial or not ([2, 10.1]). It is well known that $\mu(\sigma, \lambda) = |c(\sigma, \lambda)|^{-2}$, where $c(\sigma, \lambda)$ is Harish-Chandra's c-function. Recall that for any $\lambda \in \mathbb{C}, \sigma \in \widehat{M}, m, n \in \mathbb{Z}^\sigma$, the matrix coefficients $\Phi^{m,n}_{\sigma,\lambda}(x) = \langle \pi_{\sigma,\lambda}(x)e_m, e_n\rangle$ of the principal series representation $\pi_{\sigma,\lambda}$ are eigenfunctions of the Casimir operator Ω with eigenvalue $(\lambda^2 - 1)/4$ ([2, p. 18]). Note that the elementary spherical function ϕ_λ is indeed $\Phi^{0,0}_{\sigma,\lambda}$, where $\sigma = 0$, that is, σ is the trivial representation of M. It also follow from the action of the principal series representation ([2, 4.1]) that $\phi_{-1} \equiv 1$. For $l \in \mathbb{Z}^*$, the discrete series representation π_l is infinitesimally equivalent to an infinite dimensional subrepresentation of $\pi_{\sigma,|l|}$ where σ is determined by $l \in \mathbb{Z}^{-\sigma}$. The matrix coefficient $\Psi^{r,s}_l(x)$ of π_l is up to a scaler factor $\Phi^{r,s}_{\sigma,|l|}(x)$ and is an eigenfunctions of Ω with eigenvalue $(l^2 - 1)/4$, where e_r, e_s are as follows: $r, s \in \mathbb{Z}^\sigma$, if $l > 0$ then $r, s > l$ and if $l < 0$ then $r, s < l$. It follows easily from the definition of the principal series representation ([2, p. 15]) that for any $\lambda \in \mathbb{C}, k \in K$ and $n \in \mathbb{Z}$, the functions

$$e_{\lambda,k,n} : x \mapsto e^{-(\lambda+1)H(x^{-1}k^{-1})} e_{-n}(K(x^{-1}k^{-1}))$$

$$\text{and} \quad e_{l,k,n} : x \mapsto e^{-(|l|+1)H(x^{-1}k^{-1})} e_{-n}(K(x^{-1}k^{-1}))$$

are again eigenfunctions of Ω with eigenvalues $(\lambda^2-1)/4$ and $(l^2-1)/4$ respectively. A function is said to be of right type n if $f(xk_\theta)=f(x)e^{in\theta}$ for all $x\in G$ and $k_\theta\in K$. One knows that for a function f of right type n, $\Omega f=\Delta_n f$, where Δ_n is an explicitly computable elliptic differential operator (see [17, p. 198]). It is easy to verify that $e_{\lambda,k,n}$ as a function on G is of right type n. Hence $\Omega e_{\lambda,k,n}=\Delta_n e_{\lambda,k,n}$ for all $\lambda\in\mathbb{C}$ and $k\in K$.

For $f\in C_c^\infty(G)$, $(\lambda,\sigma)\in\mathbb{C}\times\widehat{M}$ and $n\in\mathbb{Z}^\sigma$ we define

$$\widetilde{f}(\lambda,\sigma,k,n)=\int_G f(x)e^{-(\lambda+1)H(x^{-1}k^{-1})}e_n(K(x^{-1}k^{-1}))\,dx. \tag{3.1}$$

As n determines a unique $\sigma=\sigma(n)$ by requiring $n\in\mathbb{Z}^\sigma$, we sometimes omit σ as an argument of the Fourier transform and write $\widetilde{f}(\lambda,k,n)$ for $\widetilde{f}(\lambda,\sigma,k,n)$, if no confusion arises.

For $l\in\mathbb{Z}^*$ we define,

$$\mathbb{Z}(l)=\{n\in\mathbb{Z}^{-\sigma(l)}\mid n>l \text{ if } l>0 \text{ and } n<l \text{ if } l<0\}.$$

For two symbols $0_+,0_-$ we also define

$$\mathbb{Z}(0_+)=\{n\in 2\mathbb{Z}+1\mid n>0\} \text{ and } \mathbb{Z}(0_-)=\{n\in 2\mathbb{Z}+1\mid n<0\}.$$

For $n\in\mathbb{Z}^*$, let L_n be the set

$$\left\{\begin{array}{ll} l\in\mathbb{Z}^*\mid & e_n \text{ is in the unique irreducible subrepresentation of} \\ & \pi_{\sigma,|l|} \text{ infinitesimally equivalent to } \pi_l \end{array}\right\}.$$

Precisely,

$$L_n=\{l\in\mathbb{Z}^{-\sigma(n)}\mid n\in\mathbb{Z}(l)\}.$$

Note that for every $l\in\mathbb{Z}^*\sqcup\{0_+,0_-\}$, $\mathbb{Z}(l)$ is an infinite set while for every $n\in\mathbb{Z}^*$, L_n is a finite set. (Here $\sqcup$ denotes the *disjoint union*).

For $l\in\mathbb{Z}^*\sqcup\{0_+,0_-\}$, $n\in\mathbb{Z}(l)$ and $k\in K$ we define:

$$\widetilde{f}(l,k,n)=\int_G f(x)e^{-(|l|+1)H(x^{-1}k^{-1})}e_n(K(x^{-1}k^{-1}))\,dx, \tag{3.2}$$

where by $|0_+|$ or $|0_-|$ we mean 0.

For f as above, let f_n be the projection of f in the subspace of right n-type functions, that is, $f_n(x)=\int_K f(xk_\theta)e^{-in\theta}\,dk_\theta$. The function f has an unique decomposition in right-K-types as $f=\sum_n f_n$. In fact when $f\in C^\infty(G)$ then this is an absolutely convergent series in the C^∞-topology.

(When $f \in L^p(G)$, $p \in [1, \infty)$, then the equality is in the sense of distributions.) It can be verified that $\widetilde{f}(\lambda, k, n) = \widetilde{f_n}(\lambda, k, n)$, and if g is a function of right type $m \neq n$ then $\widetilde{g}(\cdot, \cdot, n) \equiv 0$, as a function of λ and k.

A function $f \in C_c^\infty(G)$ (or $L^1(G)$) is said to be of type (m, n) (or a (m, n)-type function) if

$$f(k_\theta x k_\phi) = f(x) e^{im\theta} e^{in\phi} \text{ for all } x \in G, k_\theta, k_\phi \in K.$$

For a function $f \in L^1(G)$, let $f_{m,n}$ be its projection in the subspace of (m, n)-type functions, that is, $f_{m,n} = \int_K \int_K f(k_\theta x k_\phi) e^{-im\theta} e^{-in\phi} \, dk_\theta \, dk_\phi$. It can be verified that $f_{m,n}$ itself is a function of type (m, n). As before f can be decomposed in the sense of distribution as $f = \sum_{m,n} f_{m,n}$.

For any function space $\mathcal{F}$ of G, $\mathcal{F}_n$ will denote its subspace of right n-type functions while $\mathcal{F}_{m,n}$ will denote the subspace of (m, n)-type functions.

For a function $f \in C_c^\infty(G)$, let $\widehat{f}(\sigma, \lambda)$ and $\widehat{f}(l)$ denote its (operator valued) principal and discrete Fourier transforms at the representations $\pi_{\sigma,\lambda}$ and π_l respectively. Precisely:

$$\widehat{f}(\sigma, \lambda) = \int_G f(x) \pi_{\sigma,\lambda}(x^{-1}) dx \text{ and } \widehat{f}(l) = \int_G f(x) \pi_l(x^{-1}) dx.$$

The (m, n)-th matrix entries of $\widehat{f}(\sigma, \lambda)$ and $\widehat{f}(l)$ are denoted by $\widehat{f}(\sigma, \lambda)_{m,n}$ and $\widehat{f}(l)_{m,n}$ respectively. Thus $\widehat{f}(\sigma, \lambda)_{m,n} = \langle \widehat{f}(\sigma, \lambda) e_m, e_n \rangle = \int_G f(x) \Phi_{\sigma,\lambda}^{m,n}(x^{-1}) dx$ and $\widehat{f}(l)_{m,n} = \int_G f(x) \Psi_l^{m,n}(x^{-1}) dx$. It is easy to verify that $\int_G f(x) \Phi_{\sigma,\lambda}^{m,n}(x^{-1}) dx = \int_G f_{m,n}(x) \Phi_{\sigma,\lambda}^{m,n}(x^{-1}) dx$, that is, $\widehat{f}(\sigma, \lambda)_{m,n} = \widehat{f}_{m,n}(\sigma, \lambda)$. Similarly $\widehat{f}(l)_{m,n} = \widehat{f}_{m,n}(l)$. Henceforth we will not distinguish between $\widehat{f}(\sigma, \lambda)_{m,n}$ $(\widehat{f}(l)_{m,n})$ and $\widehat{f}_{m,n}(\sigma, \lambda)$ (respectively, $\widehat{f}_{m,n}(k)$). As mentioned earlier that the integers m, n (of the same parity) uniquely determine a $\sigma \in \widehat{M}$ by $m, n \in \mathbb{Z}^\sigma$. Therefore we may sometimes omit the obvious σ and write $\Phi_\lambda^{m,n}$ for $\Phi_{\sigma,\lambda}^{m,n}$ and $\widehat{f}_{m,n}(\lambda)$ (or $\widehat{f}(\lambda)_{m,n}$) for $\widehat{f}_{m,n}(\sigma, \lambda)$.

Starting from the usual inversion formula:

$$f_n(x) = \frac{1}{4\pi^2} \int_{i\mathbb{R}} \text{Trace } (\pi_{\sigma,\lambda}(f_n) \pi_{\sigma,\lambda}(x)) \mu(\sigma(n), \lambda) \, d\lambda + \sum_{l \in L_n} \text{Trace}(\pi_l(f_n) \pi_l(x)) \frac{|l|}{2\pi},$$

one gets the inversion formula:

$$f_n(x) = \int_{i\mathbb{R} \sqcup L_n} \int_K \widetilde{f}_n(\lambda, k, n) e^{(\lambda - 1) H(x^{-1} k^{-1})} e_{-n}(K(x^{-1} k^{-1})) \, dk \, d\nu_n. \tag{3.3}$$

We also have the following Plancherel formula:

$$\int_G |f_n(x)|^2\,dx = \int_{i\mathbb{R}\sqcup L_n}\int_K |\widetilde{f_n}(\lambda,k,n)|^2\,dk\;d\nu_n. \tag{3.4}$$

Here $d\nu_n$ restricted to $i\mathbb{R}$ is $1/4\pi^2\;\mu(\sigma(n),\lambda)\,d\lambda$, $d\lambda$ being the Lebesgue measure and $d\nu_n$ restricted to L_n is the counting measure with weight $|l|/2\pi$ on $l\in L_n$. This measure is really the "Harish-Chandra Plancherel measure" in disguise. Using the decomposition of $f=\sum_n f_n$ and noting that $\widetilde{f}(\lambda,k,n)=\widetilde{f_n}(\lambda,k,n)$, we have respectively the inversion formula and the Plancherel formula for $f\in C_c^\infty(G)$:

$$f(x)=\sum_{n\in\widehat{K}}\int_{i\mathbb{R}\sqcup L_n}\int_K \widetilde{f}(\lambda,k,n)e^{(\lambda-1)H(x^{-1}k^{-1})}e_{-n}(K(x^{-1}k^{-1}))\,dk\;d\nu_n, \tag{3.5}$$

$$\int_G |f(x)|^2\,dx=\sum_{n\in\widehat{K}}\int_{i\mathbb{R}\sqcup L_n}\int_K |\widetilde{f}(\lambda,k,n)|^2\,dk\;d\nu_n. \tag{3.6}$$

(Recall that, in the above, $\sqcup$ is the disjoint union, and enters because of the discrete series representations.)

4. The Helgason Fourier Transform for L^1 functions on G

Let

$$\mathcal{S}_1=\{\lambda\in\mathbb{C}\mid |\Re\lambda|\le 1\}$$

be the Helgason–Johnson strip, where $\Re\lambda$ is the real part of $\lambda\in\mathbb{C}$. It is well known that the elementary spherical functions $\phi_\lambda,\lambda\in\mathbb{C}$ of Harish-Chandra ([11]), are bounded if and only if $\lambda\in\mathcal{S}_1$ ([15]). In fact $\|\phi_\lambda\|_\infty\le 1$ for λ in that strip. Recall that ϕ_λ is nothing but $\Phi^{0,0}_{0,\lambda}$. By comparing with ϕ_λ it can be shown easily that for any $m,n\in\mathbb{Z}^\sigma$, the (m,n)-th matrix coefficient of the principal series representation $\pi_{\sigma,\lambda}$, $\Phi^{m,n}_{\sigma,\lambda}$ is bounded if $\lambda\in\mathcal{S}_1$.

For L^1 functions on the symmetric space $X=G/K$, the *natural domain* of definition of the Helgason Fourier transform is the strip $\mathcal{S}_1$ ([16]). In [25, 18] we made use of these facts.

Using argument similar to that in [13, 14, 18], where proofs were provided for the Helgason Fourier transform on symmetric spaces, we can prove most of the following:

Theorem 4.1. *Let f be a function in $L^1(G)$. Then there exists a subset $B_f = B$ of K of full Haar measure (depending on f), such that*

(i) $\widetilde{f}(\lambda, k, n)$ *exists for all* $k \in B$, $n \in \mathbb{Z}$ *and* $\lambda \in \mathcal{S}_1$,

(ii) $\widetilde{f}(l, k, n)$ *exists for all* $k \in B, n \in \mathbb{Z}(l)$ *and* $l \in \mathbb{Z}^* \sqcup \{0_-, 0_+\}$,

(iii) $\widetilde{f}(\lambda, \cdot, n) \in L^1(K)$ *and* $\|\widetilde{f}(\lambda, \cdot, n)\|_{L^1(K)} \leq C\|f\|_{L^1(G)}$, *for all* $\lambda \in \mathcal{S}_1$ *and* $n \in \mathbb{Z}$ *(that is, the constant* C *is independent of* n *and* λ *),*

(iv) *for each fixed* $k \in B$ *and* $n \in \mathbb{Z}$, $\lambda \mapsto \widetilde{f}(\lambda, k, n)$ *is holomorphic on* $\mathcal{S}_1^\circ$ *and continuous on* $\mathcal{S}_1$, *where* $\mathcal{S}_1^\circ$ *is the interior of* $\mathcal{S}_1$,

(v) $\|\widetilde{f}(\lambda, \cdot, n)\|_{L^1(K)} \longrightarrow 0$ *as* $|\lambda| \longrightarrow \infty$, *uniformly in* $\lambda \in \mathcal{S}_1$ *and in* $n \in \mathbb{Z}^\sigma$,

(vi) *for* $\lambda \in i\mathbb{R}$,

$$\widetilde{f}(\lambda, \cdot, n) \in L^2(K) \text{ and } ||\widetilde{f}(\lambda, \cdot, n)||_{L^2(K)} \leq ||f||_{L^1(G)}.$$

Result (v) can be considered as a Riemann–Lebesgue Lemma (contrast with [9, Theorem 5]). For another variant of this, see Section 8. We only need to justify (ii), that is, the definition of $\widetilde{f}(l, k, n)$ given in (3.2) makes sense for $n \in \mathbb{Z}(l)$ and $f \in L^1(G)$.

Suppose that $f \in L^1(G)$. We recall that for any $l \in \mathbb{Z}^*$, π_l is (infinitesimally equivalent to) a subrepresentation of $\pi_{\sigma,|l|}$, where $l \in \mathbb{Z}^{-\sigma}$. The carrier space H_l of π_l is the closed subspace of $L^2(K)$ generated by $\{e_n | n \in \mathbb{Z}(l)\}$. With respect to a suitably chosen new inner product π_l is a unitary representation and therefore $\pi_l(f)$ exists as a bounded operator on H_l. We take an element $n \in \mathbb{Z}(l)$. Then $-n \in \mathbb{Z}(-l)$. We define,

$$\widetilde{f}(l, k, n) = (\pi_{-l}(f)e_{-n})(k) = \int_G (f(x)\pi_{-l}(x)e_{-n})(k)dx = \int_G (f(x)\pi_{\sigma,|l|}(x)e_{-n})(k)\, dx.$$

(Note that although the inner product is different, the G-action is the same and hence the usual definition of $\pi_l(f)$ as a bounded linear operator used above makes sense.) This is the same as the definition (3.2) of $\widetilde{f}(l, k, n)$. This also shows that $\widetilde{f}(l, k, n)$ exists for almost every $k \in K$. Note that for $l \in \{0_-, 0_+\}$, π_l is a subrepresentation of a principal series representation $\pi_{1,0}$. But as $0 \in \mathcal{S}_1$, existence of $\widetilde{f}(l, k, n), l \in \{0_-, 0_+\}$ for almost every k follows automatically from the argument given for $\lambda \in \mathcal{S}_1$.

For $f \in L^1(G)$, $\int_K \widetilde{f}(l,k,n)e_m(k)\,dk = 0$ for $l \in \mathbb{Z}(n) \sqcup \{0_+, 0_-\}$ and $m \notin \mathbb{Z}(l)$. In particular $\int_K \widetilde{f}(l,k,n)\,dk = 0$, since $0 \notin \mathbb{Z}(l)$ for any $l \in \mathbb{Z}^* \sqcup \{0_+, 0_+\}$.

Formally, as noted earlier, the Fourier inversion formula gives us

$$\begin{aligned} f(x) \quad &\sim \quad \tfrac{1}{4\pi^2}\textstyle\sum_{n\in\widehat{K}}\int_{i\mathbb{R}}\int_K \widetilde{f}(\lambda,k,n)e^{(\lambda-1)H(x^{-1}k^{-1})}e_{-n}(K(x^{-1}k^{-1}))\,dk\;\mu(\sigma(n),\lambda)\,d\lambda \\ &+ \quad \tfrac{|l|}{2\pi}\textstyle\sum_{n\in\widehat{K}}\sum_{l\in L_n}\int_K \widetilde{f}(l,k,n)e^{(l-1)H(x^{-1}k^{-1})}e_{-n}(K(x^{-1}k^{-1}))\,dk. \end{aligned}$$

Denoting the first term and the second term of right hand side of the above by f_P and f_D respectively, it is clear that $\widetilde{f_P}(\lambda,k,n) = \widetilde{f}(\lambda,k,n)$ for $\lambda \in i\mathbb{R}, n \in \mathbb{Z}$ and for almost every $k \in K$; and $\widetilde{f_D}(l,k,n) = \widetilde{f}(l,k,n)$ for $l \in \mathbb{Z}^*, n \in \mathbb{Z}(l)$, and for almost every $k \in K$.

As in the case of the Helgason Fourier transform for symmetric spaces, we now have the following inversion formula for right K-finite functions on the group:

Theorem 4.2. *Let f be a right K-finite function in $L^1(G)$. If for all n, $\widetilde{f}(\cdot,\cdot,n) \in L^1(i\mathbb{R} \times K, \mu(\sigma(n),\lambda)\,d\lambda\,dk)$, then,*

$$f(x) = \sum_n \int_{i\mathbb{R}\sqcup L_n}\int_K \widetilde{f}(\lambda,k,n)e^{(\lambda-1)H(x^{-1}k^{-1})}e_{-n}(K(x^{-}k^{-1})\,dk\,d\nu_n,$$

for almost every $x \in G$, in particular for all Lebesgue points of f.

Remark 4.3. Note that for a right K-finite function, the sum on the right hand side of the equation above is only a finite sum. See the discussion preceding (3.3). In [18] to prove the analogue of the theorem above for symmetric spaces, we appeal to some old results of [27]. Such results can also be developed for $f * \Phi_\lambda^{n,n}(x)$ and the proof of Theorem 4.2 follows in a similar way.

5. Benedicks' Theorem

If f is an L^2-function on $\mathbb{R}^n$, Benedicks [3] proved that if both f and $\widehat{f}$ are zero almost everywhere outside sets of finite measure, then $f = 0$ almost everywhere. The exact analogue of this for the group theoretic Fourier transform for a noncommutative connected Lie group is still open, although some partial results are known (see [20, 6, 8]). In this paper we offer an analogue of Benedicks' Theorem for the Helgason Fourier transform on $\mathrm{SL}_2(\mathbb{R})$. These are related to the results in [20, 6]. Benedicks' Theorem can be viewed as a *qualitative* uncertainty principle in harmonic analysis, which asserts that both a function and its Fourier transform cannot be simultaneously concentrated.

Let m be a fixed left invariant Haar measure on G. Let us define the measure ϖ on $(i\mathbb{R}\sqcup\mathbb{Z}^*)\times K\times\mathbb{Z}$ as:

$$\begin{aligned} d\varpi(\lambda,k,n) &= 1/4\pi^2\ \mu(\sigma(n),\lambda)\,d\lambda\,dk, && (\lambda,k,n)\in i\mathbb{R}\times K\times\mathbb{Z},\\ d\varpi(l,k,n) &= |l|/2\pi\,dk, && (l,k,n)\in\mathbb{Z}^*\times K\times\mathbb{Z} \text{ and } n\in\mathbb{Z}(l)\\ &= 0, && (l,k,n)\in\mathbb{Z}^*\times K\times\mathbb{Z} \text{ and } n\notin\mathbb{Z}(l). \end{aligned}$$

Note that this measure is a slight variant of the Harish-Chandra's Plancherel measure used in [6].

Theorem 5.1. *Let f be a function in $L^2(G)$. If $m\{x\in G|f(x)\neq 0\}<\infty$ and*

$$\varpi\{(\lambda,k,n)\in(i\mathbb{R}\sqcup\mathbb{Z}^*)\times K\times\mathbb{Z}\,|\,\widetilde{f}(\lambda,k,n)\neq 0\}<\infty,$$

then $f=0$ almost everywhere.

Proof. As f is L^2 and supported on a set of finite measure , $f\in L^1(G)$.

Therefore for every fixed $n\in\mathbb{Z}$ and $k\in B$, where B is a set of full Haar measure in K, $\widetilde{f}(\lambda,k,n)$ is analytic in λ on $\mathcal{S}_1^\circ$. The given condition on $\widetilde{f}(\lambda,k,n)$ implies that for fixed $n\in\mathbb{Z}$ and almost every fixed $k\in K$, $\widetilde{f}(\lambda,k,n)=0$ for all $\lambda\in i\mathbb{R}$.

Suppose that

$$h(x)=\sum_{n\in\mathbb{Z}_F}\sum_{l\in L_n}\frac{|l|}{2\pi}\int_K\widetilde{f}(l,k,n)e^{-(l+1)H(x^{-1}k^{-1})}e_{-n}(K(x^{-1}k^{-1})\,dk,$$

where $\mathbb{Z}_F=\{n\in\mathbb{Z}\mid\widetilde{f}(\cdot,\cdot,n)\not\equiv 0\}$ is a finite subset of $\mathbb{Z}$.

As $e_{l,k,n}\ :\ x\ \mapsto\ e^{-(l+1)H(x^{-1}k^{-1})}e_{-n}(K(x^{-1}k^{-1})$ is an eigenfunction of the elliptic operator Δ_n, it is a real analytic function on G and hence $\int_K\widetilde{f}(l,k,n)e^{-(l+1)H(x^{-1}k^{-1})}e_{-n}(K(x^{-1}k^{-1})$ is real analytic on G. As $\mathbb{Z}_F$ is a finite set and for a fixed n, L_n is also finite, $h(x)=\sum_{n\in\mathbb{Z}_F}\sum_{l\in L_n}\int_K\widetilde{f}(l,k,n)e^{-(l+1)H(x^{-1}k^{-1})}e_{-n}(K(x^{-1}k^{-1})\,dk$ is real analytic in $x\in G$. Now consider the two functions $f(x)$ and $h(x)$. If f is a reasonably nice function then inversion formula holds pointwise and the two functions are identical. But since f is only assumed to be in $L^1\cap L^2$, the two functions will agree at least in the sense of distributions. But we have established that h is real analytic. As f and h are the same as distributions, they must also be equal almost everywhere. Hence f which is equal to a real analytic function almost everywhere cannot have the property $m\{x\in G|f(x)\neq 0\}<\infty$, unless it is the zero function. $\square$

In connection with the theorem above, we remark here that a careful examination of the proof in [6] leads to Theorem 5.2 below.

We call a set $E_1 \subset G$ right K-invariant if for all $x \in E_1$ and $k \in K$, $xk \in E_1$.

Theorem 5.2. *Let E_1 be a right K-invariant subset of G of positive Haar measure and E_2 be a subset of $i\mathbb{R}$ of positive Plancherel measure. If a function $f \in L^1(G) \cap L^2(G)$ is zero on E_1 while $\widetilde{f}(\lambda, \cdot, \cdot) \equiv 0$ for every $\lambda \in E_2$ then $f = 0$ almost everywhere.*

Proof. Fix a right K-type $n \in \mathbb{Z}$. As E_1 is right K-invariant, the right n-th component f_n of f is zero on E_1.

For almost every fixed k and every fixed n, $\widetilde{f}(\lambda, k, n)$ is an analytic function on $\mathcal{S}_1$. Therefore $\widetilde{f}(\lambda, k, n)$ is identically zero on $\mathcal{S}_1 \times K \times \mathbb{Z}$, that is, $f_{nP} = 0$. There are only finitely many discrete series relevant for a right n-type function. Therefore we can show as in the previous theorem that f_{nD} is analytic, since it involves only a finite sum of analytic functions. Thus f_n is equal to an analytic function almost everywhere, which contradicts the fact that f_n is zero on E_1, unless $f_n = 0$ almost everywhere. As $n \in \mathbb{Z}$ is arbitrary $f = 0$ almost everywhere. $\square$

Remark 5.3. For other groups like nilpotent Lie groups etc., the analogue of this result has been proved (see [8]).

6. Wiener–Tauberian Theorem

As noted earlier that the $\Phi_\lambda^{m,n}$ as well as the $e_{\lambda,k,n}$ are eigenfunctions of the Casimir Ω of G with eigenvalue $(\lambda^2 - 1)/4$. Thus, from the (formal) self adjointness of Ω, for sufficiently nice f,

$$(\Omega f)\widehat{\ }_{m,n}(\lambda) = \frac{(\lambda^2 - 1)}{4} \widehat{f}_{m,n}(\lambda) \qquad \text{and} \qquad (\Omega f)\widetilde{\ }(\lambda, \cdot, n) = \frac{(\lambda^2 - 1)}{4} \widetilde{f}(\lambda, \cdot, n). \tag{6.1}$$

Here $\widehat{f}_{m,n}(\lambda)$ is $\int_G f(x) \Phi_\lambda^{m,n}(x^{-1})\, dx$.

We have also noted that if $f \in L^1(G)$ and $\lambda \in i\mathbb{R}$, then $\widetilde{f}(\lambda, \cdot, n) \in L^2(K)$. For each fixed $\lambda \in i\mathbb{R}$, we have the Fourier series:

$$\widetilde{f}(\lambda, \cdot, n) =_{L^2} \sum_{i \in \mathbb{Z}^{\sigma(n)}} \widehat{f}_{i,n}(\lambda) e_i \qquad \text{and} \qquad \|\widetilde{f}(\lambda, \cdot, n)\|^2_{L^2(K)} = \sum_{i \in \mathbb{Z}^{\sigma(n)}} |\widehat{f}_{i,n}(\lambda)|^2. \tag{6.2}$$

For some $\epsilon > 0$, let $\mathcal{T}_\epsilon$ be the strip $\mathcal{T}_\epsilon = \{\lambda \in \mathbb{C} \mid |\Re\lambda| \leq 1+\epsilon\}$ and $\mathcal{T}_\epsilon^\circ$ be its interior. Let us define $\mathcal{L}_\epsilon(G)$ to be the set of measurable functions on G such that $\int_G |f(x)| e^{\epsilon d(xK,K)}\, dx < \infty$, where d is the canonical distance function for G/K coming from the Riemannian structure induced by the Cartan–Killing form restricted to $\mathfrak{p}$. Here $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{p}$ can be identified with tangent space at eK of G/K. Then,

(i) $C_c^\infty(G) \subset \mathcal{L}_\epsilon(G) \subset L^1(G)$, and hence $\mathcal{L}_\epsilon(G)$ is a dense subset of $L^1(G)$.

(ii) for $f \in \mathcal{L}_\epsilon(G)$, $\widetilde{f}(\lambda, k, n)$ exists for all $(\lambda, k, n) \in \mathcal{T}_\epsilon \times B_f \times \mathbb{Z}$, is holomorphic in $\mathcal{T}_\epsilon^\circ$ and continuous in $\mathcal{T}_\epsilon$ in λ, for each $k \in B_f$ and $n \in \mathbb{Z}^*$, where $B_f \subset K$ is a set of full Haar measure, obtained in a fashion similar to Theorem 4.1. (One needs to use some simple estimate of the elementary spherical function ([11, Proposition 4.6.1 and Theorem 4.6.4]) along with Fubini's Theorem.)

(iii) for f as above and $n \in \mathbb{Z}$, $\widetilde{f}(\lambda, \cdot, n)$ is in $L^1(K)$ for $\lambda \in \mathcal{T}_\epsilon$ and $\|\widetilde{f}(\lambda, \cdot, n)\|_{L^1(K)} \longrightarrow 0$ as $|\lambda| \longrightarrow \infty$, uniformly in $\mathcal{T}_\epsilon$.

(iv) if $\epsilon = 0$, then $\mathcal{L}_\epsilon(G) = L^1(G)$ and $\mathcal{T}_\epsilon$ reduces to $\mathcal{S}_1$.

The following analogue of the Wiener–Tauberian Theorem for (n,n)-type L^1-functions of G was proved in [22] extending a result of [4], where the result was proved for $n = 0$:

Theorem 6.1. *Fix an integer n. Let $\mathcal{F}$ be a subset of $L^1(G)_{n,n}$. Suppose that for some $\epsilon > 0$, the (n,n)-th Fourier transform $\widehat{f}_{n,n}$ of every $f \in \mathcal{F}$ can be extended holomorphically on an augmented strip $\mathcal{T}_\epsilon = \{\lambda \in \mathbb{C} \mid |\Re\lambda| \leq 1+\epsilon\}$ and $\lim_{|\lambda| \longrightarrow \infty} |\widehat{f}(\lambda)_{n,n}| = 0$ on $\mathcal{T}_\epsilon$. Assume further that $\{\widehat{f}_{n,n} \mid f \in \mathcal{F}\}$ does not have a common zero on $\mathcal{T}_\epsilon \sqcup L_n$ and that there exists $f^0 \in \mathcal{F}$ which satisfies*

$$\limsup_{|t| \longrightarrow \infty} |\widehat{f^0}(t)_{n,n}| e^{\alpha e^{|t|}} > 0 \textit{ for all } \alpha > 0.$$

Then the ideal generated by $\mathcal{F}$ in $L^1(G)_{n,n}$ is dense in $L^1(G)_{n,n}$.

With this preparation we now offer the following versions of the Wiener–Tauberian Theorem for G.

Theorem 6.2. *Let $\mathcal{F}$ be a subset of $\mathcal{L}_\epsilon(G)$ for some $\varepsilon > 0$. Suppose that*

$$Z_1 = \{(\sigma, \lambda) \in \{0,1\} \times \mathcal{T}_\epsilon \mid \widetilde{f}(\sigma, \lambda, \cdot, \cdot) \equiv 0 \textit{ for all } f \in \mathcal{F}\}$$

$$\textit{and} \quad Z_2 = \{l \in \mathbb{Z}^* \sqcup \{0_-, 0_+\} \mid \widetilde{f}(l, \cdot, \cdot) \equiv 0 \textit{ for all } f \in \mathcal{F}\}.$$

If $Z_1 \sqcup Z_2$ is empty, $\int_G f(x)\,dx \neq 0$ for some $f \in \mathcal{F}$ and there exist $f^j \in \mathcal{F}, j = 0, 1$ such that for some $n \in \mathbb{Z}^j, j = 0, 1$

$$\limsup_{\lambda \in i\mathbb{R}, |\lambda| \longrightarrow \infty} \|\widetilde{f^j}(\lambda, \cdot, n)\|^2_{L^2(K)} e^{\alpha e^{|\lambda|}} > 0, \text{ for all } \alpha > 0, \tag{6.3}$$

then the (two-sided) G-translates of the functions in $\mathcal{F}$ span a dense subspace of $L^1(G)$.

Remark 6.3. (i) $\int_G f(x)\,dx \neq 0$ is really the condition $\widetilde{f}(-1, \cdot, 0) \not\equiv 0$ as a function of $k \in K$.

(ii) In [22, 26], it was assumed that the Fourier transforms of the functions in $\mathcal{F}$ exist in an augmented strip. This essentially amounts to demanding that the functions are in a suitable weighted L^1-space.

(iii) The condition (6.3) says that $\widetilde{f}$ does not go to zero *too rapidly* at infinity.

First a few remarks about the necessity of the conditions: "nonvanishing on the Helgason–Johnson strip", Z_2 being empty and $\int_G f(x)\,dx \neq 0$.

For $f \in L^1(G)$, let W be the closed span of the (two-sided) G-translates of f. Suppose that for some $\sigma \in \widehat{M}$, $\widetilde{f}(\sigma, \lambda, \cdot, \cdot) \equiv 0$ on $K \times \mathbb{Z}^\sigma$ for some $\lambda \in \mathcal{S}_1$. That is,

$$\int_G f(x) e^{(\lambda+1)H(x^{-1}k^{-1})} e_n(K(x^{-1}k^{-1})\,dx = 0$$

for all $k \in K$ and $n \in \mathbb{Z}^\sigma$.

We use the following *symmetry property* of the matrix coefficients of the principal series representations: for $n \in \mathbb{Z}^\sigma, x, y \in G, \lambda \in \mathbb{C}$,

$$\begin{aligned}\Phi^{n,n}_{\sigma,\lambda}(y^{-1}x) &= \langle \pi_\lambda(x)e_n, \pi_{-\bar{\lambda}}(y)e_n\rangle \text{ (as } \langle \pi_{\sigma,\lambda}(x^{-1})e_m, e_n\rangle = \langle e_m, \pi_{\sigma,-\bar{\lambda}}(x)e_n\rangle \text{ [2, p. 16])} \\ &= \textstyle\int_K e^{-(\lambda+1)H(x^{-1}k^{-1})} e_{-n}(K(x^{-1}k^{-1})) e^{-(-\lambda+1)H(y^{-1}k^{-1})} e_n(K(y^{-1}k^{-1}))\,dk.\end{aligned}$$

Using this one can show that depending on σ is trivial or not, *either* $f * \phi_\lambda \equiv 0$ as well as $\phi_\lambda * f \equiv 0$ *or* $f * \Phi^{1,1}_{1,\lambda} \equiv 0$ as well as $\Phi^{1,1}_{1,\lambda} * f \equiv 0$.

For example if σ is trivial:

$$\int_G {}^y f(x)\phi_\lambda(x^{-1})\,dx = 0 \text{ and } \int_G f^y(x)\phi_\lambda(x^{-1})\,dx = 0, \text{ for all } y \in G,$$

where ${}^y f$ and f^y are respectively the left and right translates of f by $y \in G$. This amounts to saying that $\int_G g(x)\phi_\lambda(x^{-1})\,dx = 0$ for all $g \in W$, since W is the smallest closed two-sided translation invariant subspace containing all ${}^y f$ and f^y. But ϕ_λ is bounded when $\lambda \in \mathcal{S}_1$, so it defines a linear functional on

$L^1(G)$. Therefore W is a proper subspace of $L^1(G)$, and we have therefore proved the necessity of the nonvanishing condition on the Helgason–Johnson strip $\mathcal{S}_1$. The necessity of the nonvanishing condition on the discrete series can be established by an analogous argument as the matrix coefficients $\Psi_l^{m,n}$ are also bounded and hence define linear functional on $L^1(G)$.

For $\mathrm{SL}_2(\mathbb{R})$, the necessity of some kind of *not too rapidly decreasing condition* on the Fourier transform was established in [10, Lemma 7.13 and Theorem 7.2] (see also the comments after [9, Proposition 5.1]).

Proof. For a function h in $\mathcal{L}_\epsilon(G)$ we define h^* by $h^*(x) = \overline{h(x^{-1})}$. If h is of left K-type n then it is not hard to show that $h^* * h$, is of (n,n)-type and is also in $\mathcal{L}_\epsilon(G)$.

Suppose that $g = f_n^{0*} * f_n^0$ for $n \in \mathbb{Z}^0$. Then g is a (n,n)-type function in $\mathcal{L}_\epsilon(G)$ which is in the two-sided $L^1(G)$-module generated by f^0.

For every $\lambda \in \mathbb{C}$ for which $\widehat{f^0}$ can be defined, we have, $\widehat{f^{0*}}_{i,n}(-\overline{\lambda}) = \overline{\widehat{f^0}_{i,n}(\lambda)}$, since

$$\langle \pi_{\sigma,\lambda}(x^{-1})e_m, e_n\rangle = \langle e_m, \pi_{\sigma,-\bar{\lambda}}(x)e_n\rangle \text{ ([2, p. 16]).}$$

As noted earlier, for $\lambda \in i\mathbb{R}$,

$$\widetilde{f^0}(\lambda,\cdot,n) = \widetilde{f^0_n}(\lambda,\cdot,n) =_{L^2} \sum_i \widehat{f^0}_{i,n}(\lambda)e_i \qquad \text{and} \qquad \|\widetilde{f^0}(\lambda,\cdot,n)\|^2_{L^2(K)} = \sum_i |\widehat{f^0}_{i,n}(\lambda)|^2.$$

On the other hand,

$$\widehat{g}(\lambda)_{n,n} = \sum_i |\widehat{f^0}_{i,n}(\lambda)|^2 \text{ for all } \lambda \in i\mathbb{R}.$$

Therefore g satisfies for $n \in \mathbb{Z}^0$

$$\limsup_{\lambda\in i\mathbb{R}, |\lambda|\longrightarrow\infty} |\widehat{g}(\lambda)_{n,n}|e^{\alpha e^{|\lambda|}} > 0 \text{ for all } \alpha > 0. \tag{6.4}$$

If we work with f^1 instead of f^0 then we get another function say g' which satisfies the inequality above for some $n \in \mathbb{Z}^1$.

The following is essentially proved in [22]. Fix an integer n. Then n determines a $\sigma \in \{0,1\}$ by $n \in \mathbb{Z}^\sigma$. Given $\lambda \in \mathcal{T}_\epsilon$ and $f \in \mathcal{F}$ with $\widetilde{f}(\sigma,\lambda,\cdot,\cdot) \not\equiv 0$, there exists a (n,n)-type function $f_\lambda \in \mathcal{L}_\epsilon(G)$ in the two-sided $L^1(G)$-module generated by f with $\widehat{f_\lambda}(\lambda)_{n,n} \neq 0$ except for the following cases:

[i] $(\sigma,\lambda) = (0,-1)$ and $n = 0$,

[ii] $(\sigma,\lambda) = (1,0)$ and $n \in 2\mathbb{Z}+1$,

[iii] $(\sigma, \lambda) = (0, 1)$ and $n \in 2\mathbb{Z}$, $n \neq 0$.

For these we have the following remedy. For [i] we get a function f_{-1} such that $\widehat{f_{-1}}(-1)_{0,0} \neq 0$ in the $L^1(G)$-module generated by the function $f \in \mathcal{F}$ such that $\int_G f(x)\, dx \neq 0$. For [ii] a function f_0 such that $\widehat{f_0}(1)_{n,n} \neq 0$ can be found in the L^1-module generated by the function $f \in \mathcal{F}$ such that $\widetilde{f_0}(0_+, \cdot, \cdot) \neq 0$ $(\widetilde{f}(0_-, \cdot, \cdot) \neq 0)$ if $n > 0$ (respectively, $n < 0$). For [iii] a function f_1 such that $\widehat{f_1}(1)_{n,n} \neq 0$ can be found in the L^1-module generated by the function $f \in \mathcal{F}$ such that the discrete Fourier transform $\widetilde{f_1}(1, \cdot, \cdot) \neq 0$ $(\widetilde{f}(-1, \cdot, \cdot) \neq 0)$ if $n > 0$ (respectively, $n < 0$).

We also note that for a function f of type (n, n), $\int_K \widetilde{f}(\lambda, k, n) e_{-n}(k)\, dk = \widehat{f}_{n,n}(\lambda)$. Therefore by (iii) in the discussion preceding Theorem 6.1, $\widehat{f_\lambda}(\nu)_{n,n} \longrightarrow 0$ as $|\nu| \longrightarrow \infty$ uniformly for $\nu \in \mathcal{T}_\epsilon$.

Consequently, the family $\{f_\lambda\} \cup \{g\}$ satisfies the conditions of Theorem 6.1, and so the $L^1(G)_{n,n}$-module generated by the family above is dense in $L^1(G)_{n,n}$. Since $\{f_\lambda\}$ and g are contained in the $L^1(G)$-module generated by $\mathcal{F}$ in $L^1(G)$, it follows that the closed span of the two-sided G-translates of the functions in $\mathcal{F}$ contains $L^1(G)_{n,n}$ for all $n \in \mathbb{Z}$.

For every $t > 0$, let us define a (n, n)-type functions h_t by the data $\widehat{h_t}(\lambda)_{n,n} = e^{t\lambda^2}$ for $\lambda \in \mathbb{C}$ and $\widehat{h_t}(\lambda)_{r,s} = 0$ for $r \neq n$ or $s \neq n$. In view of the *embedding* of the discrete series in the principal series, the values of $\widehat{h}_{t n,n}$ at the relevant discrete series representations are automatically determined. Then $h_t \in C^p(G)$, the Harish-Chandra Schwartz space (see [2, p. 13]), for every $p \in [0, 2]$. It can be shown, as in the K-biinvariant case, that for any function $f \in L^1(G)$ of right type n, $f * h_t \longrightarrow f$ in L^1 as $t \longrightarrow 0$ (see [24]). This shows that the closed span of the two-sided G-translates of $\mathcal{F}$ contains $L^1(G)_n$ for all $n \in \mathbb{Z}$. As the smallest such closed subspace of $L^1(G)$ is $L^1(G)$ itself, the theorem is proved. □

Remark 6.4. In an attempt to understand the nature of functions which satisfy the growth condition (6.3), we see that if a function F in $\mathcal{F}$ is in $C_c^\infty(G)$, then a Phragmén–Lindelöff argument combined with the Paley–Wiener Theorem guarantees that it satisfies the required not-too-rapidly-decreasing condition. We will see below that if $|F(x)| \leq Ce^{-\alpha d(xK,K)^2}$, then we can apply an analogue of Hardy's uncertainty principle due to Cowling et. al. ([7]) to get the same conclusion about F. We also have an alternative version where the growth condition on the Fourier transform is substituted by a condition requiring that the collection $\mathcal{F}$ has functions which are not "too smooth".

Theorem 6.5. *Let $\mathcal{F}$ be a subset of $\mathcal{L}_\epsilon(G)$, for some $\varepsilon > 0$. Let the sets Z_1 and Z_2 be as in Theorem 6.2. If $Z_1 \sqcup Z_2$ is empty, $\int_G f(x)\,dx \neq 0$ for some function $f \in \mathcal{F}$ and if there exist functions F, H in $\mathcal{F}$ with nonzero even and odd part respectively such that one of the following conditions is satisfied:*

(a) $|F(x)| \leq Ce^{-\alpha d(xK,K)^2}$ *and* $|H(x)| \leq Ce^{-\alpha d(xK,K)^2}$ *for some positive constants α and C,*

(b) *one of the right K-finite components of the even (odd) part of F (respectively, H) is not equal to a real analytic function almost everywhere,*

(c) *a right K-finite component of the even (odd) part of F (respectively, H) is zero on a set E_1 (respectively, E_2) of positive measure.*

(d) *the even part of F and odd part of H are zero on right K-invariant sets of positive measures,*

then the two-sided G-translates of the functions in $\mathcal{F}$ span a dense subspace of $L^1(G)$.

Proof. If we show that F and H satisfy condition (6.3), then this theorem will follow from Theorem 6.2. (a) As F has nonzero even component, $F_{m,n} \not\equiv 0$ for some $m, n \in 2\mathbb{Z}$. Since $d(xK,K)$ is K-biinvariant, $|F_{m,n}(x)| \leq Ce^{-\alpha d(xK,K)^2}$. Now we take $\beta = 1/3\alpha$. Then it follows that

$$\limsup_{|\lambda| \longrightarrow \infty, \lambda \in i\mathbb{R}} |\widehat{F}_{m,n}(\lambda) e^{\beta|\lambda|^2}| > 0,$$

because assuming that the right hand side equal to 0 will lead to the conclusion that $F_{m,n}$ and $\widehat{F}_{m,n}$ both are *very rapidly decreasing*, which in turn will lead to the contradiction that $F_{m,n} \equiv 0$ by the analogue of Hardy's Theorem proved in [7]. This contradicts our hypothesis that F has nonzero even component.

Since $\|\widetilde{F}(\lambda, \cdot, n)\|^2_{L^2(K)} = \sum_{i \in \mathbb{Z}^{\sigma(n)}} |\widehat{F}_{i,n}(\lambda)|^2$, condition (6.3) will be guaranteed for F. A similar argument works for H.

(b) Suppose that the right n-th component of F, F_n is not equal to a real analytic function almost everywhere for some $n \in \mathbb{Z}$. If F does not satisfy (6.3), then

$$\|\widetilde{F}(\lambda, \cdot, n)\|^2_{L^2(K)} \leq Ce^{-\alpha e^{|\lambda|}} \text{ for all } \lambda \in i\mathbb{R} \tag{6.5}$$

for some constant C and some $\alpha > 0$. Then by appealing to Plancherel Theorem (3.6) and by observing that the Plancherel measure $\mu(\sigma, \lambda)$ is at most of polynomial growth on $i\mathbb{R}$ (see (6.7) below), the principal part of F_n, namely F_{nP} is clearly in $L^2(G)$.

By remarks made earlier, if g is a sufficiently nice function, for example in $C_c^\infty(G)$, then

$$(\Delta g)\tilde{}(\lambda, \cdot, n) = \frac{(\lambda^2 - 1)}{4} \widetilde{g}(\lambda, \cdot, n).$$

We consider ΩF_n in the sense of distributions. If h is an L^2-function on G of fixed right-type n, such that $(\lambda^2 - 1)/4\ \widetilde{h}(\lambda, k, n)$ is also in $L^2(\mathfrak{a}^* \times K, \mu(\lambda)\, d\lambda\, dk)$, then it is not hard to show that $\Delta_n h$, which is a priori only defined as a distribution, is actually in $L^2(G)$ and $(\Delta_n h)\tilde{}(\lambda, \cdot, n) = (\lambda^2 - 1)/4\ \widetilde{h}(\lambda, \cdot, n)$.

Now $\widetilde{F_n}(\lambda, \cdot, n)$ is very rapidly decreasing in λ, so $\Delta_n F_{nP} \in L^2(G)$ and

$$(\Delta_n F_{nP})\tilde{}(\lambda, \cdot, n) = \frac{(\lambda^2 - 1)}{4} \widetilde{F_{nP}}(\lambda, \cdot, n).$$

By repeated application of the argument above, we see that $\Delta_n^m F_{nP}$ is in $L^2(G)$ for any positive integer m and

$$(\Delta_n^m F_{nP})\tilde{}(\lambda, \cdot, n) = \left(\frac{\lambda^2 - 1}{4}\right)^m \widetilde{F_{nP}}(\lambda, \cdot, n). \tag{6.6}$$

As Δ_n is elliptic, Sobolev theory implies that F_{nP} can be taken to be C^∞ (that is, it is equal almost everywhere to a C^∞-function).

From the Plancherel Theorem (3.6), we have, for all positive integers m,

$$\|\Delta_n^m F_{nP}\|_2^2 = \frac{1}{4\pi^2} \int_{i\mathbb{R}} \|(\Delta_n^m F_{nP})\tilde{}(\lambda, \cdot, n)\|_{L^2(K)}^2\, \mu(\sigma(n), \lambda)\, d\lambda.$$

Using (6.5) and (6.6), we see that

$$\|\Delta_n^m F_{nP}\|_2^2 \le C \int_0^\infty \left(\frac{\lambda^2 - 1}{4}\right)^{2m} e^{-\alpha e^{\lambda}}\, \mu(\sigma(n), \lambda)\, d\lambda.$$

We use the following estimate of $\mu(\sigma, \lambda)$ ([2, 10.2]):

$$|\mu(\sigma, \lambda)| \le (1 + |\lambda|) \text{ for all } \lambda \in i\mathbb{R} \text{ and } \sigma \in \widehat{M}, \tag{6.7}$$

and get,

$$\|\Delta_n^m F_{nP}\|_2^2 \le C \int_0^\infty \lambda^{4m-1} e^{-\alpha\lambda^2}\, d\lambda.$$

Note that the constant C is independent of m. Thus

$$\|\Delta_n^m F_{nP}\|_2^2 \le C_1^{2m} (2m)! \tag{6.8}$$

for some constant C_1 and for all positive integers m. By an elliptic regularity theorem of Kotaké and Narasimhan ([19, Theorem 3.8.9]), F_{nP} is real analytic. A slight variation of the proof of Theorem 5.1 shows that the discrete part F_{nD} of F_n is also real analytic.

Hence F_n is real analytic, a contradiction. Thus F_n satisfies (6.3). Similarly we can show that H also satisfies (6.3) and hence the theorem is proved.

(c) We have shown above that if a function does not satisfy (6.3), then all its right K-finite components are real analytic almost everywhere. Hence none of them can be zero on a set of positive measure.

(d) If a function is zero on a right K-invariant set, then all the right K-finite components of the function are also zero on that set. Therefore we can apply (c). □

Remark 6.6. As in [18], we could have also discussed results for $L^1 \cap L^p, 1 \leq p < 2$, but for the sake of brevity, we have chosen to drop them.

7. $\mathrm{SL}_2(R)$ as the hyperbolic space $X = \mathrm{SO}_0(2,2)/\mathrm{SO}_0(1,2)$

Although the results of this section are really results of Sections 5 and 6 in a different guise, we decided to include them in the hope that the results here can be formulated and proved for other *generalized* hyperbolic spaces in the sense of [28, 21].

For integers $p \geq 0, q \geq 1$, suppose that $\mathrm{SO}(p,q) = \{A \in \mathrm{SL}_{p+q}(\mathbb{R}) \mid A^T J A = J\}$ where $J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$, A^T is the transpose of A and I_n is the $n \times n$ identity matrix. When $p = 0$, $\mathrm{SO}(0,q)$ is naturally identified with $\mathrm{SO}(q)$. Let $\mathrm{SO}_0(p,q)$ be the connected component of the identity element of $\mathrm{SO}(p,q)$. If $p \geq 1$, a matrix $A \in \mathrm{SO}_0(p-1,q)$ can be identified with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ in $\mathrm{SO}_0(p,q)$. In this way for $p \geq 1, q \geq 1$ we consider $\mathrm{SO}_0(p-1,q)$ as a closed subgroup of $\mathrm{SO}_0(p,q)$. Thus $X = \mathrm{SO}_0(p,q)/\mathrm{SO}_0(p-1,q)$ is a homogenous space under left $\mathrm{SO}_0(p,q)$-action.

For $x, y \in \mathbb{R}^{p+q}$, we define $\langle x, y\rangle = x^T J y$. Then $\langle \cdot, \cdot \rangle$ is a nondegenerate quadratic structure on $\mathbb{R}^{p+q}$ of signature (p,q) which is preserved by $\mathrm{SO}(p,q)$. Consider the noncompact hypersurface, $\{x \in \mathbb{R}^{p+q} \mid \langle x, x\rangle = 1\}$. Then $\mathrm{SO}_0(p,q)$ acts transitively on this surface and the isotropy subgroup of the point $(1, 0, \ldots, 0)$ in it, is precisely $\mathrm{SO}_0(p-1,q)$. Thus X can be identified with this hypersurface.

We will take up the particular case $p = q = 2$, that is, the homogenous space $X = \mathrm{SO}_0(2,2)/\mathrm{SO}_0(1,2)$ under the left $\mathrm{SO}_0(2,2)$-action and for $x, y \in \mathbb{R}^4$, $\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4$. For the rest of the paper G will denote $\mathrm{SO}_0(2,2)$ instead of $\mathrm{SL}_2(\mathbb{R})$. Let K be $\mathrm{SO}(2)$, as in the previous sections. Then $Y = K \times K$ is a maximal compact subgroup of G. A point $(k_\theta, k_\phi) \in Y$ is identified with a point in $\mathbb{R}^4$ as:

$$(k_\theta, k_\phi) \longleftrightarrow (\cos\psi_1, \sin\psi_1, \sin\psi_2, \cos\psi_2),$$

where $\psi_1 = \theta - \phi$ and $\psi_2 = \theta + \phi$. Every element $x \in X$ can be written as $(\mathrm{ch}\ t\cos\psi_1, \mathrm{ch}\ t\sin\psi_1, \mathrm{sh}\ t\sin\psi_2, \mathrm{sh}\ t\cos\psi_2), t \geq 0, \psi_1, \psi_2 \in [0, 2\pi]$. This is called the polar decomposition of x and we write $x = x(y,t)$, where $y = (k_\theta, k_\phi) \in Y$ and θ, ϕ are related to ψ_1, ψ_2 as above. For $x = x(y,t)$, by $|x|$ we mean t. The form $\langle\cdot,\cdot\rangle$ induces a pseudo-Riemannian structure on $\mathbb{R}^4$ and the corresponding Laplace–Beltrami operator is $\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2}$. The hypersurface X is also equipped with the pseudo-Riemannian structure inherited from $\mathbb{R}^4$ and the corresponding Laplace–Beltrami operator Δ_X is really the part of $\Box$ tangential to X. There exists a unique (up to multiplication by a positive scaler) measure dx on X which is G-invariant. The algebra of (left) G-invariant differential operators on X is generated by Δ_X. For each $\lambda \in \mathbb{C}$, $y \in Y$ and $\varepsilon \in \{0,1\}$ consider the function on X defined by:

$$e_{\varepsilon,\lambda,y} : x \mapsto |\langle x, y\rangle|^{\lambda-1}\ \mathrm{sign}^\varepsilon\langle x, y\rangle,$$

where y is now thought as a point in $\mathbb{R}^4$, and $\langle\cdot,\cdot\rangle$ is the quadratic form defined earlier. Then this is a locally integrable function on X with respect to dx on X, at least for $\lambda \in \mathbb{C}$ with $\Re\lambda \geq 1$ ([1]), and is an eigenfunction (in the sense of distributions) of Δ_X with eigenvalue $\lambda^2 - 1$.

For a suitably nice function f, $\widetilde{f}_X$ is defined on $\{0,1\} \times Y \times \mathbb{C}$ (or some subspace there of) by:

$$\widetilde{f}_X(\varepsilon, \lambda, y) = \int_X |\langle x, y\rangle|^{\lambda-1}\ \mathrm{sign}^\varepsilon\langle x, y\rangle f(x)\, dx,$$

for all those λ for which the right hand side exists. Sometimes we will also write $\widetilde{f}_X(\varepsilon, \lambda, y)$ as $\widetilde{f}_X(\varepsilon, \lambda, k_1, k_2)$ where $y = (k_1, k_2) \in K \times K = Y$. It is also possible to identify X with $\mathrm{SL}_2(\mathbb{R})$ and the left-action of G on X corresponds to the two sided action of $\mathrm{SL}_2(\mathbb{R})$ on itself. Under this correspondence dx is just the Haar measure on $\mathrm{SL}_2(\mathbb{R})$ (see [1] for details). Taking into account these identifications, the subspace $\mathcal{L}_\epsilon(\mathrm{SL}_2(\mathbb{R}))$ of $L^1(\mathrm{SL}_2(\mathbb{R}))$ defined in Section 6, carries over to a subspace

$\mathcal{L}_\epsilon(X)$ of $L^1(X)$ and the Casimir Ω of $\mathrm{SL}_2(\mathbb{R})$ corresponds to a multiple of Δ_X. Further it can be shown that for $f \in \mathcal{L}_\epsilon(X)$, $\widetilde{f}_X(\varepsilon, \lambda, y)$ exists for each y in a set of full measure on Y as a "meromorphic function" (see explanation below) on the augmented strip $\mathcal{S}_{1+\epsilon} = \{\lambda \in \mathbb{C} \mid |\Re\lambda| \leq 1 + \epsilon\}$, with possible simple poles at 0 or -1 depending on whether ε is 0 or 1. The connection between $\widetilde{f}_X(\varepsilon, \lambda, y)$ and the Helgason–Fourier transform $\widetilde{f}(\lambda, k, n)$ defined in Section 3 is given by:

$$c_\varepsilon^{s,s,+}(\lambda)\widetilde{f}(\lambda, k_1, s) = \langle \widetilde{f}_X(\varepsilon, \lambda, k_1^{-1}, \cdot), e_{-s}\rangle_{L^2(K)}, \tag{7.1}$$

for $s \in \mathbb{Z}^\varepsilon$ where $c_\varepsilon^{s,s,+}(\lambda)$ are certain c-functions (see [1] and [2, p. 23–24]). In fact the above may be viewed as the definition of $\widetilde{f}_X$. By $\widetilde{f}_X$ being meromorphic, we mean that for $k_1 \in K$ for which the left side of the expression above is defined, the expression on the right hand side is meromorphic for each $s \in \mathbb{Z}^\varepsilon$, if not always holomorphic. We also say $\widetilde{f}(\varepsilon, \lambda, \cdot)$ has a pole at λ_0 if there exists a $k \in K$ and $s \in \mathbb{Z}^\varepsilon$ such that $\lambda \mapsto \langle \widetilde{f}_X(\varepsilon, \lambda, k, \cdot), e_{-s}\rangle_{L^2(K)}$ has a pole at λ_0.

On the left hand side of (7.1), f is considered as a function on $\mathrm{SL}_2(\mathbb{R})$ while on the right hand side it is viewed as a function on X via the identification referred to above. Apart from this, there is a *discrete part* of the Fourier transform of f which in [1] enters in the guise of certain residues calculated at the poles of the c-functions described in [2]. Without going into the details, we merely remark that this can be avoided by suitably defining the discrete part $\widetilde{f}_{X,D}(l, y), l \in \mathbb{Z}^* \sqcup \{0_-, 0_+\}$. This is related to $\widetilde{f}(l, k, n)$ by

$$c_\varepsilon^{s,s,+}(\lambda)\widetilde{f}(|l|, k_1, s) = \langle \widetilde{f}_{X,D}(l, k_1, \cdot), e_s\rangle_{L^2(K)}, \tag{7.2}$$

for $s \in \mathbb{Z}(l)$. (Obviously this comes from the discrete series and the mock discrete series representations of $\mathrm{SL}_2(\mathbb{R})$. Again one can take the relation above as the definition of the discrete part of $\widetilde{f}_X$.)

Note that an element $y = (k_\theta, k_\phi)$ in Y sits inside G as $\begin{pmatrix} k_{\phi+\theta} & 0 \\ 0 & k_{\phi-\theta} \end{pmatrix}$. So, Y acts naturally on X. Suppose that $Y_0 = \{(e, k_\theta) \mid k_\theta \in K\}$, where e is the identity element of K. For a set $S \subset X$, $Y_0 S$ is the set of all (left) translations of S by elements of Y_0. We call a set $S \subset X$, Y_0-invariant if $Y_0 S = S$. Let m be a fixed G-invariant measure on X and let μ be the Plancherel measure on $\{0, 1\} \times i\mathbb{R}$ defined in [21, 1]. Then we have the following two versions of Benedicks' Theorem. These are essentially the results proved in Section 5 expressed in our new language:

Theorem 7.1. *For $f \in L^2(X)$, suppose that $A_f = \{x \in X \mid f(x) \neq 0\}$. If $m(Y_0 A_f) < \infty$ and $\mu(\{(\varepsilon, \lambda) \in \{0,1\} \times i\mathbb{R} \mid \widetilde{f}_X(\varepsilon, \lambda, .) \not\equiv 0\}) < \infty$, then $f = 0$ almost everywhere.*

Theorem 7.2. *Suppose that $f \in L^2(X)$, S is a Y_0-invariant subset of X of positive measure and S' is a subset of $i\mathbb{R}$ of positive Plancherel measure. If f is zero on S while $\widetilde{f}(\varepsilon, \lambda, \cdot)$ is zero on S' for $\varepsilon \in \{0,1\}$, then $f = 0$ almost everywhere.*

For some (carefully chosen) $\epsilon > 0$, suppose that $\mathsf{T}_\epsilon = \{0,1\} \times \mathcal{S}_{1+\epsilon} \setminus \{(0,0),(1,-1)\}$. We have observed earlier that for $f \in \mathcal{L}_\epsilon(X)$ and for fixed y in a full measure set in Y, $\lambda \mapsto \widetilde{f}_X(\varepsilon, \lambda, y)$ is analytic on $\mathcal{S}_{1+\epsilon} \setminus \{0\}$ or on $\mathcal{S}_{1+\epsilon} \setminus \{-1\}$ depending on $\varepsilon = 0$ or 1 respectively. With this preparation we are now ready to state the analogue of Wiener's Theorem on X:

Theorem 7.3. *Let $\mathcal{F}$ be a subset of $\mathcal{L}_\epsilon(X)$ for some $\epsilon > 0$. Suppose that*

$$Z_1 = \{(\varepsilon, \lambda) \in \mathsf{T}_\epsilon \mid \widetilde{f}_X(\varepsilon, \lambda, \cdot) \equiv 0 \ \textit{ for all } f \in \mathcal{F}\}$$

$$\textit{and} \quad Z_2 = \{l \in \mathbb{Z}^* \sqcup \{0_-, 0_+\} \mid \widetilde{f}_{X,D}(l, \cdot) \equiv 0 \ \textit{ for all } f \in \mathcal{F}\}.$$

If

(i) *$Z_1 \sqcup Z_2$ is empty,*

(ii) *there exist $f_0, f_1 \in \mathcal{F}$ such that $\widetilde{f_0}_X(0, \lambda, \cdot)$ has a pole at 0 and $\widetilde{f_1}_X(1, \lambda, \cdot)$ has a pole at -1,*

(iii) *there exists $g^j \in \mathcal{F}$, $j = 0, 1$ such that*

$$\limsup_{\lambda \in i\mathbb{R}^*, |\lambda| \longrightarrow \infty} \|\widetilde{g^j}_X(j, \lambda, \cdot)\|^2_{L^2(Y)} e^{\alpha e^{|\lambda|}} > 0 \textit{ for all } \alpha > 0, \tag{7.3}$$

then the left G-translates of the functions in $\mathcal{F}$ span a dense subspace of $L^1(X)$.

Remark 7.4. Condition (ii) will guarantee that there exists $r \in \mathbb{Z}^0$ and $s \in \mathbb{Z}^1$ such that $\widetilde{f_0}(0, k_1, r) \neq 0$ and $\widetilde{f_1}(1, k_2, s) \neq 0$ for some $k_1, k_2 \in K$. Because, otherwise $\widetilde{f_0}_X$ ($\widetilde{f_1}_X$) will not have a pole at 0 (respectively, at -1) when $\varepsilon = 0$ (respectively, $\varepsilon = 1$) (see (7.1)).

The only other point which may be obscure in this reformulation is why the necessary condition $\int_G f(x)\, dx \neq 0$ (see Theorem 6.2 and Theorem 6.5) is absent in the hypothesis. Indeed from the hypothesis we know that there exists $f \in \mathcal{F}$ such that $\widetilde{f}_X(0, -1, \cdot) \not\equiv 0$. Therefore there exists $s \in \mathbb{Z}^0$ and $k_1 \in K$ such that $\langle \widetilde{f}_X(0, -1, k^{-1}, \cdot), e_{-s} \rangle_{L^2(K)} \neq 0$. That is $c_0^{s,s,+}(-1) \widetilde{f}(-1, k_1, s) \neq 0$. But since

$c_0^{s,s,+}(\lambda)$ has a zero at -1 for every nonzero even s (see [2, Proposition 6.1]), we conclude that $s = 0$, that is, $\widetilde{f}(-1, k_1, 0) \neq 0$. Recall that the trivial representation is a subrepresentation of the principal series representation $\pi_{0,-1}$. Therefore $\widehat{f}(-1)_{0,n} = \int_G f(x)\Phi_{-1}^{0;n}(x)\,dx = 0$ for any nonzero n and hence $\widetilde{f}(-1, k_1, 0) \neq 0$ implies that $\widehat{f}(-1)_{0,0} \neq 0$. This is the same as $\int_G f(x)\,dx \neq 0$ because $\phi_{-1} = \Phi_{-1}^{0,0} \equiv 1$.

Remark 7.5. Condition (iii) above can be replaced by the following: there exist two functions $g_1, g_2 \in \mathcal{F}$, with nonzero even and odd part respectively, such that they satisfy one of these conditions:

a. $|g_1(x)| \leq Ce^{-\alpha|x|^2}$ and $|g_2(x)| \leq Ce^{-\alpha|x|^2}$ for some $C, \alpha > 0$,

b. $m(Y_0 A_{g_1}) < \infty$ and $m(Y_0 A_{g_2}) < \infty$, where $A_{g_i}, i = 1, 2$ are as in Theorem 7.1,

c. g_1 and g_2 are zero on a Y_0-invariant set of positive measures.

8. CONCLUDING REMARKS

As in the case of symmetric spaces, one can also define the Radon transform and relate it to the version of the Helgason Fourier transform defined in this paper. These questions will be taken up in a forthcoming paper.

We end this section by stating the following variant of the Riemann–Lebesgue Lemma (Theorem 4.1 (v)). If $f \in L^1(\mathrm{SL}_2(\mathbb{R}))$ then for any $\sigma \in \widehat{M}$ and $n \in \mathbb{Z}^\sigma$, $\widetilde{f}(\lambda, \sigma, k, n) \longrightarrow 0$ as $|\lambda| \longrightarrow \infty$ uniformly on $\mathcal{S}_1$. This can be proved using Radon transform techniques in the same manner as in [14] for symmetric spaces.

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