MULTIPLICATIVELY SPECTRUM-PRESERVING MAPS OF FUNCTION ALGEBRAS

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Abstract. Let X be a compact Hausdorff space and $\mathcal{A} \subset C(X)$ a function algebra. Assume that X is the maximal ideal space of $\mathcal{A}$. Denoting by $\sigma(f)$ the spectrum of an $f \in \mathcal{A}$, which in this case coincides with the range of f, a result of Molnár is generalized by our Main Theorem: If $\Phi : \mathcal{A} \to \mathcal{A}$ is a surjective map with the property $\sigma(fg) = \sigma(\Phi(f)\Phi(g))$ for every pair of functions $f, g \in \mathcal{A}$, then there exists a homeomorphism $\Lambda : X \to X$ such that

$$\Phi(f)(\Lambda(x)) = \tau(x)f(x)$$

for every $x \in X$ and every $f \in \mathcal{A}$ with $\tau = \Phi(1)$.

1. Introduction

Molnár [M, Theorem 5] proved the following theorem: If X is a first-countable compact Hausdorff space and $C(X)$, the algebra of complex-valued continuous functions on X, and

$$\Phi : \ C(X) \to C(X)$$

a surjective mapping such that

$$\text{for every pair of functions } f, g \in C(X), \ \sigma(fg) = \sigma(\Phi(f)\Phi(g))$$

where $\sigma(f)$ denotes the spectrum of f, which in this case would be simply $f(X)$, the range of f, then there exists a homeomorphism φ of X onto itself and a function τ, whose range is $\{-1, 1\}$ such that

$$\Phi(f)(x) = \tau(x)f(\varphi(x)) \text{ for all } x \in X \text{ and all } f \in C(X).$$

In this paper we deal with a function algebra $\mathcal{A}$ in place of $C(X)$ and regard X as the maximal ideal space of $\mathcal{A}$. X is of course compact Hausdorff but not necessarily first-countable. For this purpose, we need to recall some results of Bishop and de Leeuw [BL] concerning function algebras, peaking functions, generalized peak points etc., for which a readable exposition may be found in [Br, Chapter 2] and [P, Chapter 8].

1.1 Peaking function. A function f in $\mathcal{A}$ is said to be a peaking function if for any x in X, either $f(x) = 1$ or $|f(x)| < 1$ and the set $\{x : x \in X, f(x) = 1\}$, denoted by $P(f)$ and referred to as the **peaking set**, is non-empty.

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1.2. Generalized peak point. A point x in X is said to be a generalized peak point for the algebra $\mathcal{A}$ if, given any neighborhood V of x, there exists a peaking function f in $\mathcal{A}$ such that $P(f) \subset V, f(x) = 1$.

The set of all generalized peak points is called the *Choquet boundary* of $\mathcal{A}$ and denoted by $\partial_{\mathcal{A}}(X)$. Its closure is the so-called Shilov boundary of $\mathcal{A}$. Since any $f \in \mathcal{A}$ assumes its maximum modulus $\|f\|_\infty := \sup_{x\in X} |f(x)|$ on the Choquet boundary (see [P, Prop. 6.3]), we see that

$$\text{any peaking set meets } \partial_{\mathcal{A}}(X). \tag{1.3}$$

Also, given any $x \in X$, there exists a probability measure μ, a representing measure for x, supported on the Shilov boundary $S = \overline{\partial_{\mathcal{A}}(X)}$ such that for every $f \in \mathcal{A}$,

$$f(x) = \int_S f d\mu. \tag{1.4}$$

The following theorem will be invoked several times in the proof of our *Main Theorem* in the next section.

1.5. Theorem (Bishop). *Given any peaking set E and any $f \in \mathcal{A}$, there exists a peaking function h in $\mathcal{A}$ with $P(h) = E$ and $|f(z)h(z)| < \max_E |f|$ for any $z \notin E$.*

A proof may be found in [Br, page 102]. At one point in the next section, we shall need the fact contained in the following proposition.

1.6. Proposition. *Any family of peaking sets E_α, with finite intersection property, has a common intersection with $\partial_{\mathcal{A}}(X)$.*

Proof. The proof is a convexity argument. Let $S_{\mathcal{A}} = \{L \in \mathcal{A}^* : \|L\| = L(1) = 1\}$ be the state space of $\mathcal{A}$. We know that (see [P, page 37]) $\varphi(\partial_{\mathcal{A}}(X)) = \text{ext}(S_{\mathcal{A}})$ where $\text{ext}(S_{\mathcal{A}})$ denotes the set of extreme points of the compact convex set $S_{\mathcal{A}} \subset \mathcal{A}^*$, non-empty by the Krein-Milman theorem, and φ denotes the evaluation map $x \rightsquigarrow \varphi(x)$ that imbeds X homeomorphically into $S_{\mathcal{A}}$ with weak* topology. Each $F_\alpha :=$ weak* closed convex hull of $\varphi(E_\alpha)$, where $E_\alpha = \{x \in X : h_\alpha(x) = 1\}$ and each $h_\alpha \in \mathcal{A}$ is the associated peaking function, is a weak* closed face of $S_{\mathcal{A}}$— in fact, $F_\alpha = \{L \in S_{\mathcal{A}} : L(h_\alpha) = 1\}$. Consequently by the finite intersection property, $F := \bigcap_\alpha F_\alpha$ is a non-empty weak* closed face of $S_{\mathcal{A}}$ and therefore has an extreme point p that necessarily belongs to $\text{ext}(S_{\mathcal{A}})$ and is therefore of the form $\varphi(x)$ for some $x \in \partial_{\mathcal{A}}(X)$. But $p \in \text{ext}(F_\alpha) \subset \varphi(E_\alpha)$ for every α by the Milman theorem; hence $x \in \bigcap_\alpha E_\alpha$, and we are done. □

2. Proof of the Main Theorem

In the sequel f, g, h, k, etc. denote functions from $\mathcal{A}$ and c denotes a generic constant. Also for any $f \in \mathcal{A}$, we shall sometimes abbreviate $\|f\|_\infty$ to $\|f\|$. It is convenient to present the proof of our theorem as a sequence of remarks. We point out that the proofs of these remarks, though modelled in several instances on [M], are rendered somewhat complicated by the more general situation that is being considered here.

Remark 1. Reduction. Since $\sigma(1^2) = \sigma(\Phi(1)^2)$, we have $\Phi(1)^2 = 1$, and so by defining $\Psi f = \Phi(1)\Phi(f)$, we see that $\Psi(1) = (\Phi(1))^2 = 1$. Furthermore, $\Psi(f)\Psi(g) = \Phi(1)\Phi(f)\Phi(1)\Phi(g) = \Phi(f)\Phi(g)$ and, consequently,

$$\sigma(fg) = \sigma(\Psi(f)\Psi(g)).$$

Now if we prove the existence of a homeomorphic self-map Λ of X such that

$$\Psi(f)(\Lambda(x)) = f(x)$$

for every $x \in X$, we would have proved the theorem mentioned in the abstract. So from now on, we assume that $\Phi(1) = 1$ and so

$$\sigma(f) = \sigma(\Phi(f)) \quad \forall f \in \mathcal{A}, \tag{2.1}$$

from which it immediately follows that

$$\|f\|_\infty = \|\Phi(f)\|_\infty. \tag{2.2}$$

Remark 2. If $f, g \in \mathcal{A}$, then $|f| \le |g|$ on $\partial_{\mathcal{A}}(X)$ if and only if

$$\text{for every } c \ge 0 \text{ and every } h,\ |gh| \le c \text{ implies } |fh| \le c. \tag{2.3}$$

Proof. That $|f| \le |g|$ on $\partial_{\mathcal{A}}(X)$ implies (2.3) is obvious by (1.4). Assume that (2.3) is true but $|f| \not\le |g|$ on $\partial_{\mathcal{A}}(X)$. Hence there must exist an x_0 in $\partial_{\mathcal{A}}(X)$ such that

$$|f(x_0)| > |g(x_0)|;$$

for, otherwise, $|f| \le |g|$ on $\partial_{\mathcal{A}}(X)$.

Let $\gamma = \frac{1}{2}(|f(x_0)| + |g(x_0)|)$. So $|g(x_0)| < \gamma < |f(x_0)|$, and there exists an open neighborhood V of x_0 such that $|g(x)| < \gamma$ in V and a function h such that $h(x_0) = 1 = \|h\|$, and $|g(x)h(x)| < \gamma$ in $X \setminus V$. Such an h exists, because x_0 is a generalized peak point for $\mathcal{A}$. Therefore $|gh| < \gamma$ on all of X, but $|f(x_0)h(x_0)| = |f(x_0)| > \gamma$, a contradiction. This proves the assertion (2.3).

From (2.3), we can deduce the following:

$$\text{if } \sigma(fh) = \sigma(gh) \text{ for every } h, \text{ then on } \partial_{\mathcal{A}}(X), |f| = |g|. \tag{2.4}$$

Since $\sigma(fh) = \sigma(gh)\ \forall h \in \mathcal{A}$ we see that for any constant $c \ge 0$ and any $h \in \mathcal{A}$, $|gh| \le c$ implies $|fh| \le c$ and so (2.3) gives $|f| \le |g|$ on $\partial_{\mathcal{A}}(X)$. Since the hypothesis is symmetric in f, g, we obtain also $|g| \le |f|$ on $\partial_{\mathcal{A}}(X)$. Combining, we have (2.4).

As a consequence we have

Remark 3.

$$\text{On } \partial_{\mathcal{A}}(X),\ |f| \le |g| \Leftrightarrow |\Phi(f)| \le |\Phi(g)|\ \forall f, g \in \mathcal{A}. \tag{2.5}$$

Proof. Assume that $|f| \le |g|$ on $\partial_{\mathcal{A}}(X)$ and $|\Phi(g)k| \le c$ for some $k \in \mathcal{A}$ and $c \ge 0$. Φ being surjective, there exists an $h \in \mathcal{A}$ such that $\Phi(h) = k$. Hence we have

$$|\Phi(g)\Phi(h)| \le c.$$

But since

$$\sigma(gh) = \sigma(\Phi(g)\Phi(h)),$$

we obtain $|gh| \le c$ and so by (2.3), $|fh| \le c$. Since

$$\sigma(fh) = \sigma(\Phi(f)\Phi(h)),$$

we obtain $|\Phi(f)\Phi(h)| = |\Phi(f)k| \le c$. Now since k, c are arbitrary, from Remark 2, it follows that

$$|\Phi(f)| \le |\Phi(g)| \text{ on } \partial_{\mathcal{A}}(X).$$

Now the other implication has a similar proof.

Remark 4. For any fixed $x \in \partial_{\mathcal{A}}(X)$,

$$E := \bigcap_{f \in \mathcal{F}_x} P(f) = \{x\}, \tag{2.6}$$

where $\mathcal{F}_x$ denotes the family of all peaking functions $f \in A$ such that $f(x) = 1$.

Proof. Assume E contains a point y other than x. From (1.2) it follows that every point of $\partial_{\mathcal{A}}(X)$ is a generalized peak point for $\mathcal{A}$, which means that, given any neighborhood V of x, there exists a peaking function h in $\mathcal{A}$ such that $h(x) = 1 = \|h\|$ and $|h| < 1$ outside V, which means $P(h) \subset V$. So if we choose a neighborhood V of x that does not contain y, since $E \subset V$, $y \notin E$, a contradiction.

We now have the important

Remark 5. If $x \in \partial_{\mathcal{A}}(X)$,

$$\bigcap_{f \in \mathcal{F}_x} P(\Phi(f)) \text{ contains one and only one generalized peak point.} \tag{2.7}$$

First, because of (2.1), $\Phi(f)$ is a peaking function if and only if f is a peaking function. Also, each $P(\Phi(f))$ is compact.

Secondly, if $f_1, f_2, \dots, f_n$ belong to $\mathcal{F}_x$, then $g = f_1 f_2 \dots f_n$ belongs to $\mathcal{F}_x$. Since $|g| \leq |f_i|$, we obtain in view of (2.5),

$$|\Phi(g)| \leq |\Phi(f_i)| \text{ for each } 1 \leq i \leq n \text{ on } \partial_{\mathcal{A}}(X).$$

Since g is a peaking function, so is $\Phi(g)$, and so $\Phi(g)(\xi) = 1$ for some ξ in $\partial_{\mathcal{A}}(X)$. Then $\Phi(f_i)(\xi) = 1$ for $1 \leq i \leq n$ or

$$\bigcap_{1 \leq i \leq n} P(\Phi(f_i)) \neq \emptyset.$$

This proves that the family of sets $\{P(\Phi(f)) : f \in \mathcal{F}_x\}$ has the finite intersection property, and since each of them is compact, it must be that

$$E' = \bigcap_{f \in \mathcal{F}_x} P(\Phi(f)) \neq \emptyset.$$

Thus, E' being a non-empty intersection of peaking sets must intersect $\partial_{\mathcal{A}}(X)$ by Proposition 1.6.

Thirdly, if $y \in E' \cap \partial_{\mathcal{A}}(X)$, let k be a peaking function such that $k(y) = 1$. By surjectivity of Φ, $k = \Phi(h)$ for some peaking function $h \in \mathcal{A}$ (recall that $\sigma(k) = \sigma(h)$). We claim that $h(x) = 1$. To show this, choose any neighborhood V of x and a peaking function g such that $g(x) = 1$ and $|g| < 1$ outside V. So $g \in \mathcal{F}_x$ and hence $\Phi(g)(y) = 1$. Consider $\Phi(g)\Phi(h) = \lambda \in \mathcal{A}$. $\Phi(g), \Phi(h)$ being both peaking functions that take the value 1 at y, we see that $\lambda(y) = 1$ and λ is a peaking function. Again Φ being surjective, there exists a peaking function $\mu \in \mathcal{A}$ such that $\Phi(\mu) = \lambda$. Since $|\lambda| \leq |\Phi(g)| \wedge |\Phi(h)|$ on $\partial_{\mathcal{A}}(X)$, by (2.5) it follows that $|\mu| \leq |g| \wedge |h|$ on $\partial_{\mathcal{A}}(X)$. Hence there exists a $\xi \in \partial_{\mathcal{A}}(X)$ such that $\mu(\xi) = 1$, and so $g(\xi) = h(\xi) = 1$, which implies that $\xi \in V$. Since V is an arbitrary neighborhood of x and h is continuous, we get

$$h(x) = 1.$$

Lastly, if there is a generalized peak point z other than y in E', we can choose k in such a way that $k(y) = 1$, $|k(z)| < 1$. Φ being surjective, we obtain h' such that $\Phi(h') = k$. So by the previous paragraph, we see that h' belongs to $\mathcal{F}_x$ and so

$\Phi(h') = 1$ on E' and consequently $k(z) = 1$, which is a contradiction. This proves Remark 5. □

Let the unique point y given by Remark 5 be denoted by $\tau(x)$ since it depends on x and nothing else. We sum up what we established above as follows:

Remark 6. If $x \in \partial_{\mathcal{A}}(X)$ and $f \in \mathcal{F}_x$, then $\tau(x) \in \partial_{\mathcal{A}}(X)$ and $\Phi(f)$ belongs to $\mathcal{F}_{\tau(x)}$. Conversely, if $k \in \mathcal{F}_{\tau(x)}$ and $\Phi(h) = k$, then $h \in \mathcal{F}_x$.

We now have

Remark 7. Φ is injective and homogeneous, i.e., $\Phi(cf) = c\Phi(f)$ for any $f \in \mathcal{A}$ and $c \in \mathbb{C}$.

Proof. Suppose if possible that $\Phi(f) = \Phi(g)$ for some $f \neq g$. For any $h \in \mathcal{A}$, $\Phi(f)\Phi(h) = \Phi(g)\Phi(h)$ and consequently,

$$\sigma(\Phi(f)\Phi(h)) = \sigma(\Phi(g)\Phi(h)),$$

from which we see that

$$\sigma(fh) = \sigma(gh).$$

We deduce from (2.4) that $|f| = |g|$ on $\partial_{\mathcal{A}}(X)$. Since $f \neq g$, there exists a $y \in \partial_{\mathcal{A}}(X)$ such that $f(y) \neq g(y)$; for otherwise $f - g$ would vanish on $\partial_{\mathcal{A}}(X)$, and so $f = g$ on X by (1.4). We may assume that $f(y) \neq 0$ because if $f(y) = 0$, then, since $|f(y)| = |g(y)|$, it would follow that $g(y) = 0 = f(y)$. Therefore we can choose a neighborhood V of y and a peaking function h such that $1 = h(y)$, $|h(z)| < 1$ outside V. Then $E := P(h) \subset V$. By (1.5), we can modify h so that it would still be a peaking function that peaks on E and moreover satisfies the following:

$$\begin{aligned} |f(z)h(z)| &< \max_E |f| = \max_X |fh|, \\ |g(z)h(z) &< \max_E |g| = \max_X |gh| \end{aligned} \tag{2.8}$$

for all z outside E.

There exists $\xi \in E$ such that $|f(\xi)| = \max_E |f| = \|fh\|_\infty$. Since $\sigma(fh) = \sigma(gh)$, $f(\xi) = f(\xi)h(\xi) = g(z)h(z)$ for some $z \in X$. If $z \notin E$, then $|g(z)h(z)| < \max_E |g| = \|gh\|_\infty = \|fh\|_\infty = |f(\xi)|$, a contradiction. So $z \in E$ and $f(\xi) = g(z)$ where both ξ, z lie in V. Since V is an arbitrary neighborhood of y and f, g are continuous, we get $f(y) = g(y)$, again a contradiction.

Thus

$$\sigma(fh) = \sigma(gh) \forall h \Leftrightarrow f = g$$

and Φ is injective.

Now for the homogeneity. Notice that

$$\sigma(\Phi(cf)\Phi(h)) = \sigma(cfh) = c\sigma(fh) = c\sigma(\Phi(f)\Phi(h)) = \sigma(c\Phi(f)\Phi(h)).$$

Since Φ is bijective, we see that $\Phi(cf) = c\Phi(f) \forall f \in \mathcal{A}$. □

Remark 8.

$$|f(x)| = |\Phi(f)(\tau(x)))| \quad \forall f \in \mathcal{A}, \quad \forall x \in \partial_{\mathcal{A}}(X). \tag{2.9}$$

Proof. Take $f \in \mathcal{A}$ and assume first that $f(y) \neq 0, y \in \partial_{\mathcal{A}}(X)$. In this case, for any given neighborhood V of y, we can find a function h such that $h(y) = 1 = \|h\|$ and fh attains its maximum modulus in V. (To find h, let k be a peaking function with $P(k) \subset V$, and let $h = k^n$ for some sufficiently large positive integer n.) There exists a ξ in V such that

$$|f(\xi)h(\xi)| = \|fh\|_\infty.$$

But $\sigma(\Phi(f)\Phi(h)) = \sigma(fh)$ from which it follows that $|\Phi(f(\tau(y)))\Phi(h(\tau(y)))| \leq |f(\xi)h(\xi)|$. Since $\Phi(h(\tau(y))) = h(y) = 1$ (Remark 6),
$|h(\xi)| \leq 1$, we get

$$|\Phi(f)(\tau(y)))| \leq |f(\xi)|.$$

V being arbitrary and f continuous, we have

$$|\Phi(f)(\tau(y)))| \leq |f(y)|.$$

If, on the other hand, $f(y) = 0$, we could ensure that h satisfies $h(y) = 1 = \|h\|_\infty$ and $\|fh\|_\infty < \epsilon$ for some preassigned $\epsilon > 0$. Hence once again because $\sigma(\Phi(f)\Phi(h)) = \sigma(fh)$, we see that $\|\Phi(f)\Phi(h)\| < \epsilon$ by (2.2) and so

$$|\Phi(f)(\tau(y))\Phi(h)(\tau(y))| < \epsilon,$$

and since $\Phi(h)(\tau(y)) = 1$, we get

$$|\Phi(f)(\tau(y))| < \epsilon,$$

which proves $f(y) = \Phi(f)(\tau(y)) = 0$.

Now let V be any neighborhood of $\tau(y)$, and assume that $\Phi(f)(\tau(y)) \neq 0$. We can, as before, choose h' with $h'(\tau(y)) = 1 = \|h'\|$ and $\Phi(f)h'$ attains its maximum modulus at a point ξ in V. Since Φ is surjective, let $\Phi(h) = h'$. By Remark 6, $h(y) = 1$ and since $f(y)h(y)$ belongs to $\sigma(fh) = \sigma(\Phi(f)\Phi(h))$, we get

$$f(y) = \Phi(f)(\xi')\Phi(h)(\xi')$$

for some ξ' in X. So $|f(y)| \leq |\Phi(f)(\xi)|$. By continuity, we see that

$$|f(y)| \leq |\Phi(f)(\tau(y))|.$$

If $\Phi(f)(\tau(y)) = 0$, we can repeat an argument similar to the one in the last paragraph and obtain $f(y) = 0$.

Putting all these facts together, we see that the proof of Remark 8 is complete. □

Remark 9. τ is a homeomorphism of $\partial_{\mathcal{A}}(X)$ onto itself.

Proof. We observe first that τ is injective: if $\tau(x) = \tau(y)$, then $|\Phi(f)(\tau(x))| = |\Phi(f)(\tau(y))|$ and this implies that $|f(x)| = |f(y)|$ for all $f \in \mathcal{A}$ by Remark 8. Since $\mathcal{A}$ separates points of X, it is easily seen that there exist functions f such that $f(x) = 0, f(y) = 1$ proving that $x = y$. Next we show that τ is continuous. Choose any $x \in X$ and a neighborhood V of $\tau(x)$ and a peaking function h such that

$$h(\tau(x)) = 1, \ |h(y)| \leq 1/2 \quad \forall y \in X \setminus V.$$

Φ being surjective, there exists a g such that $\Phi(g) = h$. Since $|g| \equiv |\Phi(g(\tau))|$ by Remark 8, if we let $W = \{\xi : |g(\xi)| > 1/2\}$, then $\tau(W) \subset V$ because if $\xi \in W$, then

$$|h(\tau(\xi))| = |\Phi(g)(\tau(\xi))| = |g(\xi)| > 1/2.$$

Since $|h(\tau(x))| = |\Phi(g)(\tau(x))| = |g(x)| = 1$, W is a neighborhood of x in $\partial_{\mathcal{A}}(X)$. Thus we have proved that τ is injective and continuous.

Now since Φ is a bijection, we see that Φ^{-1} has the same properties as Φ. Thus there would exist an injective continuous map $\psi : \partial_{\mathcal{A}}(X) \to \partial_{\mathcal{A}}(X)$ such that

$$|g(x)| \equiv |\Phi^{-1}(g)(\psi(x))| \forall x \in \partial_{\mathcal{A}}(X), \forall g \in \mathcal{A}.$$

Now let $g = \Phi(h)$. Then $|\Phi(h)(x)| = |g(\psi(x))|$. Now let $x = \tau(y)$. Then $|g(y)| = |\Phi(h)(\tau(y))| = |g(\psi(\tau(y)))|$ by Remark 8. Since functions of type $|g|$ separate points of $\partial_{\mathcal{A}}(X)$, we get $\psi(\tau(y)) \equiv y$ and by a similar argument, we also obtain $\tau(\psi(y)) \equiv y$. Thus we proved that τ is a self-homeomorphism of $\partial_{\mathcal{A}}(X)$. □

Remark 10.

$$f(x) = \Phi(f)(\tau(x)) \text{ for all } x \text{ in } \partial_{\mathcal{A}}(X) \text{ and for all } f \text{ in } \mathcal{A}. \tag{2.10}$$

Choose any point x in $\partial_{\mathcal{A}}(X)$. Let V be any open neighborhood of x. Since x is in $\partial_{\mathcal{A}}(X)$, there exists a peaking function h such that $h(x) = 1$ and the peaking set $P(h) = E$ is contained in V. Now by Bishop's theorem 1.5, we can modify h so that it has the same properties as before but, in addition,

$$|f(z)h(z)| < \max_E |f| \text{ for all } z \text{ outside } E. \tag{2.11}$$

Thus there exists a ξ in E such that $|f(\xi)| = \max_E |f| = \|fh\|_\infty$. Since $\sigma(fh) = \sigma(\Phi(f)\Phi(h))$, we have $\|fh\| = \|\Phi(f)\Phi(h)\|$ and so there exists a point z such that $f(\xi)h(\xi) = \Phi(f)(z)\Phi(h)(z)$. We may assume that $z \in \partial_{\mathcal{A}}(X)$ since the set where $\Phi(f)\Phi(h)$ assumes the value $f(\xi)h(\xi)$ is a peaking set and every peaking set meets $\partial_{\mathcal{A}}(X)$.

Since τ is surjective, $z = \tau(\eta)$ for some η in $\partial_{\mathcal{A}}(X)$. Now by (2.9) we notice that

$$|\Phi(f)(\tau(\eta))\Phi(h)(\tau(\eta))| = |f(\eta)h(\eta)|.$$

Now η must be in E because otherwise $|f(\eta)h(\eta)| < |f(\xi)|$ by (2.11). Thus we have found ξ, η in E such that $f(\xi) = \Phi(f)(\tau(\eta))$, since $\Phi(h)(\tau(\eta)) = h(\eta) = 1$ by Remark 6. Since ξ, η lie in V and V is an arbitrary open neighborhood of x, we get by continuity of τ, f, and $\Phi(f)$ that $f(x) = \Phi(f)(\tau(x))$. This completes the proof of (2.10).

Remark 11. Φ is an algebra isomorphism of $\mathcal{A}$ onto itself.

Proof. We already saw that it is a bijection and homogeneous. Let $f, g \in \mathcal{A}$. By (2.10) for any x in $\partial_{\mathcal{A}}(X)$,

$$f(x) = \Phi(f)(\tau(x)), g(x) = \Phi(g)(\tau(x))$$

and

$$f(x)g(x) = \Phi(fg)(\tau(x)), f(x) + g(x) = \Phi(f+g)(\tau(x)).$$

Thus

$$\Phi(fg)(\tau(x)) = \Phi(f)(\tau(x))\Phi(g)(\tau(x)), \Phi(f+g)(\tau(x)) = \Phi(f)(\tau(x)) + \Phi(g)(\tau(x)).$$

Since τ is surjective, we get

$$\Phi(f)(x)\Phi(g)(x) = \Phi(fg)(x), \Phi(f+g)(x) = \Phi(f)(x) + \Phi(g)(x)$$

on all of $\partial_{\mathcal{A}}(X)$ and then by the maximum principle on all of X. This completes the proof of Remark 11. The algebraic isomorphism $\Phi : \mathcal{A} \to \mathcal{A}$ gives rise to a weak* homeomorphism $\Phi^* : \mathcal{A}^* \to \mathcal{A}^*$, which in turn induces a homeomorphism Λ of X (the maximal ideal space of $\mathcal{A}$) onto itself and hence we can state

Remark 12. There exists a self-homeomorphism Λ of X onto itself such that

$$\Phi(f)(\Lambda(x)) = f(x) \text{ on all of } X.$$

But in view of (2.10), we see that $\Lambda(x) = \tau(x)$ for all x in $\partial_{\mathcal{A}}(X)$. This completes the proof of the Main Theorem announced in the abstract. □

Conclusion. We conclude this paper by observing:

If X is a compact Hausdorff space (not necessarily first countable), then our Main Theorem clearly holds for $C_{\mathbb{R}(X)}$ — the Choquet boundary being X and the peaking functions being those given by Urysohn's lemma — and it follows that Theorem 6 in [M] is valid in this general setting with the same proof as given there.

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