isid/ms/2003/14
June 12, 2003
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Small Asymmetric Fractional Factorial Plans for Main Effects and Specified Two-factor Interactions

Ashish Das
Aloke Dey
Paramita Saha

Indian Statistical Institute, Delhi Centre
7, SJSS Marg, New Delhi–110 016, India

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Ashish Das, Aloke Dey and Paramita Saha
Indian Statistical Institute, New Delhi 110 016, India

Fractional factorial plans represented by orthogonal arrays of strength two are known to be optimal in a very strong sense under a model that includes the mean and all the main effects, when all interactions are assumed to be absent. When a fractional factorial plan given by an orthogonal array of strength two is not saturated, one might think of entertaining some two-factor interactions also in the model. In such a situation, it is of interest to examine which of the two-factor interactions can be estimated via a plan represented by an orthogonal array, as also to study the overall efficiency of the plan when some interactions are in the model alongwith the mean and all main effects. In this paper, an attempt has been made to examine these issues by considering some practically useful plans for asymmetric (mixed level) factorials with small number of runs.

KEY WORDS : Fractional factorial plans; Asymmetric orthogonal arrays; D- and A-efficiency.

1. INTRODUCTION AND PRELIMINARIES

Fractional factorial plans with factors at two levels each are used quite often in scientific and engineering experiments due to the run size economy provided by such plans. However, practical considerations often dictate the desirability of including some factors at more than two levels. A quantitative factor like temperature may affect the response in a non-monotone fashion and only two settings of the temperature will not be able to capture the curvilinear relationship between response and temperature. Similarly, there may be more than two settings of a qualitative factor like machine type and it is necessary to include all the settings, as, the response at one level of a qualitative factor cannot be used to infer about the response to another level. In such situations, use of asymmetric (or, mixed level) factorial experiments becomes necessary. For instance, consider the experiment reported by Wang and Wu (1992) in their Example 1. There are five factors, one of which (say, G) is at three levels, corresponding to the three sources of gear, and the others (say, F_1, F_2, F_3, F_4) are at two levels each, these factors being temperature and time in furnace, quench-oil type and temperature. The primary interest of the experimenter is to find which of the factors have large effects. Due to budget and time constraints, an experiment with at most 12 runs can be performed. Under such a scenario, what is the best design that the experimenter can choose? It is well known that a fractional factorial plan represented by an orthogonal array of strength two (a definition of an orthogonal array appears later in this section) is universally optimal in the sense of Kiefer (1975) and Sinha and Mukerjee (1982), under a model that includes the mean and complete sets of orthonormal contrasts belonging to all the main effects, assuming the absence of all 2-factor and higher order interactions. Recall that a universally optimal plan is also optimal according to the commonly used criteria like A-, D- and E- . Therefore, if the mean and all main effects are of primary interest, one would ideally look for an appropriate orthogonal array of strength two, if one existed. Fortunately, a 12-rowed orthogonal array of strength two with one column having three symbols and four columns having two symbols each exists (see Table 1) and a fractional factorial plan represented by this orthogonal array (with columns of the array representing the factors and rows, the treatment combinations or, runs) can therefore be used for the above experiment.

Table 1. *An* $OA(12, 5, 2^4 \times 3, 2)$

F_1	F_2	F_3	F_4	G
0	0	0	0	0
0	1	0	1	0
1	0	1	1	0
1	1	1	0	0
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	1
0	0	1	0	2
0	1	0	1	2
1	0	0	0	2
1	1	1	1	2

However, if the experimenter is not prepared to assume the absence of *all* two-factor interactions and, in fact suspects that the 3-level factor G might interact with two of the four 2-level factors, then can she/he use the same plan to estimate the mean, all main effects and the two specified interactions, assuming the absence of all other factorial effects? Since the 12-run plan is *not* saturated, there being 5 degrees of freedom (d.f.) unused after the estimation of the mean and all the main effects, this is a natural question to ask. It turns out that the answer to the above question is in the affirmative; the 12-run plan allows the estimability of the interactions F_1G and F_2G, apart from that of the mean and all main effects. Alternatively, if the two-factor interactions among 2-level factors are considered important, then as we shall see in Section 2, as many as five of the possible six of such interactions can be estimated via the same plan, apart from the mean and complete sets of orthonormal contrasts belonging to the main effects.

The above example shows that some fractional factorial plans represented by (asymmetric) orthogonal arrays of strength two can be used to estimate specified 2-factor interactions, alongwith the mean and all main effects. Recall that such plans have been traditionally used for the estimation of main effects alone. It is therefore important from a practical view-point to assess the plans based on orthogonal arrays of strength two in respect of their capacity to allow the estimability of certain 2-factor interactions, apart from the mean and all main effects.

For completeness, we first recall the definition of an orthogonal array.

Definition. An orthogonal array, $OA(N, n, m_1 \times \cdots \times m_n, g)$ is an $N \times n$ matrix with symbols in the ith column from a finite set of $m_i (\geq 2)$ symbols, $1 \leq i \leq n$, such that in every $N \times g$ submatrix, all possible combinations of symbols appear equally often as a row.

Orthogonal arrays with $m_1 = \cdots = m_n = m$ (say) are called symmetric and we denote such arrays by $OA(N, n, m, g)$; otherwise, the array is called asymmetric. The integer g, $2 \leq g < n$, is called the *strength* of the array. Clearly, the rows of an $OA(N, n, m_1 \times \cdots \times m_n, g)$ can be visualized as the runs of an $m_1 \times \cdots \times m_n$ factorial and the array itself can then be regarded as an N-run fraction of an $m_1 \times \cdots \times m_n$ experiment. Orthogonal arrays have been studied extensively and for comprehensive accounts, the reader is referred to Hedayat, Sloane and Stufken (1999) and Dey and Mukerjee (1999a).

Consider an $m_1 \times \cdots \times m_n$ factorial experiment involving the factors $F_1, \ldots, F_n$ with F_i appearing at $m_i (\geq 2)$ levels, $i = 1, \ldots, n$. Let $\mathcal{D}_N$ be the collection of all N-run plans for the experiment, such that each member of $\mathcal{D}_N$ allows the estimability of all the factorial effects under a model that includes the mean, the main effects $F_1, \ldots, F_n$ and a specified set of two-factor interactions, say, $F_{i_1}F_{j_1}, \ldots, F_{i_k}F_{j_k}$, all other factorial effects being assumed negligible. Let $\mathcal{I}_d$ denote the information matrix of a plan $d \in \mathcal{D}_N$ under the stated model. Recall that an arbitrary N-run plan d is in $\mathcal{D}_N$ if and only if $\mathcal{I}_d$ is a positive definite matrix of order α (*cf.* Dey and Mukerjee 1999a, Theorem 2.3.1), where

$$\alpha = 1 + \sum_{i=1}^{n}(m_i - 1) + \sum_{u=1}^{k}(m_{i_u} - 1)(m_{j_u} - 1). \tag{1.1}$$

From Lemma 2.5.1 in Dey and Mukerjee (1999a), it follows that for a plan $d \in \mathcal{D}_N$ and under the stated model,

$$\mathrm{tr}(\mathcal{I}_d) = \alpha N / v, \tag{1.2}$$

where $v = \prod_{i=1}^{n} m_i$ and $\mathrm{tr}(\cdot)$ denotes the trace of a square matrix. The A-value of a plan $d \in \mathcal{D}_N$ is given by $A_d = \alpha^{-1}\mathrm{tr}(\mathcal{I}_d^{-1}) \geq \alpha(\mathrm{tr}(\mathcal{I}_d))^{-1}$ and it follows from (1.2) that

$$A_d \geq v/N. \tag{1.3}$$

The A-efficiency of the plan d is defined as

$$E_A(d) = A_{d^*}/A_d, \tag{1.4}$$

where A_{d^*} is the A-value of an A-optimal plan $d^* \in \mathcal{D}_N$. From (1.3), a *lower bound* to the A-efficiency of a plan $d \in \mathcal{D}_N$ is given by

$$e_A(d) = (v/N)/A_d = v/(NA_d). \tag{1.5}$$

On similar lines, a lower bound to the D-efficiency of the plan d is given by

$$e_D(d) = (\det(\mathcal{I}_d))^{1/\alpha} v/N, \tag{1.6}$$

where det(·) denotes the determinant of a square matrix.

The purpose of this communication is to examine the issue of estimability of two-factor interactions alongwith the mean and all main effects in some plans for *asymmetric* (mixed level) experiments represented by asymmetric orthogonal arrays of strength two. We mostly consider only those asymmetric orthogonal arrays which are *maximal* in the sense that no more columns can be added to these arrays, retaining orthogonality of the array. The study is restricted to plans with at most 36 runs and factors having at most 7 levels. In the following sections, we consider unsaturated N-run plans (under the mean and main effects model) represented by asymmetric orthogonal arrays of strength two with $N = 12, 18, 20, 24, 28$ and 36 and examine the issue of estimability of the mean, all main effects and a specified set of two-factor interactions under each plan. For each case considered, we also evaluate lower bounds to the overall D- and A-efficiency of the plans. Several plans are seen to have high D- and A-efficiencies under models with specified interactions.

In what follows, we shall often use the term "D-(A-)efficiency" to mean a *lower bound* to these efficiencies, as given by (1.5) and (1.6). Also, we shall call a model that includes the mean, all main effects and a specified set of 2-factor interactions *admissible* if the information matrix of a given plan under the stated model is positive definite. Finally, the term "interaction" will invariably mean a two-factor interaction.

2. TWELVE RUN PLANS

There are only two (maximal) asymmetric orthogonal arrays of strength two involving 12 rows; these are an $OA(12, 3, 2^2 \times 6, 2)$ and an $OA(12, 5, 2^4 \times 3, 2)$. The first of these is shown in Table 2 (in transposed form) and the second one has already been displayed in Table 1. Both the arrays give rise to plans that are unsaturated under a mean plus main effects model. It is known (Wang and Wu, 1992) that one cannot add more 2-symbol columns in either of the arrays, retaining orthogonality.

2.1. $OA(12, 3, 2^2 \times 6, 2)$

We first consider a plan represented by the array in Table 2. Note that any other orthogonal array $OA(12, 3, 2^2 \times 6, 2)$ is isomorphic to the one given in Table 2 (two orthogonal arrays are isomorphic if one can be obtained from the other by a permutation of rows and columns and symbol changes). In the case of the plan represented by this orthogonal array, there are 4 d.f. unused after the estimation of the mean and all main effects. Let us denote the 6-level factor by G and the 2-level factors by F_1 and F_2. Clearly, if one wishes to estimate an interaction in addition to the mean and *all* main effects, it has necessarily to be the interaction F_1F_2. It turns out that for the plan represented by an $OA(12, 3, 2^2 \times 6, 2)$, the interaction F_1F_2 cannot be included in the model, alongwith the mean and all main effect contrasts, for, inclusion of the interaction in the model alongwith all the main effects and the mean gives rise to a singular information matrix. Thus, this orthogonal array is incapable of providing information on the interaction F_1F_2, when the mean and all main effects are already in the model.

Table 2. An $OA(12, 3, 2^2 \times 6, 2)$

F_1	000	111	111	000
F_2	000	000	111	111
G	012	345	012	345

However, it is possible to estimate certain components of the interactions F_1G or F_2G. A choice of a complete set of five orthonormal contrasts belonging to the main effect of G is as follows :

$$\begin{aligned} G_1 &= \frac{1}{\sqrt{70}}(-5, -3, -1, 1, 3, 5) \\ G_2 &= \frac{1}{\sqrt{84}}(5, -1, -4, -4, -1, 5) \\ G_3 &= \frac{1}{\sqrt{180}}(-5, 7, 4, -4, -7, 5) \\ G_4 &= \frac{1}{\sqrt{28}}(1, -3, 2, 2, -3, 1) \\ G_5 &= \frac{1}{\sqrt{252}}(-1, 5, -10, 10, -5, 1). \end{aligned}$$

Obviously, one can contemplate including at most four interactions of the types F_1G_i or, F_2G_i, $i = 1, \ldots, 5$ in the model alongwith the mean and *all* main effects. It turns out that there are six admissible models, that include the mean, all main effects and one of the following sets of interactions :

$$(F_iG_1, F_iG_2, F_iG_3, F_iG_4), (F_iG_1, F_iG_2, F_iG_4, F_iG_5), (F_iG_2, F_iG_3, F_iG_4, F_iG_5),$$

$i = 1, 2$. The overall D- efficiencies under these models are 0.82, 0.85 and 0.98 respectively. The corresponding A-efficiencies are 0.39, 0.48 and 0.95 respectively.

2.2. $OA(12, 5, 2^4 \times 3, 2)$

Next, we consider a plan represented by the array in Table 1. It is known (Wang and Wu, 1992) that upto isomorphism, the $OA(12, 5, 2^4 \times 3, 2)$ exhibited in Table 1, is unique. For the plan represented by this orthogonal array, there are 5 d.f. unused after the estimation of the mean and all the main effects. As in Section 1, the 3-level factor is denoted by G and the 2-level factors by F_i, $1 \leq i \leq 4$. Thus, we can possibly include either (a) at most five interactions involving the 2-level factors in the model, or, (b) the interaction F_iG for some $i, 1 \leq i \leq 4$ and at most three interactions of the type F_jF_k, where F_j or F_k could be the same as F_i, or, (c) F_iG, F_jG for some i, j, $i \neq j$ and at most one interaction of the type $F_kF_{k'}$, where $F_k(F_{k'})$ could be the same as F_i or F_j. We examine these three cases separately.

In case (a), among the six 2-factor interactions F_iF_j, $1 \leq i < j \leq 4$, if any five interactions are taken simultaneously in the model, then the plan allows the estimability of all the five interactions, alongwith that of the mean and all the main effects. However, for the six distinct choices of the five interactions that can be included in the model, the plan under consideration is *not* equally efficient as per the D- and A- criteria. It turns out that for the sets of interactions $(F_1F_2, F_1F_3, F_1F_4, F_2F_3, F_3F_4)$ or, $(F_1F_2, F_1F_4, F_2F_3, F_2F_4, F_3F_4)$, the lower bounds to the D- and A-efficiencies are respectively 0.70 and 0.23, while for any other set of 5 interactions, the lower bounds to the D- and A-efficiencies are appreciably lower. When fewer than five interactions are considered important, the best efficiencies are obtained when the interactions listed in Table 3 (a) are included in the model.

In case (b), there are 80 possible models with four interactions, among which only 32 are admissible. Among the admissible models, there are 16 models which include the interaction F_iG for some i, $1 \le i \le 4$ and any three out of the four interactions $(F_1F_2, F_1F_4, F_2F_3, F_3F_4)$ and for each of these sets, the D-efficiencies are all equal to 0.73. The A-efficiencies range between 0.26-0.36. The remaining sets include the interactions (i) $(F_iG,\ i = 2 \text{ or } 4,\ F_2F_4)$ and any one of the sets (F_1F_2, F_1F_4), (F_1F_2, F_3F_4), (F_1F_4, F_2F_3), (F_2F_3, F_3F_4), or, (ii) $(F_iG, i = 1 \text{ or } 3,\ F_1F_3)$ and any one of the sets (F_1F_2, F_2F_3), (F_1F_2, F_3F_4), (F_1F_4, F_2F_3), (F_1F_4, F_3F_4). For each of these sets of interactions, the D-efficiencies are all equal to 0.65. The A-efficiencies however differ from one set to the other, these being in the range 0.19-0.24. If less than four interactions are considered important in case (b), the best efficiencies are obtained when the interactions listed in Table 3 (b) are included in the model. For other models, the range of D-efficiencies are given in the last column of Table 3 (b). With only one interaction in the model, the interactions listed in Table 3 (b) are the only ones under which the model is admissible.

Table 3 (a). *Efficiencies Under Different Models*

No. of Interactions	Interactions	Lower Bound to D-Eff. (A-Eff.)
4	$(F_1F_2, F_1F_4, F_2F_3, F_3F_4)$	0.88(0.74)
3	Any three of the above four	0.91(0.80)
2	Any two of the above four with a common factor	0.94(0.88)
2	Any two of the above four with no common factor	0.92(0.83)
1	Any one of the above four	0.97(0.93)

Table 3 (b). *Efficiencies Under Different Models*

No. of Interactions	Interactions	Lower Bound to D-Eff. (A-Eff.)	Range of D-Eff.
3	$(F_iG, F_jF_2, F_jF_4), i = 2, 4, j = 1, 3$; $(F_iG, F_jF_1, F_jF_3), i = 1, 3, j = 2, 4$	0.81(0.52)	0.69-0.78
2	$(F_1G, F_iF_3), (F_3G, F_iF_1), i = 2, 4$; $(F_2G, F_iF_4), (F_4GF_iF_2), i = 1, 3$	0.84(0.64)	0.72-0.84
1	$F_iG, i = 1, 2, 3, 4$	0.89(0.69)	-

In case (c), it turns out that among the various possibilities, the interactions (F_2G and F_4G) or, (F_1G and F_3G) cannot be included simultaneously in the model, as inclusion of either set leads to a singular information matrix. The interactions that can be estimated involve (i) $(F_2G, F_iG,\ i = 1$ or 3) and $F_iF_j,\ (i,j) \neq (2,4)$ or $(1,3)$ or, (ii) $(F_4G, F_iG,\ i = 1$ or 3) and $F_iF_j,\ (i,j) \neq (2,4)$ or $(1,3)$. The D-efficiencies are all equal to 0.76 and the A-efficiencies range between 0.38-0.50. The inclusion of two interactions (F_2G, F_iG), or $(F_4G, F_iG), i = 1, 3$ results in a D-efficiency lower bound of 0.82, the corresponding lower bound to the A-efficiency being 0.58.

3. EIGHTEEN RUN PLAN

Starting from an $OA(18, 7, 3^6 \times 6, 2)$ (see e.g., Dey and Mukerjee, 1999a, p. 62), one can obtain an $OA(18, 8, 2 \times 3^7, 2)$, by replacing the 6-symbol column, say H, by two columns, one having 2 symbols and the other, 3 symbols (see Table 4). Denote these two columns by F and G_1 respectively. The plan represented by the latter array is unsaturated, there being 2 d.f. unused after the estimation of the mean and all main effects.

Table 4. *An* $OA(18, 8, 2 \times 3^7, 2)$

F	000	111	000	111	000	111
G_1	012	012	012	012	012	012
G_2	022	011	100	122	211	200
G_3	121	020	202	101	010	212
G_4	112	002	220	110	001	221
G_5	000	000	111	111	222	222
G_6	201	012	012	120	120	201
G_7	210	021	021	102	102	210

Clearly, one cannot include an interaction among 3-level factors, if the mean and all main effects are already in the model. Suppose one decides to include the interaction FG_1 in the model, apart from the mean and all main effects. The 18-run plan allows the estimability of FG_1 alongwith the mean and all the main effects. This fact has also been observed by Wu and Hamada (2000, p. 313); see also Wang and Wu (1995). Furthermore, from a result of Dey and Mukerjee (1999b), such a plan is universally optimal for the estimation of the mean, all main effects and the 2-factor interaction FG_1 if the level combinations of the following sets of factors occur equally often :

$$(G_i, G_j),\ i, j = 2, \ldots, 7,\ i < j;$$

$$(F, G_1, G_j),\ j = 2, \ldots, 7.$$

It is not hard to see that these conditions are satisfied by the plan under consideration and thus, the plan is universally optimal under a model that includes the mean, all main effects and the interaction FG_1. It is also saturated. However, we find somewhat surprisingly that none of the other interactions $FG_j,\ 2 \leq j \leq 7$ is estimable when the mean and all main effects are already in the model.

4. TWENTY RUN PLAN

An $OA(20, 9, 2^8 \times 5, 2)$ is exhibited (in transposed form) in Table 5. It may be noted that this array is maximal (Wang and Wu, 1992), that is, no further 2-symbol columns can be added to the above array, without disturbing the orthogonality of the array. Let the 5-level factor be denoted by G and the 2-level factors by $F_1, \ldots, F_8$. The plan represented by this array has 7 d.f. unused, after the estimation of the mean and all main effects. Therefore one can contemplate including either (a) at most seven interactions of the type F_iF_j, $1 \leq i < j \leq 8$ or, (b) an interaction of the type F_iG for some $i, i = 1, \ldots, 8$ and at most three interactions of the type F_jF_k.

Table 5. *An* $OA(20, 9, 2^8 \times 5, 2)$

F_1	0011	0011	0011	0011	0011
F_2	0101	0101	0101	0101	0101
F_3	0110	0011	1100	0101	1010
F_4	0101	1001	0011	1100	1010
F_5	0101	1010	1001	0011	1100
F_6	0011	0101	1010	1001	1100
F_7	0110	1100	1001	1001	0011
F_8	0110	1001	1010	0110	0101
G	0000	1111	2222	3333	4444

In case (a) above, it turns out that if we include seven of the 28 interactions among the 2-level factors in the model alongwith the mean and all main effects, then among the $\binom{28}{7}$ models, not all are admissible. Among the admissible models, the highest D- and A-efficiencies are obtained when the interactions $F_2F_7, F_2F_8, F_3F_6, F_3F_8, F_4F_6, F_5F_6, F_5F_8$ are included. The D- and A-efficiencies are respectively 0.72 and 0.40. The D-efficiencies for other models with seven interactions range between 0.40-0.72. If fewer than seven interactions are deemed important, the best efficiencies are obtained when the interactions listed in Table 6 are included in the model. For other models, the range of D-efficiencies are given in the last column of Table 6.

In Table 6, there are $\binom{28}{3} = 3276$ possible models with three interactions of the type F_jF_k, out of which 3174 models are admissible. There are $\binom{28}{2} = 378$ possible models when two of the interactions among 2-level factors are considered. Finally, if only one interaction among the 2-level factors is important, then the plan ensures the estimability of any one interaction. Next, consider case (b) above. If all the interactions listed under case (b) above

Table 6. *Efficiencies Under Different Models*

No. of Ints.	Interactions	Lower Bound to D-Eff. (A-Eff.)	Range of D-Eff.
6	$(F_2F_7, F_3F_6, F_3F_8, F_4F_6, F_4F_7, F_5F_8)$	0.77(0.51)	0.43-0.77
5	$(F_2F_7, F_3F_6, F_3F_8, F_4F_6, F_5F_8)$;		
	$(F_2F_4, F_2F_7, F_3F_6, F_4F_6, F_5F_8)$;		
	$(F_2F_7, F_3F_6, F_3F_8, F_4F_6, F_4F_7)$;		
	$(F_2F_4, F_2F_7, F_3F_6, F_4F_6, F_4F_7)$	0.82(0.59)	0.46-0.81
4	$(F_1F_2, F_1F_8, F_3F_5, F_5F_8)$;		
	$(F_2F_7, F_3F_6, F_4F_6, F_5F_8)$;		
	$(F_1F_2, F_1F_8, F_3F_5, F_4F_7)$;		
	$(F_2F_7, F_3F_6, F_4F_6, F_4F_7)$	0.86(0.67)	0.53-0.86
3	(F_1F_2, F_1F_8, F_3F_5); (F_2F_7, F_3F_6, F_4F_6)	0.91(0.79)	0.62-0.90
2	(F_3F_5, F_3F_6); (F_1F_2, F_2F_7);		
	(F_1F_2, F_3F_5); (F_2F_7, F_3F_6)	0.96(0.92)	0.72-0.93
1	F_1F_2; F_2F_7; F_3F_5; F_3F_6	0.98(0.96)	0.79-0.94

are included in the model, the highest D- and A-efficiencies are obtained when either of the following two sets are included: $(F_2G, F_1F_2, F_2F_3, F_2F_7)$ or, $(F_3G, F_2F_3, F_3F_5, F_3F_6)$, the overall D- and A-efficiencies under either of these models being 0.74 and 0.34 respectively. Other models of the same type result in lower D-efficiencies, these being in the range 0.53-0.72.

If one includes an interaction among the 5-level factor and a 2-level factor alongwith two interactions among 2-level factors, accounting for 19 d.f., then for each of the sets of interactions (F_2G, F_1F_2, F_2F_7) or, (F_3G, F_3F_5, F_3F_6) in the model, the D- and A-efficiencies are equal to 0.77 and 0.37 respectively. With respect to the D-criterion, inclusion of either of these two sets results in the highest efficiency. For the remaining models, the D-efficiency ranges between 0.55-0.76.

If an interaction between G and F_i and an interaction of the type F_jF_k, accounting for 18 d.f., are deemed important, consideration of the sets of interactions (F_7G, F_2F_7) or, (F_5G, F_3F_5) results in the highest D-efficiency of 0.80. The A-efficiency is 0.44 in either case. For the remaining models, the D-efficiencies range between 0.64-0.79.

Finally, if only one interaction of the type $F_jG,\ j = 1, \ldots, 8$ is important, then the plan is equally efficient as per the D- and A-criteria, no matter which 2-level factor figures in the interaction. The D- and A-efficiencies are respectively, 0.80 and 0.44.

5. TWENTY FOUR RUN PLANS

There are several 24-rowed asymmetric orthogonal arrays of strength two. The ones that we consider are (i) $OA(24, 13, 2^{11}\times 4\times 6, 2)$, (ii) $OA(24, 15, 2^{13}\times 3\times 4, 2)$, (iii) $OA(24, 15, 2^{14}\times 6, 2)$, and (iv) $OA(24, 17, 2^{16}\times 3, 2)$. The array (iii) is obtained from (i) by replacing the symbols in the 4-symbol column by the rows of a symmetric orthogonal array $OA(4, 3, 2, 2)$. Similarly, array (ii) is obtained from (iv) by replacing three specific 2-symbol columns by a 4-symbol column;

we provide more details on this replacement later in this section. The plans represented by these four arrays are considered below separately.

5.1. $OA(24, 13, 2^{11} \times 4 \times 6, 2)$

An $OA(24, 13, 2^{11} \times 4 \times 6, 2)$ is displayed (in transposed form) in Table 7. It is not as yet known whether more 2-symbol columns can be added to this array, retaining the orthogonality of the array. The plan represented by this array has 4 d.f. unused, after the estimation of the mean and all main effects. Therefore, one can contemplate including either (a) at most four interactions among the 2-level factors, or, (b) an involving the 4-level factor and a 2-level factor alongwith at most one interaction involving 2-level factors. Let us denote the 6-level factor by H, the 4-level factor by G and the 2-level factors by $F_1, \ldots, F_{11}$. These two cases are treated separately.

Table 7. An $OA(24, 13, 2^{11} \times 4 \times 6, 2)$

F_1	100011	101001	011100	010110
F_2	110001	010101	001110	101010
F_3	111000	100011	000111	011100
F_4	110100	111000	001011	000111
F_5	101010	110100	010101	001011
F_6	101001	011010	010110	100101
F_7	101100	001101	010011	110010 .
F_8	110010	001110	001101	110001
F_9	100101	100110	011010	011001
F_{10}	100110	010011	011001	101100
F_{11}	000111	111000	000111	111000
G	222222	111111	333333	000000
H	012345	012345	012345	012345

We consider case (b) first. Interestingly, it is observed that the inclusion of any interaction of the type F_jG, $1 \le j \le 11$, in the model alongwith the mean and all main effects gives rise to a singular information matrix. Thus interactions of the types in case (b) above are not estimable, when the mean and all main effects are also in the model. This in turn means that *only* interactions among 2-level factors can be included in the model alongwith the mean and all main effects, which is the setup of case (a).

Consider now case (a) above. It turns out that the inclusion of any one interaction of the type F_jF_{11}, $1 \le j \le 10$, in the model alongwith the mean and all main effects gives rise to a singular information matrix. Thus, among the 55 possible interactions involving 2-level factors, only 45 can be considered for inclusion in the model. In view of the above, there are $\binom{45}{4}$ possible models, taking four interactions at a time. The best choice of the four interactions, in terms of the highest D- and A-efficiencies, is provided by the set $(F_4F_6, F_4F_9, F_4F_{10}, F_6F_8)$, the D-and A-eficiencies being 0.91 and 0.78 respectively. If fewer than four interactions are considered important, the best efficiencies are obtained when the interactions listed in Table 8 are included in the model.

Table 8. *Efficiencies Under Different Models*

No. of Ints.	Interactions	Lower Bound to D-Eff. (A-Eff.)	Range of D-Eff.
3	(F_1F_6, F_2F_3, F_4F_6); (F_2F_3, F_2F_5, F_4F_6); (F_2F_3, F_3F_8, F_4F_6); (F_2F_3, F_4F_6, F_4F_9)	0.95(0.87)	0.74-0.95
2	(F_2F_3, F_4F_6)	0.98(0.94)	0.81-0.97
1	F_2F_3; F_4F_6	0.99(0.99)	0.90-0.97

5.2. $OA(24, 15, 2^{14} \times 6, 2)$

Now consider the array $OA(24, 15, 2^{14} \times 6, 2)$, obtained by replacing the symbols in the 4-symbol column in the $OA(24, 1, 2^{11} \times 4 \times 6, 2)$ by the rows of a symmetric $OA(4, 3, 2, 2)$. A symmetric $OA(4, 3, 2, 2)$ has the rows (0,0,0), (0,1,1), (1,0,1) and (1,1,0). Replacing the symbols 0, 1, 2, and 3 under the 4-symbol column in the array in Table 7 by the four rows of the symmetric orthogonal array according to the scheme $0 \to (0, 0, 0)$, $1 \to (0, 1, 1)$, $2 \to (1, 0, 1)$, $3 \to (1, 1, 0)$, we get an $OA(24, 15, 2^{14} \times 6, 2)$. A plan represented by this array has 4 d.f. unused, after the estimation of the mean and all main effects. The 2-level factors in such a plan are denoted by $F_1, \ldots, F_{14}$ and the 6-level factor by H. Clearly, one can include at most four interactions among the 2-level factors in the model, apart from the mean and all main effects. Among a very large number of possible models with four interactions, the inclusion of the set of interactions $(F_5F_{14}, F_6F_{14}, F_8F_{14}, F_{10}F_{14})$ results in the highest overall D-efficiency of 0.92; the A-efficiency in that case is 0.79. For other models, the D-efficiencies range between 0.68-0.91. If fewer than four interactions are considered important, the best efficiencies are obtained when the interactions listed in Table 9 are included in the model.

Table 9. *Efficiencies Under Different Models*

No. of Ints.	Interactions	Lower Bound to D-Eff. (A-Eff.)	Range of D-Eff.
3	$(F_4F_6, F_6F_{14}, F_6F_8)$; $(F_4F_6, F_6F_{14}, F_5F_{14})$	0.95(0.88)	0.74-0.95
2	(F_4F_6, F_6F_{14}); (F_2F_3, F_2F_{13})	0.98(0.95)	0.81-0.98
1	F_2F_3; F_4F_6	0.99(0.99)	0.90-0.98

In Table 9, when only one interaction is included in the model, only 65 of the 91 possible models are admissible.

5.3. $OA(24, 17, 2^{16} \times 3, 2)$

Next, let us consider an $OA(24, 17, 2^{16} \times 3, 2)$, displayed in Table 10. In a plan represented by this array, there are 5 d.f. unused after the estimation of the mean and all main effects. Let us denote the 3-level factor by G and the 2-level factors by $F_1, \ldots, F_{16}$. We can thus contemplate inclusion of either (a) at most five interactions of the type F_iF_j or, (b) an interaction F_iG for some i, $1 \leq i \leq 16$ and at most three interactions of the type F_iF_j or, (c) two interactions of the type F_iG, F_jG and at most one interaction of the type F_iF_j.

Table 10. *An* $OA(24, 17, 2^{16} \times 3, 2)$

F_1	0000	0000	0000	1111	1111	1111
F_2	0011	0011	0011	1100	1100	1100
F_3	0101	0101	0101	1010	1010	1010
F_4	0011	1100	1001	1100	0011	0110
F_5	0110	1010	0101	1001	0101	1010
F_6	0100	0110	1011	1011	1001	0100
F_7	0010	0111	1100	1101	1000	0011
F_8	0101	1011	1000	1010	0100	0111
F_9	0110	1101	0010	1001	0010	1101
F_{10}	0001	1110	0110	1110	0001	1001
F_{11}	0000	1001	1111	1111	0110	0000
F_{12}	0111	0000	1110	1000	1111	0001
F_{13}	0011	0011	0011	0011	0011	0011
F_{14}	0101	0101	0101	0101	0101	0101
F_{15}	0011	1100	1001	0011	1100	1001
F_{16}	0110	1010	0101	0110	1010	0101
G	0000	1111	2222	0000	1111	2222

Consider case (a) above. If five interactions among the 2-level factors are included in the model, the highest D- and A-efficiencies are obtained when the following interactions are included in the model : $F_6F_{12}, F_{13}F_{14}, F_{13}F_{16}, F_{14}F_{15}, F_{15}F_{16}$. The D- and A-efficiencies are respectively 0.88 and 0.70. For other models of the same type, the D-efficiencies range between 0.59-0.0.86. If fewer than five interactions are considered important, the best efficiencies are obtained when the interactions listed in Table 11(a) are included in the model. For other models, the range of D-efficiencies are given in the last column of Table 11(a).

Table 11(a). *Efficiencies Under Different Models*

No. of Ints.	Interactions	Lower Bound to D-Eff. (A-Eff.)	Range of D-Eff.
4	$(F_{13}F_{14}, F_{13}F_{16}, F_{14}F_{15}, F_{15}F_{16})$	0.94(0.86)	0.64-0.90
3	$(F_{13}F_{14}, F_{13}F_{16}, F_{14}F_{16})$;$(F_2F_5, F_{14}F_{15}, F_{15}F_{16})$; $(F_3F_4, F_4F_5, F_{13}F_{14})$	0.96(0.90)	0.70-0.95
2	(F_1F_6, F_1F_i), $i = 7, 8, 9, 10$; (F_1F_7, F_1F_i), $i = 8, 9$; (F_1F_i, F_1F_{10}), $i = 8, 9$	0.98(0.96)	0.76-0.98
1	$F_1F_i,\ i = 6, 7, \ldots, 10$	0.99(0.98)	0.87-0.99

In Table 11 (a), if only three interactions are considered important, then there are 17 sets of interactions with highest D- and A-efficiencies. Among these 17 sets, there are three non-isomorphic models as indicated in Table 11 (a) (two models are called isomorphic if one can be obtained from the other by a renaming of the factors).

Now consider the models under (b) above. There are eight models, under each of which the overall D- and A-efficiencies are the same. These are: $(F_{13}G, F_8F_{10}, F_{13}F_{16}, F_{15}F_{16})$, $(F_{13}G, F_7F_9, F_{13}F_{14}, F_{14}F_{15})$, $(F_{14}G, F_9F_{10}, F_{13}F_{14}, F_{13}F_{16})$, $(F_{14}G, F_7F_8, F_{14}F_{15}, F_{15}F_{16})$, $(F_{15}G, F_7F_9, F_{13}F_{14}, F_{14}F_{15})$, $(F_{16}G, F_7F_8, F_{14}F_{15}, F_{15}F_{16})$, $(F_{16}G, F_9F_{10}, F_{13}F_{14}, F_{13}F_{16})$, $(F_{15}G, F_8F_{10}, F_{13}F_{16}, F_{15}F_{16})$. The D- and A-efficiency lower bounds for these models are 0.87 and 0.64. For other models of the same type, the D-efficiencies range between 0.68-0.85. If fewer than four interactions listed under (b) above are important, the best efficiencies are obtained when the interactions listed in Table 11 (b) are included in the model.

Table 11 (b). *Efficiencies Under Different Models*

No. of Ints.	Interactions	Lower Bound to D-Eff. (A-Eff.)	Range of D-Eff.
3	$(F_{16}G, F_3F_4, F_4F_{11})$; $(F_{16}G, F_4F_{11}, F_{14}F_{15})$	0.90(0.75)	0.74-0.90
2	$(F_{14}G, F_1F_6)$; $(F_{13}G, F_1F_8)$	0.93(0.78)	0.81-0.93
1	$F_{13}G$; $F_{14}G$; $F_{15}G$; $F_{16}G$	0.95(0.84)	-

In Table 11 (b), if only three interactions are important, the inclusion of the two non-isomorphic models listed in Table 11 (b) result in the highest D-efficiency lower bound. If only one interaction of the type F_iG is included, then only four of the possible 16 models are admissible.

In case (c), there are three non-isomorphic models. These are $(F_{13}G, F_{14}G, F_7F_{10})$, $(F_{15}G, F_{16}G, F_7F_{10})$ and $(F_{14}G, F_{15}G, F_{13}F_{16})$. The lower bounds to the D-efficiency under either of the models is 0.87 and the same under A-criterion is 0.67. For other models of the same type, the D-efficiencies are lower, these ranging between 0.82-0.87. If two of the three interactions of the types included in (c) above are important, the inclusion of the sets $(F_{13}G, F_{14}G)$ or, $(F_{13}G, F_{16}G)$ or, $(F_{14}G, F_{15}G)$ or, $(F_{15}G, F_{16}G)$ result in the same D- and A-efficiency, these being 0.91 and 0.74 respectively.

5.4. $OA(24, 15, 2^{13} \times 3 \times 4, 2)$

An $OA(24, 15, 2^{13} \times 3 \times 4, 2)$ can be obtained from the array $OA(24, 17, 2^{16} \times 3, 2)$, displayed in Table 10, by deleting column F_{14} and replacing the combinations under columns F_2 and F_3 by four distinct symbols according to the following scheme: $(0,0) \to 0$, $(0,1) \to 1$, $(1,0) \to 2$, $(1,1) \to 3$. A plan represented by this array leaves 5 d.f. unused, after the estimation of the mean and all main effects. Let the 4-level factor be denoted by H, the 3-level factor by G and the 2-level factors by $F_1, \ldots, F_{13}$.

Two general facts are observed: When the mean and all the main effects are already in the model, (1) no interaction involving H is estimable, and, (2) models which include F_iG, $i = 1, \ldots, 10$ are inadmissible. Thus, one can contemplate the inclusion of the following types of interactions in the model, alongwith the mean and all main effects: (a) $(F_iG, F_jG, F_kF_{k'})$, (b) F_iG and at most three interactions among 2-level factors, (c) at most five interactions among 2-level factors; of course, in (a) and (b) above, $i, j = 11, 12$ or, 13.

First consider case (a). The highest D- and A-efficiencies are obtained when $(F_iG, F_{12}G, F_4F_{10})$ or $(F_iG, F_{12}G, F_6F_7)$, $i = 11, 13$ are in the model. The D- and A-efficiencies are respectively, 0.87 and 0.67. The D-efficiency lower bounds for other models range between 0.82-0.87. If only two of the interactions of the type F_iG are considered important, then there are only two admissible models that include either $(F_{11}G, F_{12}G)$ or, $(F_{12}G, F_{13}G)$. The plan is equally efficient under either of these models, the D- and A-efficiencies being 0.91 and 0.74 respectively.

The highest efficiencies under the models specified in (b) are obtained when $(F_{12}G, F_3F_9, F_4F_8, F_{12}F_{13})$ are in the model. The D- and A-efficiencies are respectively, 0.87 and 0.64. The D-efficiency lower bounds for other models range between 0.68-0.87. If fewer than four interactions listed under (b) above are important, the best efficiencies are obtained when the interactions listed in Table 12 (a) are included in the model. The range of D-efficiencies for other modelsd are also given in Table 12 (a).

Table 12 (a). *Efficiencies Under Different Models*

No. of Ints.	Interactions	Lower Bound to D-Eff. (A-Eff.)	Range of D-Eff.
3	$(F_{11}G, F_2F_3, F_2F_9)$; $(F_{11}G, F_2F_9, F_{12}F_{13})$	0.90(0.75)	0.74-0.90
2	$(F_{12}G, F_1F_9)$; $(F_{13}G, F_{11}F_{12})$	0.93(0.80)	0.81-0.93
1	$F_iG,\ i = 11, 12, 13$	0.95(0.84)	-

Consider finally case (c) above. The highest efficiencies are obtained when $(F_4F_{10}, F_5F_6, F_7F_9, F_{11}F_{12}, F_{12}F_{13})$ are in the model. The D- and A-efficiencies are respectively, 0.84 and 0.59. The D-efficiency lower bounds for other models range between 0.59-0.84. If fewer than five interactions listed under (c) above are important, the best efficiencies are obtained when the interactions listed in Table 12 (b) are included in the model.

Table 12 (b). *Efficiencies Under Different Models*

No. of Ints.	Interactions	Lower Bound to D-Eff. (A-Eff.)	Range of D-Eff.
4	$(F_3F_8, F_3F_{10}, F_{11}F_{12}, F_{12}F_{13})$	0.90(0.75)	0.64-0.90
3	(F_1F_2, F_1F_9, F_2F_3); $(F_1F_9, F_{11}F_{12}, F_2F_3)$	0.95(0.90)	0.70-0.95
2	$(F_{11}F_{12}, F_{12}F_{13})$; $(F_2F_3, F_{11}F_{12})$;		
	$(F_1F_2, F_{12}F_{13})$; (F_1F_2, F_2F_3)	0.98(0.94)	0.76-0.97
1	F_1F_2	0.99(0.97)	0.87-0.99

In Table 12 (b), if only three interactions among 2-level factors are considered important, then instead of F_2F_3 in either of the models, above can include the interaction $F_{12}F_{13}$, without sacrificing the efficiencies. If only one interaction among 2-level factors is included, among the 78 possible models, only 48 models are admissible. The highest D- and A-efficiencies are obtained when F_1F_2 (or, some models isomorphic to this) is included.

6. TWENTY EIGHT RUN PLAN

An $OA(28, 13, 2^{12} \times 7, 2)$ was reported by Dey and Midha (1996). This is displayed (in transposed form) in Table 13. In a plan represented by this array, there are 9 d.f. unused. Let the 7-level factor be denoted by G and the 2-level factors by $F_i, i = 1, \ldots, 12$. One can think of including interactions of the following types: (a) at most nine interactions of the type F_iF_j or, (b) one interaction of the type F_iG and at most three interactions of the type F_iF_j. We treat these two cases separately.

We first consider case (b). Among the possible models, the inclusion of the interactions $(F_7G, F_5F_8, F_6F_{11}, F_6F_{12})$ results in the highest D-efficiency of 0.66, the corresponding A-efficiency being 0.21. For other models, the D-efficiencies range between 0.50-0.66.

If the interaction F_iG for some i and only two interactions among F_i's are considered important, the highest D-efficiency is obtained when $(F_6G, F_3F_8, F_4F_{11})$ or, (F_3G, F_5F_7, F_6F_9) are included in the model, the D-efficiency being 0.69; the A-efficiency in such a case is 0.20. For other models of the same type, the D-efficiencies range between 0.52-0.69. If one interaction of the type F_iG and one of the type F_iF_j are to be included in the model, the highest D-efficiency is obtained when (F_3G, F_6F_9) or, (F_6G, F_3F_8) are in the model, the D- and A-efficiencies being 0.71 and 0.24 respectively. For other models of the same type, the D-efficiencies range between 0.59-0.71. Finally, if one interaction of the type F_iG is considered important, all the 12 models are admissible. The D-efficiency when $F_iG,\ 1 \leq i \leq 12$ is included in the model is 0.71 and the A-efficiency is 0.23.

Table 13. *An* $OA(28, 13, 2^{12} \times 7, 2)$

F_1	1100	1100	1100	1100	1100	1100	1100
F_2	1100	0011	1001	0101	1010	1010	0101
F_3	1100	1010	0011	0101	1001	0101	1010
F_4	1100	1001	0110	0011	0101	1010	0110
F_5	1100	0101	0011	1010	0011	1100	1001
F_6	1100	0110	0110	1100	0110	0011	0011
F_7	1010	1001	0110	1001	1010	0101	0101
F_8	1010	1001	1001	1100	0101	1001	0011
F_9	1010	0101	1100	0101	0011	0110	1010
F_{10}	1010	0110	0101	0011	1100	1100	0011
F_{11}	1010	0101	0011	0110	1100	0011	1100
F_{12}	1010	1010	1010	1010	1010	1010	1010
G	0000	1111	2222	3333	4444	5555	6666

Under case (a), we limit the number of interactions to five. The efficiencies under different models are summarized in Table 14.

Table 14. *Efficiencies Under Different Models*

No. of Ints.	Interactions	Lower Bound to D-Eff. (A-Eff.)	Range of D-Eff.
5	$(F_5F_{12}, F_6F_9, F_6F_{11}, F_6F_{12}, F_8F_9)$	0.89(0.74)	0.75-0.88
4	$(F_5F_8, F_6F_9, F_6F_{11}, F_8F_9)$	0.91(0.80)	0.62-0.91
3	$(F_3F_8, F_4F_9, F_4F_{11})$; (F_5F_7, F_5F_8, F_6F_9)	0.95(0.88)	0.70-0.95
2	F_3F_8 and one among $\{F_4F_9, F_4F_{11}, F_6F_{12}\}$;		
	F_6F_9 and one among $\{F_3F_{10}, F_5F_7, F_5F_8\}$	0.97(0.92)	0.80-0.97
1	F_2F_{11}; F_3F_8; F_1F_7; F_6F_9	0.99(0.98)	0.87-0.98

7. THIRTY SIX RUN PLANS

Among the several 36-rowed orthogonal arrays, we consider the following : (i) $OA(36, 15, 2^2 \times 3^{12} \times 6, 2)$, (ii) $OA(36, 17, 2^4 \times 3^{13}, 2)$, (iii) $OA(36, 14, 3^{13} \times 4, 2)$, (iv) $OA(36, 10, 3^7 \times 6^3, 2)$ and (v) $OA(36, 11, 2^8 \times 6^3, 2)$. We consider plans represented by these arrays one by one. To begin with, an $OA(36, 13, 12 \times 3^{12}, 2)$ is displayed in Table 15, from which arrays (i) - (iii) can be obtained on replacing the symbols in the 12-symbol column H by the rows of a suitable orthogonal array with 12 rows. In Table 15, the last two levels of the 12-level factor are denoted by a and b.

Table 15. An $OA(36, 13, 12 \times 3^{12}, 2)$

H	012345	6789*ab*	012345	6789*ab*	012345	6789*ab*
G_2	000000	000000	111111	111111	222222	222222
G_3	000011	112222	111122	220000	222200	001111
G_4	001222	011120	112000	122201	220111	200012
G_5	100221	022011	211002	100122	022110	211200
G_6	120002	211021	201110	022102	012221	100210
G_7	002101	220121	110212	001202	221020	112010 .
G_8	021012	020211	102120	101022	210201	212100
G_9	102012	202110	210120	010221	021201	121002
G_{10}	020122	102101	101200	210212	212011	021020
G_{11}	200102	121210	011210	202021	122021	010102
G_{12}	201021	201201	012102	012012	120210	120120
G_{13}	010020	221112	121101	002220	202212	110001

7.1 $OA(36, 15, 2^2 \times 3^{12} \times 6, 2)$

An $OA(36, 15, 2^2 \times 3^{12} \times 6, 2)$ can be constructed by replacing the symbols in column H by the rows of an $OA(12, 5, 2^2 \times 6, 2)$ displayed in Table 2. A plan represented by this orthogonal array has 4 d.f. unused. However, it turns out that this plan does not allow the estimabilty of any two-factor interaction.

7.2. $OA(36, 17, 2^4 \times 3^{13}, 2)$

An $OA(36, 17, 2^4 \times 3^{13}, 2)$ can be constructed by replacing the symbols in column H by the rows of an $OA(12, 5, 2^4 \times 3, 2)$ displayed in Table 1, whose columns are denoted by F_1, F_2, F_3, F_4 and G_1. A plan represented by this array has 5 d.f. unused. The 3-level factors are denoted by $G_1, \ldots, G_{13}$ and the 2-level factors by $F_1, \ldots, F_4$. The plan allows the estimability of any one interaction among 2-level factors, i.e., any one of the interactions $F_iF_j,\ 1 \leq i < j \leq 4$. Also, the plan ensures the estimability of interactions $F_iG_{13},\ 1 \leq i \leq 4$ with respective D- and A-efficiencies of 0.97 and 0.89. No other interaction between a 3-level factor and a 2-level factor is estimable via this plan.

The following models can be envisaged : (a) interactions $(F_iG_{13}, F_jG_{13}, F_kF_{k'})$,
(b) $(F_iG_{13}, F_jF_{j'}, F_kF_{k'}, F_lF_{l'})$, (c) five interactions involving 2-level factors.

Consider case (a) above. Among the possible models of this type, only 16 are admissible. Under each of these admissible models, the plan has the same D-efficiency of 0.91. The highest A-efficiency of 0.75 is obtained when either $(F_1G_{13}, F_2G_{13}, F_3F_4)$ or, $(F_1G_{13}, F_4G_{13}, F_2F_3)$ or, $(F_2G_{13}, F_3G_{13}, F_1F_4)$ or, $(F_3G_{13}, F_4G_{13}, F_1F_2)$ are included in the model. If only two interactions, each involving a 3-level and a 2-level factor are important, the interactions that can be included are only either (F_1G_{13}, F_2G_{13}) or, (F_1G_{13}, F_4G_{13}) or, (F_2G_{13}, F_3G_{13}) or, (F_3G_{13}, F_4G_{13}). The D- and A-efficiencies under each of the models are 0.94 and 0.81 respectively.

Now consider case (b). Among the 32 admissible models, the highest D- and A-efficiencies of 0.90 and 0.63 respectively are obtained when any of the following sets is included : $(F_1G_{13}, F_1F_i, F_2F_3, F_3F_4), i = 2, 4$, $(F_2G_{13}, F_2F_i, F_1F_4, F_3F_4), i = 1, 3$, $(F_3G_{13}, F_3F_i, F_1F_2, F_1F_4),\ i = 2, 4$ or, $(F_4G_{13}, F_4F_i, F_1F_2, F_2F_3),\ i = 1, 3$. If only three interactions, with one of them involving a 3-level factor, are to be included in the model, then among the 40 admissible cases, the highest D- and A-efficiencies of 0.94 and 0.78 respectively are obtained when any of the following sets is included : $(F_iG_{13}, F_1F_2, F_2F_3),\ i = 1, 3$, $(F_iG_{13}, F_1F_4, F_3F_4),\ i = 1, 3$, $(F_iG_{13}, F_2F_3, F_3F_4),\ i = 2, 4$ or, $(F_iG_{13}, F_1F_2, F_1F_4),\ i = 2, 4$. If two interactions, with one of them involving a 3-level factor, are considered important, then among 20 admissible models, the highest D- and A-efficiencies of 0.95 and 0.86 respectively are obtained when any of the following sets is included: $(F_iG_{13}, F_1F_2),\ i = 3, 4$, $(F_iG_{13}, F_1F_4),\ i = 2, 3$, $(F_iG_{13}, F_2F_3),\ i = 1, 4$ or, $(F_iG_{13}, F_3F_4),\ i = 1, 2$.

Finally, consider case (c). There are six interactions among the four 2-level factors, out of which a subset of five interactions can be included in the model at a time. The plan allows the estimability of all such subsets of five interactions, alongwith those of the mean and all main effects. The highest D- and A-efficiencies of 0.89 and 0.47 respectively are obtained when the following sets are included in the model: $(F_1F_2, F_1F_4, F_2F_3, F_3F_4, F_iF_j),\ (i, j) = (1, 3)$ or $(2, 4)$.

If only four interactions among the 2-level factors are considered important, the highest D- and A-efficiencies of 0.96 and 0.90 respectively are obtained when $(F_1F_2, F_1F_4, F_2F_3, F_3F_4)$ are in the model. When only three interactions are to be included, inclusion of any three of the four listed above leads to the highest efficiencies, the D- and A-efficiencies being 0.97 and 0.93 respectively. For two interactions in the model, the highest D- and A-efficiencies 0.98 and 0.96 respectively are obtained when any two of the four interactions $(F_1F_2, F_1F_4, F_2F_3, F_3F_4)$ are included in the model, except the combinations (F_1F_2, F_3F_4) and (F_1F_4, F_2F_3), under which the efficiencies are slightly smaller. Finally, if only one interaction is to be included, all six possible models are admissible. The highest D- and A-efficiencies are respectively 0.99 and 0.98, except when the interactions F_1F_3 or F_2F_4 are included, the D- and A-efficiencies under these two cases being 0.93 and 0.67 respectively.

7.3. $OA(36, 14, 3^{13} \times 4, 2)$

An $OA(36, 14, 3^{13} \times 4, 2)$ can be obtained from an $OA(36, 13, 3^{12} \times 12, 2)$ in Table 15 by replacing the 12-symbol column H by two columns, one with 4 symbols and the other with 3 symbols. A plan represented by this array has 6 d.f. unused. Let the 4-level factor be denoted by K and the 3-level factors by $G_1, \ldots, G_{13}$. Therefore, one can contemplate the following types of interactions in the model, alongwith the mean and all main effects: (i) an interaction among G_i and G_j, (ii) an interaction of the type G_iK. It turns out that the plan does not ensure the estimability of *any* interaction among G_i and G_j. Furthermore, only one interaction, G_1K can be included in the model and inclusion of any interaction of the type G_iK, $i \neq 1$ renders the model inadmissible. Arguing as in Section 3, the plan can in fact be shown to be universally optimal under a model that includes the mean, all main effects and the interaction G_1K. The plan is also saturated.

7.4. $OA(36, 10, 3^7 \times 6^3, 2)$

An $OA(36, 10, 3^7 \times 6^3, 2)$ was reported by Finney (1982) and displayed in Wu and Hamada (2000, Table 7C.8, p. 340). A plan represented by this array has 6 d.f. unused. Let the 3-level factors be denoted by $G_1, \ldots, G_7$ with column $3 + i$ in Table 7C.8 in Wu and Hamada (2000) corresponding to $G_i, 1 = 1, \ldots, 7$. Clearly, only one interaction among the 3-level factors can be included in the model. Out of 21 possible models involving the interactions G_iG_j, only 6 are admissible. These are the ones that include (G_iG_j), $i = 1, 7, j = 2, 4, 6$. The plan is equally efficient under each of these models. The D- and A-efficiencies are 0.76 and 0.23 respectively.

7.5. $OA(36, 11, 2^8 \times 6^3, 2)$

An $OA(36, 11, 2^8 \times 6^3, 2)$, reported by Finney (1982), is displayed in Wu and Hamada (2000, Table 7C.9, pp.340-341). A plan represented by this array has 12 d.f. unused, after the estimation of the mean and all main effects. Let the 6-level factors be denoted by H_1, H_2 and H_3 and the 2-level factors by $F_1, \ldots, F_8$ with the first 3 columns in Table 7C.9 corresponding to H_1, H_2, H_3 respectively and column $3 + i$ corresponding to $F_i, i = 1, \ldots, 8$. One can then contemplate including the following types of interactions in the model: (a) two interactions between a 6-level and a 2-level factor alongwith at most two interactions among the 2-level factors, (b) one interaction between a 6-level and a 2-level factor alongwith at most seven interactions among the 2-level factors, (c) at most 12 interactions among the 2-level factors.

In case (a), inclusion of the interactions $(F_8H_1, F_8H_3, F_3F_5, F_6F_7)$ results in the highest efficiency; the D-efficiency lower bound in this case is 0.63 and that under A-criterion is 0.18. For other models of the same type, the D-efficiencies range between 0.46-0.63. If only three interactions are deemed important, then the highest efficiency is obtained when the interactions (F_2H_2, F_7H_2, F_2F_6) are in the model, the D- and A-efficiencies being 0.67 and 0.27 respectively. The D-efficiencies for other models range between 0.49-0.67. If only two interactions between a 6-level and a 2-level factor are to be included, then the highest efficiency is obtained when either of the sets (F_2H_i, F_7H_i) $i = 2, 3$ are included in the model. The D- and A-efficiencies are then 0.72 and 0.37 respectively. The D-efficiencies for other similar models range between 0.58-0.71.

Under cases (b) and (c) above, we present results when at most five interactions are included in the model on the presumption that not too many interactions are considered important. The best efficiencies obtained under different models with five or fewer interactions in case (b) are presented in Table 16 (a).

Table 16 (a). *Efficiencies Under Different Models*

No. of Ints.	Interactions	Lower Bound to D-Eff. (A-Eff.)	Range of D-Eff.
5	$(F_4H_3, F_2F_5, F_3F_5, F_4F_7, F_5F_8)$	0.68(0.27)	0.56-0.68
4	$(F_4H_3, F_3F_7, F_5F_6, F_5F_8)$	0.72(0.37)	0.60-0.72
3	(F_4H_3, F_1F_3, F_5F_8)	0.76(0.44)	0.64-0.76
2	(F_1H_1, F_3F_5); (F_8H_1, F_3F_5)	0.81(0.49)	0.72-0.81
1	F_1H_1; F_8H_1	0.86(0.63)	0.81-0.84

Finally, consider case (c). The best efficiencies with five or fewer interactions in the model under case (c) are sumnmarized in Table 16 (b).

Table 16 (b). *Efficiencies Under Different Models*

No. of Ints.	Interactions	Lower Bound to D-Eff. (A-Eff.)	Range of D-Eff.
5	$(F_2F_8, F_3F_7, F_5F_6, F_5F_8, F_6F_i)$, $i = 2, 7$	0.75(0.38)	0.67-0.75
4	$(F_3F_4, F_4F_7, F_5F_6, F_5F_8)$	0.80(0.46)	0.71-0.80
3	$(F_1F_3, F_5F_8, F_iF_j), i = 4, 6, j = 2, 7$	0.86(0.60)	0.78-0.86
2	(F_1F_3, F_5F_8)	0.91(0.73)	0.83-0.90
1	F_1F_3; F_5F_8	0.95(0.84)	0.91-0.95

8. CONCLUDING REMARKS

Fractional factorial plans represented by orthogonal arrays of strength two are know to be universally optimal (and hence in particular A-, D- and E-optimal) in the class of all plans with same number of runs under a model that includes the mean and all main effects, all other factorial effects being assumed negligible. Because of this, such plans are used quite often in practice. However, when a plan represented by an orthogonal array of strength two is not saturated, one can think of entertaining some two-factor interactions also in the model. The issue of estimability of factorial effects, i.e., the mean, all main effects and a specified set of interactions has been examined thoroughly in this paper with respect to fractional factorial plans for asymmetric factorials represented by asymmetric orthogonal arrays of strength two with small number of runs. It is found that while in most cases, one or more interactions can be estimated, alongwith the mean and all main effects, there are a few plans based on asymmetric orthogonal arrays that do not permit the estimability of any two-factor interaction, when the mean and all main effects are already in the model. When the factorial effects of interest are estimable via a plan represented by an orthogonal array, we find lower bounds to the overall D- and A-efficiencies of the plan under a model that includes the mean, all main effects and a specified set of two-factor interactions. It turns out that many of the plans have high overall efficiencies under the D- and A-criterion.

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