A class of Diophantine equations involving Bernoulli polynomials

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Let a, b be nonzero rational numbers and $C(y)$ a polynomial with rational coefficients. We study the Diophantine equations

$$aB_m(x) = bf_n(y) + C(y)$$

and

$$af_m(x) = bB_n(y) + C(y)$$

with $m \geqslant n > \deg C + 2$ for solutions in integers x, y. Here $f_n(x) = x(x+1)\cdots(x+n-1)$ and the Bernoulli polynomials $B_n(x)$ are defined by the generating series

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.$$

Then, $B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i} x^i$ where $B_r = B_r(0)$ is the rth Bernoulli number. In fact, B_r are rational numbers defined recursively by $B_0 = 1$ and $\sum_{i=0}^{n-1} \binom{n}{i} B_i = 0$ for all $n \geqslant 2$. The odd Bernoulli number $B_r = 0$ for r odd > 1 and the first few are:

$$B_0 = 1, \quad B_1 = -1/2, \quad B_2 = 1/6, \quad B_4 = -1/30.$$

The Bernoulli polynomials B_n are related to the sums of nth powers of the first few natural numbers as follows. For any $n \geq 1$, the sum $1^n + 2^n + \cdots + k^n$ is a polynomial function $S_n(k)$ of k and $S_n(x) = (B_{n+1}(x+1) - B_{n+1})/(n+1)$.

One says that an equation $f(x) = g(y)$ has infinitely many rational solutions with bounded denominator if there exist a positive integer λ such that $f(x) = g(y)$ has infinitely many rational solutions x, y satisfying $x, y \in \frac{1}{\lambda}\mathbb{Z}$ and, more generally, we look for rational solutions with bounded denominators.

Earlier, we have studied the equations of the type $f(x) = g(y)$ for:

(i) $f(x) = x(x+1)\cdots(x+m-1)$ and a general $g(y)$ [2,4] and
(ii) $f(x) = aB_m(x)$, $g(y) = bB_n(y) + C(y)$ where $m \geq n > \deg(C) + 2$ [5].

Here, we prove the following two theorems:

Theorem 1. *For $m \geq n > \deg(C) + 2$, the equation*

$$aB_m(x) = bf_n(y) + C(y)$$

has only finitely many rational solutions with bounded denominator except in the following situations:

(i) *$m = n$, $m+1$ is a perfect square, $a = b(\sqrt{m+1})^n$,*
(ii) *$m = 2n$, $(n+1)/3$ is a perfect square, $a = b(\frac{n}{2}\sqrt{\frac{n+1}{3}})^n$.*

In each case, there is a uniquely determined polynomial C for which the equation has infinitely many rational solutions with a bounded denominator. Further, C is identically zero when $m = n = 3$ and has degree $n - 4$ when $n > 3$.

Theorem 2. *For $m \geq n > \deg(C) + 2$, the equation*

$$af_m(x) = bB_n(y) + C(y)$$

has only finitely many rational solutions with bounded denominator excepting the following situations when it has infinitely many:

$$m = n, \quad m+1 \text{ is a perfect square}, \quad b = a\left(\sqrt{m+1}\right)^m.$$

In these situations, the polynomial C is also uniquely determined to be

$$C(x) = af_m\left(\left(\pm\sqrt{m+1}\right)x + \frac{1-m+\sqrt{m+1}}{2}\right) - bB_m(x)$$

and has degree $m - 4$.

Remarks. (a) The condition $n > \deg(C) + 2$ in the two theorems is sharp as can be seen from the fact that the equation

$$B_4(y+2) = f_4(y) + 2y^2 + 6y + \frac{119}{30}$$

holds for all y.

(b) A (common) particular case of the theorems was proved in [1].

(c) In the exceptional cases (i) and (ii) in the first theorem, the unique polynomial C for which the equation has infinitely many solutions, is given as follows: In case (i),

$$C(x) = aB_m\left(\frac{x + (m \pm \sqrt{m+1} - 1)/2}{\pm\sqrt{m+1}}\right) - bf_m(x).$$

In case (ii), writing $n + 1 = 3u^2$ and writing $\phi(x)$ for the unique polynomial of degree n for which $\phi(x^2) = B_{2n}(x + 1/2)$,

$$C(x) = a\phi\left(\frac{2x + 6u^3 + 24u^2 + 6u - 16}{u(3u^2 - 1)}\right) - bf_{3u^2-1}(x).$$

(d) It should be noted that when $a = b$, the computations are much easier and yield in all cases that there are only finitely many solutions.

(e) Evidently, one may assume $a = 1$ by replacing b by b/a and $C(y)$ by $C(y)/a$.

We shall make extensive use of the following theorem of Bilu and Tichy [3]:

Theorem A. *For non-constant polynomials $f(x)$ and $g(x) \in \mathbb{Q}[x]$, the following are equivalent:*

(a) *The equation $f(x) = g(y)$ has infinitely many rational solutions with a bounded denominator.*

(b) *We have $f = \phi(f_1(\lambda))$ and $g = \phi(g_1(\mu))$ where $\lambda(x), \mu(x) \in \mathbb{Q}[X]$ are linear polynomials, $\phi(x) \in \mathbb{Q}[X]$, and $(f_1(x), g_1(x))$ is a standard pair over $\mathbb{Q}$ such that the equation $f_1(x) = g_1(y)$ has infinitely many rational solutions with a bounded denominator.*

Standard pairs are defined as follows. In what follows, a and b are nonzero elements of some field, m and n are positive integers, and $p(x)$ is a nonzero polynomial (which may be constant).

STANDARD PAIRS

A standard pair of the first kind is

$$\left(x^t, ax^r p(x)^t\right) \quad \text{or} \quad \left(ax^r p(x)^t, x^t\right)$$

where $0 \leqslant r < t$, $(r, t) = 1$ and $r + \deg p(x) > 0$.

A standard pair of the second kind is

$$\left(x^2, (ax^2+b)p(x)^2\right) \quad \text{or} \quad \left((ax^2+b)p(x)^2, x^2\right).$$

A standard pair of the third kind is

$$\left(D_k(x, a^t), D_t(x, a^k)\right)$$

where $(k, t) = 1$. Here D_t is the tth Dickson polynomial

$$D_t(x, a) = \sum_{i=0}^{\lfloor t/2 \rfloor} \frac{t}{t-i}\binom{t-i}{i}(-a)^i x^{t-2i}.$$

A standard pair of the fourth kind is

$$\left(a^{-t/2}D_t(x, a), b^{-k/2}D_k(x, a)\right)$$

where $(k, t) = 2$.

A standard pair of the fifth kind is

$$\left((ax^2-1)^3, 3x^4-4x^3\right) \quad \text{or} \quad \left(3x^4-4x^3, (ax^2-1)^3\right).$$

By a standard pair over a field k, we mean that $a, b \in k$, and $p(x) \in k[x]$.

The theorem of Bilu and Tichy above shows the relevance of the following definition:

A *decomposition* of a polynomial $F(x) \in \mathbf{C}[x]$ is an equality of the form $F(x) = G_1(G_2(x))$, where $G_1(x), G_2(x) \in \mathbf{C}[x]$. The decomposition is called *nontrivial* if $\deg G_1 > 1$, $\deg G_2 > 1$.

Two decompositions $F(x) = G_1(G_2(x))$ and $F(x) = H_1(H_2(x))$ are called *equivalent* if there exist a linear polynomial $l(x) \in \mathbf{C}[x]$ such that $G_1(x) = H_1(l(x))$ and $H_2(x) = l(G_2(x))$. The polynomial called *decomposable* if it has at least one nontrivial decomposition, and *indecomposable* otherwise.

We shall also use the following result due to Bilu et al. [1]:

Theorem B. *Let $m \geqslant 2$. Then,*

(i) *$B_m(x)$ is indecomposable if m is odd and,*
(ii) *if $m = 2k$, then any nontrivial decomposition of $B_m(x)$ is equivalent to $B_m(x) = h((x-1/2)^2)$.*

The equation $S_m(x) = S_n(y)$ has been studied in [1]. This is a particular case of our result.

We first consider the first theorem. Evidently, we may assume $a = 1$ and we look at the equation $B_m(x) - bf_n(y) = C(y)$ where $f_n(x) = x(x+1)\cdots(x+n-1)$ and $m \geqslant n > \deg(C) - 2$.

Proof of Theorem 1. As remarked in the beginning (remark (e)), we may assume that $a = 1$.

Case I: Let us first consider the case when $m = n = 2d$.

If the equation has infinitely many solutions, the Bilu–Tichy theorem gives $B_{2d} = \phi \circ f_1 \circ \lambda$ and $bf_{2d} + C = \phi \circ g_1 \circ \mu$ where λ, μ are linear polynomials over $\mathbb{Q}$ and (f_1, g_1) is a standard pair over $\mathbb{Q}$. Since we know from [1] that the only nontrivial decomposition of B_{2d} up to equivalence where has $f_1(x) = (x - 1/2)^2$, it follows that either:

(a) $\deg \phi = 1$, or
(b) $\deg \phi = d$ and $B_{2d}(x) = \phi((l_0 + l_1 x - 1/2)^2)$ and $bf_{2d}(x) + C(x) = \phi(kx^2 + lx + t)$ and the equation $(x - 1/2)^2 = ky^2 + ly + t$ has infinitely many solutions, or
(c) $\deg \phi = 2d$ in which case

$$B_{2d}(rx + s) = bf_{2d}(x) - C(x).$$

First, suppose (a) holds, i.e., $\deg \phi = 1$. This means that (f_1, g_1) is a standard pair with $\deg f_1 = \deg g_1 = 2d > 2$. This is impossible as seen by looking at the conditions on the degrees of standard pairs.

Next, we consider (b), i.e., the possibility where ϕ has degree d.

We use the following observation, see [5]:

Lemma. *If $B_{2d}(rx + s) = \phi((x - 1/2))^2$ for some $r, s \in \mathbb{Q}$ with $r \neq 0$, then $(r, s) = (1, 0)$ or $(-1, 1)$. In particular, $B_{2d}(x) = \phi((x - 1/2)^2)$.*

Therefore, $B_{2d}(x) = \phi((x - 1/2)^2)$ and $bf_{2d}(x) + C(x) = \phi(kx^2 + lx + t)$.

Considering the coefficients of $x^{2d}, x^{2d-1}, x^{2d-2}$ and x^{2d-3} of the second equation, we get the following expressions.

Coefficient of x^{2d} is $b = \phi_d k^d = k^d$ (the fact that $\phi_d = 1$ we know from the first equation).

Coefficient of x^{2d-1} gives $l = k(2d - 1)$.

Coefficient of x^{2d-2} gives $t = k(d - 1)(2d - 1)/3 - (2d - 1)/12$.

Coefficient of x^{2d-3} gives

$$c_{2d-3} + b\frac{d^2(d-1)(2d-1)^2(2d-3)}{6} = d(d-1)k^{d-2}lt + \binom{d}{3}k^{d-3}l^3 + \phi_{d-1}(d-1)k^{d-2}l$$

where c_{2d-3} is the coefficient of x^{2d-3} in $C(x)$.

From the equation $B_{2d}(x) = \phi((x - 1/2)^2)$, we obtain $\phi_d = 1$ and $\phi_{d-1} = -d(2d - 1)/12$. Using this and the values of b, k, l, t, we obtain $c_{2d-3} = 0$. Thus, $\deg C < 2d - 3$.

We now proceed to show that d must be of a special form and in that case C must be determined uniquely to be of degree $2d - 4$.

The infinitude of the number of solutions of

$$\begin{aligned}(x-1/2)^2 &= ky^2+ly+t\\ &= ky^2+k(2d-1)y+\frac{k(d-1)(2d-1)}{3}+\frac{2d-1}{12}\\ &= k(y+d-1/2)^2-\frac{k(2d+1)(2d-1)}{12}+\frac{2d-1}{12}\end{aligned}$$

forces that $k(2d+1)=1$ and that k is a square in $\mathbb{Q}$. Therefore, we get $d=2r(r+1)$ for some natural number r.

Then C is uniquely determined to be

$$C(x)=B_{4r(r+1)}\left(\frac{x+2r^2+3r}{2r+1}\right)-\frac{1}{(2r+1)^{4r(r-1)}}f_{4r(r+1)}(x).$$

The claim that $\deg(C)=2d-4$ when $d=2r(r+1)$, etc., is seen as follows.

We use the property $B_{2d}(x+1)-B_{2d}(x)=2dx^{2d-1}$ of the Bernoulli polynomials. We have

$$\begin{aligned}(*)\qquad 4r(r+1)\left(\frac{x+2r^2+3r}{2r+1}\right)^{4r^2+4r-1} &= C(x+2r+1)-C(x)\\ &+\frac{1}{(2r+1)^{4r(r+1)}}\left(f_{4r(r+1)}(x+2r+1)-f_{4r(r+1)}(x)\right)\cdots.\end{aligned}$$

Already, from this one can see that C cannot be a constant; otherwise a comparison with $x=0$ gives

$$(2r+2)(2r+3)\cdots(4r^2+6r)=4r(r+1)(2r^2+3r)^{4r^2+4r-1}.$$

The last identity is impossible since a prime p exists with $2r^2+3r<p\leqslant 4r^2+6r$ and this divides the left side and not the right.

To use the above identity $(*)$ to find the coefficient of $x^{2d-4}=x^{4r^2+4r-4}$ of $C(x)$, we find the coefficient of x^{4r^2+4r-5} on both sides. Clearly, on the left side, it is $(4r^2+4r-4)(2r+1)C_{4r^2+4r-4}$. Thus, we need to check that the coefficient of x^{4r^2+4r-5} is nonzero. This is computed to be

$$\frac{4r(r+1)}{(2r+1)^{4r^2+4r-1}}\binom{4r^2+4r-1}{4}(2r^2+3r)^4-\frac{u(r)}{(2r+1)^{4r(r+1)}}$$

where $u(r)$ is the coefficient of x^{4r^2+4r-5} in $f_{4r(r+1)}(x-2r+1)-f_{4r(r+1)}(x)$, i.e., $u(r)$ is the coefficient of x^{4r^2-4r-5} in $(x+2r+1)(x+2r+2)\cdots(x+4r^2+6r)-x(x-1)\cdots(x+4r^2+4r-1)$.

Let $v(r)=(2r+1)(4r^2+4r)(2r^2+3r)^4\binom{4r^2-4r-1}{4}$.

Using MAPLE, we can explicitly compute $u(r)$ and $v(r)$ as polynomials in r.

$$\begin{aligned} u(r) = {} & \frac{4096}{3}r^{19} + \frac{47104}{3}r^{18} + \frac{231424}{3}r^{17} + 206848r^{16} + \frac{14069248}{45}r^{15} \\ & + \frac{655616}{3}r^{14} - \frac{2556544}{45}r^{13} - \frac{10018816}{45}r^{12} - \frac{6033008}{45}r^{11} \\ & + \frac{6376}{5}r^{10} + \frac{146144}{9}r^{9} - \frac{433384}{45}r^{8} - \frac{126929}{45}r^{7} + \frac{643973}{90}r^{6} \\ & + \frac{157321}{36}r^{5} + \frac{7211}{8}r^{4} + \frac{30647}{360}r^{3} + \frac{1091}{360}r^{2} + \frac{1}{5}r, \end{aligned}$$

$$\begin{aligned} v(r) = {} & \frac{4096}{3}r^{19} + \frac{47104}{3}r^{18} + \frac{231424}{3}r^{17} + 206848r^{16} + \frac{937984}{3}r^{15} \\ & + \frac{656384}{3}r^{14} - 54912r^{13} - \frac{645056}{3}r^{12} - 114864r^{11} + \frac{97544}{3}r^{10} \\ & + 47120r^{9} + 4524r^{8} - 6336r^{7} - 864r^{6} + 324r^{5}. \end{aligned}$$

Thus, in fact, the first four coefficients of $u(r)$ and $v(r)$ match!

However, MAPLE shows that they are never equal because

$$\begin{aligned} v(r) - u(r) = {} & \frac{r(2r+1)(r^2+r-1)}{360}\big(2048r^{11} + 43008r^{10} + 278528r^{9} \\ & + 976640r^{8} + 2152320r^{7} + 3022208r^{6} + 2589888r^{5} \\ & + 1250288r^{4} + 297852r^{3} + 29844r^{2} + 1019r + 72\big) \end{aligned}$$

which is obviously positive for all positive r.

Thus, $C_{4r^2+4r-4} \neq 0$, i.e., $\deg C = 2d-4$.

Finally, we consider the possibility (c), i.e.,

$$B_{2d}(rx+s) = bf_{2d}(x) + C(x).$$

Comparing the coefficients of x^{2d}, x^{2d-1} and x^{2d-2} we get

$$r^{2d} = b, \quad 2s - 1 = r(2d-1), \quad s^2 - s + \frac{1}{6} = \frac{r^2(d-1)(6d-1)}{6}.$$

This gives

$$(4d+2)s^2 - (4d+2)s - 2d^2 + 3d = 0.$$

This is possible for a rational number s if, and only if, $2d-1$ is a perfect square, say $(2u+1)^2$. We obtain

$$r = \pm\frac{1}{2u+1}, \quad s = \frac{1}{2} \pm \frac{4u^2+4u-1}{2(2u+1)}, \quad b = \frac{1}{(2u+1)^{4u^2+4u}}.$$

With these values of r, s, we find that C is the same as it was for case (b). Therefore, the same computation shows that C has degree $2d-4$.

This completes the case I when $m = n$ is even.

Case II: Let $m - n$ be odd and $> \deg C + 2$.

As before, infinitude of solutions implies the existence of a decomposition

$$B_n(x) = \phi \circ f_1 \circ \lambda(x), \qquad bf_m(x) + C(x) = \phi \circ g_1 \circ \mu(x)$$

with λ, μ linear. Now, as m is odd, B_m is indecomposable. Hence either $\deg \phi = m$, $\deg f_1 = 1$ or $\deg \phi = 1$, $\deg f_1 = m$.

First, let us suppose that $\deg \phi = 1$. Then $\deg f_1 = m = \deg g_1$. The standard pair (f_1, g_1) must, therefore, be of the first kind. So, for some $r, s \in \mathbb{Q}$ with $r \neq 0$, we have either

$$B_n(rx + s) = \phi_0 + \phi_1 x^m$$

or

$$bf_m(rx + s) + C(rx + s) = \phi_0 + \phi_1 x^m.$$

If the first possibility occurs, we equate the coefficients of x^{m-2}, and get $6s^2 - 6s + 1 - 0$, $s \in \mathbb{Q}$, which is not possible.

Suppose the second possibility occurs. Let us compare the coefficients of x^m, x^{m-1} and x^{m-2}. We have

$$br^m = \phi_1, \qquad v = \frac{1-m}{2}, \qquad v^2 + (m-1) + \frac{(m-1)(2m-1)}{6} = 0,$$

respectively. Substituting the value of v into the last equation, one gets $m^2 = 1$ which is impossible.

Thus, we suppose that $\deg \phi = m$. Then, we have $u, v \in \mathbb{Q}$ with $u \neq 0$ such that

$$C(x) = B_n(ux + v) - bf_m(x).$$

Comparing the coefficients of x^m, x^{m-1}, x^{m-2} on both sides and noting that the left side does not contribute anything, we have:

$$u^m = b, \qquad v = \frac{m-1}{2}u + \frac{1}{2}, \qquad u^2 = \frac{1}{m-1}.$$

Thus, first of all, this forces m to be such that $m - 1$ is a perfect square, say, $4r^2$. This also determines u, v in terms of r as $u = +1/(2r)$ and $v = (2r^2 - 1)u - 1/2$.

Hence C is uniquely determined to be the polynomial

$$C(x) = B_{4r^2-1}\left(\pm\frac{x + 2r^2 + r - 1}{2r}\right) - \frac{1}{(2r)^{4r^2-1}} f_{4r^2-1}(x).$$

Notice that the expression for C we obtained in case I and the expression here have the common form

$$C(x) = aB_n\left(\frac{x - (n \mp \sqrt{n+1} - 1)/2}{\sqrt{n+1}}\right) - bf_m(x).$$

A calculation exactly as in the case of even m shows that the coefficient of x^{m-3} on the right side is zero. Therefore, C must either be zero or have degree smaller than $m-3$.

If $m=3$, we must have $C\equiv 0$ and

$$f_3(x)=-8B_3\left(\frac{-x}{2}\right).$$

Let $m>3$.

Of course, one can easily check as in the even case that C cannot be a constant. Indeed, if it were, we would have

$$(2r-1)\left(2r^2+r-1\right)^{4r^2-2}=(2r+2)(2r+3)\cdots\left(4r^2+2r-2\right).$$

But, if $r>1$ (which is the case when $m>3$), there is a prime p with $2r^2+r-1<p\leq 4r^2+2r-2$; this divides the right hand side and not the left hand side. In fact, the polynomial C has degree $m-4$. To see this, we may proceed as in the m even case using the property $B_m(x+1)-B_m(x)=mx^{m-1}$.

Case III: Let m be odd and $>n>\deg C+2$.

As before writing $B_m=\phi\circ f_1\circ\lambda$, we have either $\deg\phi=1$ or $=m$. Since $bf_n+C=\phi\circ g_1\circ\mu$ has degree $n<m$, the degree of ϕ must be 1. Thus, the standard pair (f_1,g_1) must be of either the first or the third kind.

If it is of the first kind, the above argument for $m=n$ carries over verbatim to give $n^2=1$, which is a contradiction.

If it is the third kind, we have $B_m(rx+s)=D_m(x,a^n)$ and we have already derived a contradiction by concluding $m=9/2$ in this case.

Finally, we are left with:

Case IV: Let m be even and $>n>\deg C+2$.

Writing $B_m=\phi\circ f_1\lambda$ and $bf_n=\phi\circ g_1\circ\mu$, we must have either $\deg\phi=m$ or $\deg\phi=1$ or $\deg\phi=m/2$ and $f_1=(x-1/2)^2$.

Note that in the last case $n=m/2$ since $m>n$ and n is a multiple of $\deg\phi=m/2$. Also, then $\deg g_1=1$.

Since $m>n\geq\deg\phi$, the possibility $\deg\phi=m$ cannot occur.

Now, if $\deg\phi=1$, then (f_1,g_1) is a standard pair with $\deg f_1=m$, $\deg g_1=n$.

We have already seen in case II that if this pair is of the first kind, we get a contradiction to either of the equations

$$B_m(rx+s)=\phi_0+\phi_1x^n$$

or

$$bf_n(rx-s)+C(rx+s)=\phi_0+\phi_1x^n.$$

Since $m,n>2$, this standard pair cannot be of the second kind.

Suppose it is of the third kind. Then,

$$f_1(x)=D_m\left(x,a^n\right),\ g_1(x)=D_n\left(x,a^m\right)$$

where $(m, n) = 1$. Now, $B_m(rx + s) = \phi_0 + \phi_1(D_m(x, \alpha^n)$.

This means

$$\sum_{i=0}^{m} \binom{m}{i} B_{m-i}(rx + s)^i = \phi_0 + \phi_1 \sum_{i=0}^{\lfloor m/2 \rfloor} d_{m,i}\left(x^{m-2i}\right),$$

$$\text{where } d_{m,i} = \frac{m}{m-i}\binom{m-i}{i}\left(-\alpha^n\right)^i.$$

We will compare the coefficients on both sides.

Equating the coefficients of x^m on both sides, we have $r^m = \phi_1$.

The coefficient of x^{m-1} on the right-hand side is zero and, so we get $\binom{m}{1} r^{m-1} s + \binom{m}{m-1} B_1 r^{m-1} = 0$.

This gives $s = 1/2$.

The coefficients of x^{m-2} give

$$\frac{m(m-1)}{12} r^{m-2}\left(6s^2 - 6s + 1\right) = \frac{m}{m-1}\binom{m-1}{1}\left(-\alpha^n\right)\phi_1$$

which on simplification yields $r^2\alpha^n = (m-1)/24$.

By considering the coefficients of x^{m-4} and on using the values of ϕ_1, $r^2\alpha^n$, we get $m = 9/2$ which is a contradiction. Hence (f_1, g_1) can not be a standard pair of the third kind also.

The same argument goes through if the pair is of the fourth kind as the number ϕ_1 above is simply replaced by $\alpha^{-m/2}\phi_1$.

Finally, if (f_1, g_1) is of the fifth kind, then $m = 6, n = 4$ and

$$f_1(x) = \left(\alpha x^2 - 1\right)^3, \qquad g_1(x) = 3x^4 - 4x^3.$$

So

$$B_6(x) = \phi_0 + \phi_1\left(\alpha(rx + s)^2 - 1\right)^3.$$

This means that the derivative $B_6'(x)$ has a multiple root; however, $B_6'(x) = 6B_5(x)$ and one knows that $B_{\text{odd}}(x)$ has only simple roots by a result of Brillhart.

Alternatively, even by direct computation, comparison of coefficients of x^6, x^5 and x^4 gives $r^2 = 12/5\alpha, s = -r/2, \phi_1 = (5/12)^3$ and then the coefficients of x^2 do not match.

Now, we are left with the case $\deg\phi = m/2$ and $f_1 = (x - 1/2)^2$; so $m = 2n$ and g_1 is linear. Clearly, $f_1(x) = g_1(y)$ has infinitely many rational solutions with a bounded denominator.

Now $B_{2n}(ux + v) = \phi((x - 1/2)^2)$ and by the lemma observed while discussing case I, we know that we must have $B_{2n}(ux + v) = B_{2n}(x)$.

Hence we have $B_{2n}(x) = \phi((x - 1/2)^2)$ and $bf_n(rx - s) + C(rx + s) = \phi(x)$ for some $r, s \in \mathbb{Q}$ with $r \neq 0$. Thus, we have

$$B_{2n}(x) = bf_n\left(r(x - 1/2)^2 - s\right) + C\left(r(x - 1/2)^2 + s\right).$$

Using the identity $B_{2n}(x+1) - B_{2n}(x) = 2nx^{2n-1}$, we have, for some $r, t \in \mathbb{Q}$ with $r \neq 0$,

$$2nx^{2n-1} = bf_n(rx^2 - rx + t) - bf_n(rx^2 - rx + t) + C(rx^2 + rx + t) - C(rx^2 - rx + t).$$

In fact, $t = r/4 + s$.

The coefficients of x^{2n-1} and x^{2n-3} give:

$$br^n = 1, \qquad t = \frac{1-n}{2} \cdot \frac{r}{n}.$$

Comparing the coefficients of x^{2n-5} and substituting the above value of t, we have

$$r^2 = \frac{n^2(n+1)}{12}.$$

In other words $(n+1)/3$ must be a square in $\mathbb{Q}$.

Note that since $n > \deg C + 2 \geqslant 2$, this means $n \geqslant 11$. Writing $n+1 = 3u^2$ with $u \geqslant 2$, we have

$$r = \frac{u(3u^2-1)}{2}, \qquad t = 1 - \frac{u}{2} - \frac{3u^2}{2},$$

$$s = 1 - \frac{3u}{8} - \frac{3u^2}{2} - \frac{3u^3}{8}, \qquad b = \left(\frac{2}{u(3u^2-1)}\right)^{3u^2-1}.$$

Also, the coefficient of x^{n-3} in $C(x) = \phi((x-s)/r) - bf_n(x)$ is seen to be zero by substituting the values of $\phi_n, \phi_{n-1}, \phi_{n-2}, \phi_{n-3}$ obtained from the equation $B_{2n}(x) = \phi((x-1/2)^2)$.

$\deg C$ is found to be $n-4$.

Therefore, Theorem 1 is proved. $\square$

Proof of Theorem 2. Once again, we may assume $a = 1$ and look at the equation

$$f_m(x) = bB_n(y) + C(y).$$

We shall use our earlier general result on equations of the form $f_m(x) = g(y)$ for an arbitrary polynomial:

Theorem C *(cf. [4]). Suppose $f_m(x) = g(y)$ has infinitely many rational solutions x, y with a bounded denominator. Then we are in one of the following cases:*

(1) *$g(y) = f_m(g_1(y))$ for some $g_1(y) \in \mathbf{Q}[Y]$.*
(2) *m even and $g(y) = \phi(g_1(y))$ where $\phi(X) = (X - (1/2)^2)(X - (3/2)^2) \cdots (X - ((m-1)/2)^2)$ and $g_1(y) \in \mathbf{Q}[Y]$ is a polynomial whose square-free part has at most two zeroes.*
(3) *$m = 4$ and $g(y) = 9/16 + b\delta(y)^2$ where δ is a linear polynomial.*

Here, $g(y) = bB_n(y) - C(y)$ where $m \geqslant n > \deg(C) + 2$.

The last inequality shows that $n > 2$ and so, we are not in case (3) above.

If we are in case (1), then again $m \geqslant n$ shows that $m = n$. Then, we have $r, s \in \mathbb{Q}$ with $r \neq 0$ so that

$$bB_n(x) + C(x) = f_n(rx + s)$$

where $n > \deg(C) + 2$.

Therefore, we have

$$b\sum_{i=0}^{n}\binom{n}{i}B_{n-i}x^i + C(x) = (rx - s)(rx - s + 1)\cdots(rx + s + n - 1).$$

Comparing the coefficients of x^n, x^{n-1}, x^{n-2}, we get

$$b = r^n, \quad r = -2s - n + 1,$$

respectively, and a straightforward calculation gives

$$r^2 = n + 1.$$

Thus $n - 1$ has to be a perfect square.

Therefore, the equation

$$f_n(x) = bB_n(y) + C(y)$$

has infinitely many solutions if, and only if, $n + 1$ is a square, $r = \sqrt{n+1}$, $b = r^n$ and C is the polynomial

$$C(x) = f_n\left(rx - \frac{1}{2}\frac{n}{r}\right) - r^n B_n(x).$$

In fact, it turns out that C has degree $n - 4$; a comparison of the coefficients of x^{n-3} yields $c_{n-3} = 0$ and that of x^{n-4} is not zero.

Finally, suppose we are in case (2). Then, either $m = n$ and g_1 has degree 2 or $m = 2n$ and g_1 is linear.

Let us consider the former possibility first. Then, m is even, and $f_m(x) = \phi(f_1(x))$ where

$$f_1(x) = \left(x - \frac{m-1}{2}\right)^2 \quad \text{and}$$

$$\phi(x) = \left(x - \left(\frac{1}{2}\right)^2\right)\left(x - \left(\frac{3}{2}\right)^2\right)\cdots\left(x - \left(\frac{m-1}{2}\right)^2\right).$$

Therefore, writing $g_1(y) = k(y - l)^2 + t$ and assuming that $f_1(x) = g_1(y)$ has infinitely many solutions with a bounded denominator, it follows that $t = 0$ and

k is a square; that is, $g_1(y)$ is the square of a polynomial. Hence, we have $r, s \in \mathbb{Q}$ with $r \neq 0$ and

$$f_n(rx+s) = bB_n(x) + C(x).$$

This is exactly the same expression considered in case (1). Thus, in this case also, we must have that $n+1$ is a perfect square and C is determined uniquely to be a polynomial of degree $n-4$.

Let us now consider the latter possibility; that is, suppose $m = 2n$ and $\deg g_1 = 1$. Then,

$$bB_n(x) - C(x) = \left(rx + s - \left(\frac{1}{2}\right)^2\right)\left(rx + s - \left(\frac{3}{2}\right)^2\right)\cdots \left(rx + s - \left(\frac{2n-1}{2}\right)^2\right).$$

Comparing the coefficients of x^n, x^{n-1} and x^{n-2}, we get $b = r^n$,

$$-6r = 12s - (2n-1)(2n-1)$$

and

$$\frac{n(n-1)}{2}r^2 = \frac{n(n-1)}{2}s^2 - \frac{(n-1)n(2n+1)(2n-1)}{12}s + \frac{n^2(2n+1)^2(2n-1)^2}{2^5 3^2} - \frac{n(48n^4-40n^2-7}{480},$$

respectively, and a straightforward calculation gives

$$r^2 = \frac{4(n+1)(2n+1)(2n-1)}{15}.$$

We claim that this gives a contradiction. Indeed, we assert:

Claim. $(n-1)(2n-1)(2n-1)/15$ *is not a square in* $\mathbb{Q}$.

Let us write $n+1 = au^2, 2n+1 = bv^2, 2n-1 = cw^2$ where a, b, c are square-free. Note that $2n+1$ is coprime to $n+1$ as well as to $2n-1$ and that the two numbers $n-1, 2n-1$ have greatest common divisor 1 or 3. Thus, if $(n+1)(2n+1)(2n-1)/15$ is a square, a, b, c are pairwise coprime and $abc = 15$. A number of cases are possible.

Case I: Suppose $15/b$.

Then, $a = c = 1, b = 15$. This gives

$$n+1 = u^2, \qquad 2n-1 = w^2.$$

Hence $2u^2 - 3 = w^2 = 15v^2 - 2$. So w is odd which means

$$-w^2 \equiv 15v^2 - w^2 + 2 \equiv 3 \bmod 8$$

which is impossible.

Case II: Suppose $3 \mid b$ but $5 \nmid b$.

Then, $b = 3$ and either (i) $a = 5$, $c = 1$ or (ii) $a = 1$, $c = 5$.

In case (i), $5u^2 - 1 = 3v^2 = w^2 + 2$, which means that v, w must be odd. Hence u is even, say $u = 2u_1$. This gives

$$20u_1^2 = 3v^2 + 1 \equiv 1 \bmod 3$$

an impossibility.

In case (ii), $3v^2 - 5w^2 = 2$ means v, w are odd. But then

$$2 = 3v^2 - 5w^2 \equiv -2 \bmod 8$$

a contradiction.

Case III: $3 \nmid b$ but $5 \mid b$.

Again, $b = 5$ and either (i) $a = 3$, $c = 1$ or (ii) $a = 1$, $c = 3$.

In case (i), $6u^2 - 1 = 5v^2 = w^2 + 2$. So, v is even, say $v = 2v_1$. Thus,

$$w^2 + 2 = 20v_1^2 \equiv 0 \bmod 4$$

which gives a contradiction.

In case (ii), $2u^2 - 1 = 5v^2 = 3w^2 + 2$. This gives v, w are odd. So,

$$2u^2 = 5v^2 + 1 \equiv 6 \bmod 8$$

an impossibility.

Case IV: $3 \nmid b$, $5 \nmid b$.

Then, $b = 1$ and either (i) $a = 3$, $c = 5$ or (ii) $a = 5$, $c = 3$ or (iii) $a = 15$, $c = 1$ or (iv) $a = 1$, $c = 15$.

In case (i),

$$v^2 = 5w^2 + 2 \equiv 2 \text{ or } 3 \bmod 4$$

an impossibility.

In case (ii),

$$v^2 = 3w^2 + 2 \equiv 2 \text{ or } 5 \bmod 8$$

an impossibility.

In case (iii), $2 = v^2 - w^2$ is impossible mod 4.

Finally, in case (iv), $v^2 - 15w^2 = 2$, which is impossible mod 3.

Therefore, we have shown the claim.

Theorem 2 is proved. □

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