A characterization and some properties of the Banzhaf–Coleman–Dubey–Shapley sensitivity index

Rana Barua [a], Satya R. Chakravarty [b,*], Sonali Roy [b], Palash Sarkar [c]

[a] *Statistics–Mathematics Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata 700108, India*
[b] *Economic Research Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata 700108, India*
[c] *Applied Statistics Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata 700108, India*

Received 3 September 2002

Available online 24 March 2004

Abstract

A sensitivity index quantifies the degree of smoothness with which it responds to fluctuations in the wishes of the members of a voting body. This paper characterizes the Banzhaf–Coleman–Dubey–Shapley sensitivity index using a set of independent axioms. Bounds on the index for a very general class of games are also derived.

Keywords: Voting game; The Banzhaf–Coleman–Dubey–Shapley sensitivity index; Characterization; Bounds

1. Introduction

A sensitivity index is a measure of the extent of volatility in a decision rule (voting body). It is an indicator of the degree of ease with which it responds to the fluctuations in the wishes of the members of the voting body. It can as well be regarded as a democratic participation index measuring sensitivity to the desires of the voting body members.

Dubey and Shapley (1979) considered the sum of the numbers of swings of different voters in a voting game as a sensitivity index, where the number of swings of a voter is the number of winning coalitions from which the defection of the voter makes them losing.

* Corresponding author.
E-mail address: satya@isical.ac.in (S.R. Chakravarty).

(A coalition is called winning if the sum of the 'yea' votes of the members of the coalition can guarantee the passage of a resolution. A coalition is called 'losing' if it is not winning.) Thus, this index gives the numbers of possibilities in which different voters are in the critical position of being able to change the voting outcome by changing their votes. Since a critical voter's exit from a winning coalition makes it losing, it gives an indication that even a single voter could tip the scales. A normalized version of the Dubey–Shapley index was considered by Felsenthal and Machover (1998) for measuring sensitivity. We refer to this normalized formula, which is the sum of one of the Banzhaf (1965) and Coleman (1971) indices of power of different voters in the game, as the Banzhaf–Coleman–Dubey–Shapley (BCDS) sensitivity index.

Dubey and Shapley (1979) investigated several properties of their index, including determination of lower and upper bounds. A feasible and desirable direction of research along this line is to study additional/alternative properties of the BCDS index and characterize it uniquely. This is the objective of this paper. More precisely, we first discuss some properties and develop an axiomatic characterization of the BCDS sensitivity index. It is shown that the set of axioms used in the characterization theorem is minimal, that is, no proper subset of this set can characterize the index. Equivalently, we say that axioms belonging to this minimal set are independent. Then using Fourier transform analysis, we derive some additional properties and bounds for the BCDS index for a class of games, which is much more general than the class considered by Dubey and Shapley (1979). This may raise the interest of mathematicians dealing with Fourier transform in the theory of voting games.

The next section of the paper sets out the background material. Section 3 defines the BCDS index and discusses some of its properties. Section 4 derives the index axiomatically and demonstrates independence of the properties employed in the axiomatization exercise. In Section 5 we discuss some additional properties of the index, including derivation of bounds, using Fourier transform. Finally, Section 6 concludes the paper.

2. The background

It is possible to model a voting situation as a coalitional form game, the hallmark of which is that any subgroup of players can make contractual agreements among its members independently of the remaining players. Let $N = \{1, 2, \ldots, n\}$ be a set of players. The power set of N, that is, the collection of all subsets of N, is denoted by 2^N. Any member of 2^N is called a coalition. A coalitional form game with player set N is a pair $(N; V)$, where $V : 2^N \to R$ such that $V(\phi) = 0$, where R is the real line. For any coalition S, the real number $V(S)$ is the worth of the coalition, that is, this is the amount that S can guarantee to its members. For any set S, $|S|$ will denote the number of elements in S.

We frame a voting system as a coalitional form game by assigning the value 1 to any coalition which can pass a bill and 0 to any coalition which cannot. In this context, a player is a voter and the set $N = \{1, 2, \ldots, n\}$ is called the set of voters. Throughout the paper we assume that voters are not allowed to abstain from voting. A coalition S will be called winning or losing according as it can or cannot pass a resolution.

Definition 1. Given a set of voters N, a voting game associated with N is a pair $(N; V)$, where $V: 2^N \to \{0, 1\}$ satisfies the following conditions:

(i) $V(\phi) = 0$;
(ii) $V(N) = 1$;
(iii) if $S \subseteq T$, $S, T \in 2^N$, then $V(S) \leqslant V(T)$.

The above definition formalizes the idea of a decision-making committee in which decisions are made by vote. It follows that the empty coalition ϕ is losing (condition (i)) and the grand coalition N is winning (condition (ii)). All other coalitions are either winning or losing. Condition (iii) can be regarded as a monotonicity principle. It ensures that if a coalition S can pass a bill, then any superset T of S can pass it as well. A game $G = (N; V)$ is called proper if for $S, T \in 2^N$, $V(S) = V(T) = 1$ implies that $S \cap T \neq \phi$. According to this condition, two winning coalitions cannot be disjoint. The collection of all voting games is denoted by $\mathbf{F}$. For any $G = (N; V)$, we write $\mathbf{W}_G(\mathbf{L}_G)$ for the set of all winning (losing) coalitions associated with G. Thus, for any $S \subseteq N$, $V(S) = 1(0)$ is equivalent to the condition that $S \in \mathbf{W}_G(\mathbf{L}_G)$.

Definition 2. A voting game $G = (N; V)$ is called

(i) decisive if for all $S \in 2^N$, $V(S) + V(N - S) = 1$,
(ii) balanced if $|\mathbf{W}_G| = |\mathbf{L}_G| = 2^{|N|-1}$.

Clearly, a decisive game is balanced.

Definition 3. The unanimity game $(M; U_N)$ associated with a given set of voters M is the game whose only minimal winning coalition is the coalition $N \subseteq M$.

Definition 4. Given a set of voters N, let $(N; V)$ be a voting game.

(i) For any coalition $S \in 2^N$, we say that $i \in N$ is swing in S if $V(S) = 1$ but $V(S - \{i\}) = 0$.
(ii) A coalition $S \in 2^N$ is said to be minimal winning if $V(S) = 1$ but there does not exist $T \subset S$ such that $V(T) = 1$.

Thus, voter i is swing, also called pivotal or key, in the winning coalition S if his deletion from S makes the resulting coalition $S - \{i\}$ losing. For any game $G = (N; V) \in \mathbf{F}$, and $i \in N$, we write $m_i(G)$ to denote the number of winning coalitions in which voter i is swing. It is often said that $m_i(G)$ is the number of swings of voter i. We will indicate the total number of swings $\sum_{i=1}^{|N|} m_i(G)$ in G by $m(G)$.

Definition 5. For a set of voters N, let $(N; V)$ be a voting game. A voter $i \in N$ is called a dummy in $(N; V)$ if he is never swing in the game. A voter $i \in N$ is called a nondummy in $(N; V)$ if he is not dummy in $(N; V)$.

Following Felsenthal and Machover (1998) we have

Definition 6. For a voting game $(N; V)$ with the set of voters N, a voter $i \in N$ is called a dictator if $\{i\}$ is the sole minimal winning coalition in the game.

By definition, a dictator in a game is unique. If a game has a dictator, then he is the only swing voter in the game.

A very important voting game is a weighted majority game.

Definition 7. For a set of voters $N = \{1, 2, \ldots, n\}$, a weighted majority game is a quadruplet $G = (N; V; \mathbf{w}; q)$, where $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is the vector of nonnegative weights of the $|N|$ voters in N, q is a positive real number quota such that $q \leqslant \sum_{i=1}^{|N|} w_i$ and for any $S \in 2^N$,

$$V(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geqslant q, \\ 0 & \text{otherwise.} \end{cases}$$

That is, the ith voter casts w_i votes and q is the quota of votes needed to pass a bill. A weighted majority game will be proper if $(\frac{1}{2}) \sum_{i=1}^{|N|} w_i < q$. Note that a weighted majority game satisfies conditions (i)–(iii) of Definition 1. (See Felsenthal and Machover, 1998 for further discussions on Definitions 1–7.)

3. The Banzhaf–Coleman–Dubey–Shapley sensitivity index

For any $G = (N; V) \in \mathbf{F}$, we call $m_i(G)$ the first Banzhaf–Coleman index of voting power of $i \in N$. The second and third Banzhaf–Coleman indices of voting power are given respectively by $m_i(G)/2^{|N|-1}$ and $m_i(G)/m(G)$.

Earlier, Shapley and Shubik (1954) suggested an index of voting power defined as the number of orderings in which the concerned voter is swing divided by the total number of orderings of the voters. (See Dubey and Shapley, 1979; Felsenthal and Machover, 1995, and Burgin and Shapley, 2001, for further discussion.) Alternatives and variations of these indices were suggested, among others, by Deegan and Packel (1978), Johnston (1978) and Barua et al. (2002).

Dubey and Shapley (1979) suggested the use of

$$D(G) = m(G) \tag{1}$$

as a sensitivity index, where $G \in \mathbf{F}$ is arbitrary. The Felsenthal and Machover's (1998) version of this index is given by

$$B(G) = \frac{m(G)}{2^{|N|-1}}. \tag{2}$$

Since $B(G)$ is the sum of the second Banzhaf–Coleman indices of different voters in a game, we refer to $B(G)$ as the BCDS index of sensitivity. It 'reflects the "volatility" or degree of suspense in the voting body' (Dubey and Shapley, 1979). Suppose in a voting game each voter's probability of voting for or against a bill is selected independently from a uniform distribution [0,1]. Then $m_i/2^{|N|-1}$ becomes the probability p_i that other voters will vote such that the bill will pass or fail according as i votes for or against it (Straffin, 1977). The index $B(G)$ is simply $\sum_{i=1}^{|N|} p_i$.

The index B possesses the following interesting properties.

(a) *Anonymity*. Let $G = (N; V)$ and $G' = (N'; V') \in \mathbf{F}$ be two isomorphic games. That is, there exists a bijection h of N onto N' such that for all $S \subseteq N$, $V(S) = 1$ if and only if $V'(h(S)) = 1$, where $h(S) = \{h(x)\colon x \in S\}$. Then $B(G) = B(G')$.

(b) *Increasingness*. Let $G = (N; V)$ and $\overline{G} = (\overline{N}; \overline{V}) \in \mathbf{F}$ be two games such that $N = \overline{N}$ and $m_i(G) \geqslant m_i(\overline{G})$ for all $i \in N$ with > for at least one $i \in N$. Then $B(G) > B(\overline{G})$.

(c) *Dummy independence principle*. For any $G = (N; V) \in \mathbf{F}$ and for any dummy $d \in N$, $B(G) = B(G_{-d})$, where G_{-d} is the game obtained from G by excluding d.
Likewise, $B(G) = B(G_{+d})$, where G_{+d} is the game obtained from $G \in \mathbf{F}$ by including d as a dummy.

(d) *Maximality*. For any $G = (N; V) \in \mathbf{F}$, $B(G)$ attains its maximal value $r(\binom{|N|}{r})/2^{|N|-1}$ if and only if all coalitions with more than $|N|/2$ voters win and all coalitions with less than $|N|/2$ voters lose, where $r = [|N|/2] + 1$, with $[x]$ being the largest integer $\leqslant x$ (Dubey and Shapley, 1979).

(e) *Duality*. For any $G = (N; V) \in \mathbf{F}$, let $G^* = (N; V^*)$ be the dual of G; that is, $V^*(S) = V(N) - V(N - S)$ for all $S \in 2^N$. Then $B(G) = B(G^*)$ (Dubey and Shapley, 1979).

Anonymity says that a reordering of the voters does not change the sensitivity index B. Thus, all characteristics other than swings of the voters, e.g., their living conditions, are irrelevant to the measurement of sensitivity. Increasingness requires the index B to be an increasing function of the number of swings, given that the voter set remains unaltered. To understand increasingness, let us consider the weighted majority game $\widehat{G}_0 = (N; V; 1, 2, 2; 4)$ obtained from $G_0 = (N; V; 1, 2, 2; 3)$ by augmenting the quota from 3 to 4. Given that the set of voters $N = \{1, 2, 3\}$ is the same in the two games, we get $m_2(G_0) = m_2(\widehat{G}_0) = 2$, $m_3(G_0) = m_3(\widehat{G}_0) = 2$ and $m_1(G_0) = 2 > m_1(\widehat{G}_0) = 0$. We thus have $B(G_0) > B(\widehat{G}_0)$. Since a dummy is not able to influence the voting outcome, we can argue that B should satisfy the dummy independence principle. Given that the second Banzhaf-Coleman voting power index $m_i(G)/2^{|N|-1}$ remains invariant under inclusion or exclusion of a dummy (Owen, 1978; Felsenthal and Machover, 1995, 1998; Barua et al., 2002), B also satisfies this invariance condition. Maximality specifies the necessary and sufficient condition for B to achieve the maximum value and duality shows that the values of B for a voting game and its dual are the same.

Dubey and Shapley (1979) showed that for any $G = (N; V) \in \mathbf{F}$,

$$B(G) \geqslant \theta \frac{[|N| - \log_2 \theta]}{2^{|N|-1}}, \tag{3}$$

where θ is the minimum of the numbers of winning and losing coalitions in G. Hart (1976) suggested a stronger but more complicated lower bound for $B(G)$. Dubey and Shapley (1979) also noted that if G is a decisive game, then a lower bound of $B(G)$ is 1.

Examples of sensitivity indices other than $B(G)$ which satisfy properties (a)–(e) are $(B(G))^c$, $c > 0, c \neq 1$, and $\exp(B(G))$. However, because of its probabilistic interpretation, expositional and computational ease, $B(G)$ appears to be more attractive than such indices. Furthermore, in the next section we show that a characterization of $B(G)$ can be developed using a set of intuitively reasonable axioms. These therefore make $B(G)$ a desirable index of sensitivity.

4. The characterization result

In order to present the axioms that characterize the BCDS index, we need the following definitions.

Definition 8. Given $G_1 = (N_1; V_1)$, $G_2 = (N_2; V_2) \in \mathbf{F}$, where N_1 and N_2 need not be disjoint, we define $G_1 \vee G_2$ as the game with the set of voters $N_1 \cup N_2$, where a coalition $S \subseteq N_1 \cup N_2$ is winning if and only if $V_1(S \cap N_1) = 1$ or $V_2(S \cap N_2) = 1$.

Definition 9. Given $G_1 = (N_1; V_1)$, $G_2 = (N_2; V_2) \in \mathbf{F}$, we define $G_1 \wedge G_2$ as the game with the set of voters $N_1 \cup N_2$, where a coalition $S \subseteq N_1 \cup N_2$ is winning if and only if $V_1(S \cap N_1) = 1$ and $V_2(S \cap N_2) = 1$.

Thus, in order to win in $G_1 \vee G_2$ a coalition must win in either G_1 or G_2, whereas to win in $G_1 \wedge G_2$ it has to win in both G_1 and G_2. Clearly, given $G_1, G_2 \in \mathbf{F}$; $G_1 \vee G_2$, $G_1 \wedge G_2 \in \mathbf{F}$.

Finally, we have

Definition 10. Given $(N; V) \in \mathbf{F}$, suppose that the voters $i, j \in N$ are amalgamated into one voter ij. Then the post-merger voting game is the pair $(N'; V') \in \mathbf{F}$, where

$$N' = N - \{i, j\} \cup \{ij\} \quad \text{and} \quad V'(S) = \begin{cases} V(S) & \text{if } S \subseteq N' - \{ij\}, \\ V((S - \{ij\}) \cup \{i, j\}) & \text{if } ij \in S. \end{cases}$$

We are now in a position to present three axioms on a general sensitivity index $P : \mathbf{F} \to R$ that will uniquely isolate the BCDS index. The first axiom is taken from Dubey (1975) (see also (Dubey and Shapley, 1979)). It shows how the sensitivity levels in the games $G_1 \vee G_2$ and $G_1 \wedge G_2$ are related to individual sensitivities in G_1 and G_2.

Axiom A1 (*Sum principle*). For any $G_1, G_2 \in \mathbf{F}$,

$$P(G_1 \vee G_2) + P(G_1 \wedge G_2) = P(G_1) + P(G_2). \tag{4}$$

This axiom, which is also referred to as linearity/union-intersection property in the literature, is quite similar to the condition characterizing additive measures (in measure theoretic sense), such as probabilities. If A_1 and A_2 are two events in a probability space and $\vee$ and $\wedge$ are the disjunction and conjunction operations respectively, then $P(A_1 \vee A_2) + P(A_1 \wedge A_2) = P(A_1) + P(A_2)$, where P denotes probability.

The next axiom captures the change in sensitivity levels under a merger of any two voters in an unanimity game. In an unanimity game the number of swings of each nondummy voter is only one. Now, in a voting game the power of a voter is determined by his swings only. Since the number of swings across voters in unanimity games is a constant, an important source of difference between the extents of sensitivity in two such games is the number of nondummy voters. One way of reflecting this difference is to assume that the ratio between sensitivity levels in an unanimity game and a new game obtained by merging two voters in this game is proportional to the ratio of the numbers of nondummy voters in them. The following axiom gives a formulation along this direction.

Axiom A2 (*Proportionality principle*). Let $G' \in \mathbf{F}$ be the game obtained from $G = (M; U_N) \in \mathbf{F}$ by merging two voters $i, j \in N$. Then

$$\frac{P(G)}{P(G')} = \frac{1}{2}\frac{|N|}{|N'|}. \tag{5}$$

The third axiom is a normalization condition that states the value of the index if the game has a dictator.

Axiom A3 (*Normalization*). If $G = (N; V) \in \mathbf{F}$ has a dictator, then $P(G) = 1$.

Since the BCDS index is obtained directly from the second Banzhaf–Coleman index, a comparison of our axioms with some existing axiom systems that characterize the latter will be worthwhile. Lehrer (1988) characterized the second Banzhaf–Coleman index using the sum criterion A1 and a two-voter superadditivity property, which is similar to, but weaker than A2, along with Shapley's (1953) dummy axiom and an equal treatment principle. An alternative characterization of this index was developed by Nowak (1997) using a version of Lehrer's (1988) superadditivity property, equal treatment principle, dummy axiom and a postulate of Young (1985) which says that the power is (in some sense) determined by the marginal contributions of voters. An important difference of Nowak's axiomatization with Lehrer's exercise (and also ours) is that the former does not make use of the additivity axiom A1.

We now have

Theorem 1. *A sensitivity index P satisfies axioms* A1–A3 *if and only if it is the Banzhaf–Coleman–Dubey–Shapley sensitivity index B given by* (2).

Proof. We first demonstrate that B satisfies A1–A3. Let both $G_1 = (N_1; V_1)$ and $G_2 = (N_2; V_2) \in \mathbf{F}$. Assuming that $N_1 - N_2 \neq \phi$, take $i \in N_1 - N_2$. Now, any coalition $S' \subseteq N_2 - N_1$ can be appended to a swing coalition $S \subseteq N_1$ for $i \in N_1$ to obtain a swing coalition $S \cup S'$ for $i \in N_1 \cup N_2$ unless $(S \cup S') \cap N_2$ is winning in G_2. Hence the number of swings of voter $i \in N_1 - N_2$ is

$$\begin{aligned} m_i(G_1 \vee G_2) &= m_i(G_1)2^{|N_2-N_1|} - m_i(G_1 \wedge G_2) \\ &= m_i(G_1)2^{|N_2-N_1|} + m_i(G_2)2^{|N_1-N_2|} - m_i(G_1 \wedge G_2), \end{aligned} \tag{6}$$

since $m_i(G_2) = 0$ for $i \notin N_2$. The same expression for $m_i(G_1 \vee G_2)$ will be obtained if $i \in N_2 - N_1$ and $i \in N_1 \cap N_2$. Therefore,

$$\begin{aligned}
B(G_1 \vee G_2) &= \sum_{i=1}^{|N_1 \cup N_2|} \frac{m_i(G_1 \vee G_2)}{2^{|N_1 \cup N_2|-1}} \\
&= \sum_{i=1}^{|N_1 \cup N_2|} \left(\frac{m_i(G_1)2^{|N_2 - N_1|}}{2^{|N_1 \cup N_2|-1}} + \frac{m_i(G_2)2^{|N_1 - N_2|}}{2^{|N_1 \cup N_2|-1}} - \frac{m_i(G_1 \wedge G_2)}{2^{|N_1 \cup N_2|-1}} \right) \\
&= \sum_{i=1}^{|N_1 \cup N_2|} \left(\frac{m_i(G_1)}{2^{|N_1|-1}} + \frac{m_i(G_2)}{2^{|N_2|-1}} - \frac{m_i(G_1 \wedge G_2)}{2^{|N_1 \cup N_2|-1}} \right) \\
&= B(G_1) + B(G_2) - B(G_1 \wedge G_2). \qquad (7)
\end{aligned}$$

Thus, B satisfies A1.

To check satisfaction of A2 by B, consider the unanimity game $G_N = (M; U_N) \in \mathbf{F}$. Let $G_{N'} = (M'; U_{N'})$ be the game obtained from G_N by merging any two voters i, $j \in N$. Then,

$$B(G_N) = \frac{|N|}{2^{|N|-1}} \quad \text{and} \quad B(G_{N'}) = \frac{|N'|}{2^{|N'|-1}}, \quad \text{from which we have}$$
$$\frac{B(G_N)}{B(G_{N'})} = \frac{1}{2}\frac{|N|}{|N'|}, \quad \text{since} \quad |N'| = |N| - 1.$$

Thus, B verifies A2.

If a game $G = (N; V) \in \mathbf{F}$ has a dictator i, then i is the only swing voter in the game, that is, m_i is maximized, which means that $m_i = 2^{|N|-1}$ and $m_j = 0$ for all $j \neq i$. Hence,

$$B(G) = \frac{m_i}{2^{|N|-1}} = \frac{2^{|N|-1}}{2^{|N|-1}} = 1, \qquad (8)$$

which shows that B fulfils A3.

We will now demonstrate that if a sensitivity index fulfils A1–A3, then it must be the BCDS index. Note that a sensitivity index P satisfying A1 is uniquely determined on unanimity games. This is because for any game $G \in F$, $G = G_{S_1} \vee G_{S_2} \vee \cdots \vee G_{S_k}$, where $S_1, S_2, \ldots, S_k$ are minimal winning coalitions of G and G_{S_i} is the unanimity game corresponding to S_i, $i = 1, 2, \ldots, k$. Thus, by A1, P is determined if $P(G_{S_1})$, $P(G_{S_2} \vee \cdots \vee G_{S_k})$ and $P(G_{S_1} \wedge (G_{S_2} \vee \cdots \vee G_{S_k}))$ are known. But $G_{S_1} \wedge (G_{S_2} \vee \cdots \vee G_{S_k}) = G_{S_1 \cup S_2} \vee G_{S_1 \cup S_3} \vee \cdots \vee G_{S_1 \cup S_k}$ and hence by induction hypothesis on k, both $P(G_{S_2} \vee \cdots \vee G_{S_k})$ and $P(G_{S_1} \wedge (G_{S_2} \vee \cdots \vee G_{S_k}))$ are determined. So $P(G)$ is determined.

In view of the above discussion we can say that it is enough to determine $P(M; U_N)$ for any unanimity game $(M; U_N)$. We will now show by induction on $|N|$ that $P(M; U_N) = |N|/2^{|N|-1}$. If $|N| = 1$, then $(M; U_N)$ has a dictator and hence by A3, $P(M; U_N) = 1 = B(M; U_N)$. Therefore assume $|N| > 1$ and the result for all games $(M; U_{\overline{N}})$, where $|\overline{N}| < |N|$. Let $(M'; U_{N'})$ be the game obtained by merging two voters i and j in N. Then by induction hypothesis, $P(M'; U_{N'}) = |N'|/2^{|N'|-1} = B(M', U_{N'})$. By A2,

$$P(M; U_N) = \frac{1}{2}\frac{|N|}{|N'|}\frac{|N'|}{2^{|N'|-1}} = \frac{|N|}{2^{|N|-1}}, \quad \text{since} \quad |N| = |N'| + 1.$$

This demonstrates that P coincides with B on any unanimity game and hence on all games in $\mathbf{F}$. □

Theorem 1 specifies a set of necessary and sufficient conditions for identifying the BCDS index B uniquely.

Now, in order to illustrate how the BCDS index B can be calculated from minimal winning coalitions, let us consider the weighted majority game $\widetilde{G} = (N; V; 1, 2, 3; 4)$ with the voter set $N = \{1, 2, 3\}$. The minimal winning coalitions in this game are $S_1 = \{1, 3\}$ and $S_2 = \{2, 3\}$. Hence $B(\widetilde{G}) = B(\widetilde{G}_{S_1}) + B(\widetilde{G}_{S_2}) - B(\widetilde{G}_N)$, where $\widetilde{G}_{S_i}$ is the unanimity game corresponding to S_i, $i = 1, 2$ and $\widetilde{G}_N = (N; U_N)$. Then $B(\widetilde{G}) = 2/2^{2-1} + 2/2^{2-1} - 3/2^{3-1} = 1.25$.

We will now show that axioms A1–A3 are independent. Demonstration of independence requires that if one of these three axioms is dropped, then there will exist a sensitivity index that will satisfy the two remaining axioms but not the dropped one.

Theorem 2. *Axioms* A1–A3 *are independent.*

Proof. Let $G = (N; V) \in \mathbf{F}$ be arbitrary. Then consider the sensitivity indices given by

$$P_1(G) = \sum_{i=1}^{|N|} m_i \Big/ 2^{|N|}, \tag{9}$$

$$P_2(G) = \sum_{i=1}^{|N|} m_i \Big/ 2^{|N|} + \frac{1}{2} \tag{10}$$

$$P_3(G) = \sum_{i \in \overline{D}} |W_i| \Big/ 2^{|N|-1}, \tag{11}$$

where W_i is the set of winning coalitions containing i and $\overline{D}$ is the set of nondummies. It is easy to see that P_1 verifies A1 and A2 but not A3, whereas P_2 verifies A1 and A3 but not A2. One can also check that P_3 fulfils A2 and A3 but not A1. □

5. Fourier transform analysis of the Banzhaf–Coleman–Dubey–Shapley sensitivity index

In this section we analyze voting games using tools from Boolean function literature. Before embarking on the details of the analysis, we discuss the connection of games to Boolean functions and the main results that we obtain.

An n-variable Boolean function is a map $f : \{0, 1\}^n \to \{0, 1\}$, where $\{0, 1\}^n$ is the n-fold Cartesian product of $\{0, 1\}$. With the conventional identification of n-bit strings and subsets of N, we can also take the domain to be 2^N, where $N = \{1, 2, \ldots, n\}$. Therefore, Boolean functions can be regarded as indistinguishable from general games $(N; V)$, considered by Owen (1978), where the domain and the range of V are 2^N and $\{0,1\}$ respectively. We denote the set of all such games by $\mathbf{F}^*$. Thus, if $G = (N; V) \in \mathbf{F}^*$, then V is a Boolean function as well. Since $\mathbf{F} \subset \mathbf{F}^*$, our analysis is also applicable to any game in $\mathbf{F}$.

Boolean functions have been studied quite extensively in other areas such as computer science and engineering. Several analytical tools have been developed for this purpose.

The most important of these tools is the Walsh transform, which is essentially the Fourier transform of $(-1)^{f(x)}$. In this section we use the Walsh transform to obtain bounds on the index. Since we will be performing the analysis on a general Boolean function (or game) we first generalize the concept $m_i(G)$ in the following manner.

Definition 11. For any game $G = (N; V) \in \mathbf{F}^*$, the associated complement game is $\overline{G} = (N; \overline{V}) \in \mathbf{F}^*$, where for any $S \subseteq N$, $V(S) = 1$ if and only if $\overline{V}(S) = 0$. Further, G is said to be balanced if the number of winning coalitions in G and $\overline{G}$ are equal.

For any $G = (N; V) \in \mathbf{F}^*$ and $i \in N$, we write

$$M_i(G) = m_i(G) + m_i(\overline{G}). \tag{12}$$

Also we set $M(G) = \sum_{i=1}^{n} M_i(G)$. We first show that given a general game $G \in \mathbf{F}^*$, it becomes a simple voting game (i.e. $G \in \mathbf{F}$) if and only if $M(G) = m(G)$. Then we go on to obtain upper and lower bounds for $M(G)$ which immediately provide upper and lower bounds for $m(G)$. The main results that we obtain are the following.

(1) If G in $\mathbf{F}^*$ is a balanced n-player game, then $M(G) \geqslant 2^{n-1}$. Consequently, for any balanced n-player game G in $\mathbf{F}$, we have $m(G) \geqslant 2^{n-1}$. Further, equality is attained if there is a dictator.

(2) If $G \in \mathbf{F}^*$ is an n-player game and $w = |\mathbf{W}_G|$ is the number of winning coalitions, then

$$\frac{w(2^n - w)}{2^{n-1}} \leqslant M(G) \leqslant n\frac{w(2^n - w)}{2^{n-1}}.$$

Further, both the upper and lower bounds are attained.
Consequently, for any n-player game $G \in \mathbf{F}$ we have

$$\frac{w(2^n - w)}{2^{n-1}} \leqslant m(G) \leqslant n\frac{w(2^n - w)}{2^{n-1}}.$$

Remarks. (a) From result (1) above, it follows that for a decisive voting game, a lower bound of $B(G)$ is 1. As stated earlier, Dubey and Shapley (1979) derived 1 as the lower bound of B for decisive voting games. Evidently, Corollary 16 presents a lower bound for a more general class of games viz., balanced voting games. Moreover, Dubey and Shapley's (1979, p. 108) claim that the lower bound can only be derived by using Hart's (1976) bound does not appear to be true.

(b) It is known (Felsenthal and Machover, 1998, p. 56) that for $G \in \mathbf{F}$, $m(G) \geqslant n$. Result (2) above provides a lower bound on $m(G)$ for monotone games. This lower bound depends on the number of winning coalitions. Though this can be lower than n, in general it is going to be a sharper lower bound. In fact, our lower bound $w(2^n - w)/2^{n-1}$ is greater than n if

$$w > 2^{n-1} - 2^{n-1}\sqrt{1 - \frac{n}{2^{n-1}}} \cong \frac{n}{2}.$$

To obtain these results, we first prove a relation (Lemma 9) between $M_i(G)$ and the autocorrelation function of the Boolean function associated with G. (See Eq. (17) below for the definition of the autocorrelation function.) Thus, autocorrelation function becomes a helpful technique in studying swings in general voting games. Further algebraic analysis is performed using the Walsh transform which ultimately leads to the desired results. This in turn establishes the role of the Walsh transform in proving results in general voting games. Since the Walsh transform can be expressed in matrix form using the Hadamard matrix, this motivates the use of the Hadamard matrix.

For the sake of convenience, we divide this section into two subsections.

5.1. Basics of Fourier transform analysis

In this subsection, we present the mathematical preliminaries necessary for understanding the Fourier transform analysis of games.

Let F_2 be the field $\langle\{0,1\},\oplus,\cdot\rangle$, where $\oplus$ and $\cdot$ denote modulo 2 addition and multiplication. We thus consider the domain of a Boolean function to be the vector space $\langle F_2^n,\oplus\rangle$ over F_2, where, as stated, $\oplus$ is the addition operator on F_2 and also on F_2^n. The inner product of two vectors $u=(u_1,\dots,u_n)$, $v=(v_1,\dots,v_n)\in F_2^n$ is $\sum_{i=1}^n u_iv_i$ and will be denoted by $\langle u,v\rangle$. The weight of an n-bit vector u is the number of ones in u and will be denoted by $wt(u)$.

The Fourier transform is the most widely used tool in the analysis of Boolean functions. In most cases it is convenient to apply Fourier transform to $(-1)^{f(x)}$ instead of $f(x)$. The resulting transform is called the Walsh transform of $f(x)$. More precisely, the Walsh transform of $f(x)$ is an integer-valued function $W_f:\{0,1\}^n\to[-2^n,2^n]$ defined by (see, for example, Ding et al., 1978)

$$W_f(u)=\sum_{w\in F_2^n}(-1)^{f(w)\oplus\langle u,w\rangle}. \tag{13}$$

The Walsh transform is called the spectrum of f. Note that the spectrum measures the cross-correlations between a function and the set of linear functions. Another way of looking at the spectrum is via Hadamard matrices. Let H_n be the Hadamard matrix of order 2^n defined recursively as (see MacWilliams and Sloane, 1977)

$$H_1=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \tag{14}$$

where $H_n=H_1\otimes H_{n-1}$ for $n>1$, and $\otimes$ denotes the Kronecker product of two matrices. For example,

$$H_2=\begin{bmatrix}H_1 & H_1\\ H_1 & -H_1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}.$$

Considering the rows and columns of H_n to be indexed by the elements of F_2^n, we obtain $[H_n]_{(u,v)}=(-1)^{\langle u,v\rangle}$. Using this fact, the Walsh transform can be written as

$$\left\lfloor(-1)^{f(0)},\dots,(-1)^{f(2^n-1)}\right\rfloor H_n=\left[W_f(0),\dots,W_f\left(2^n-1\right)\right], \tag{15}$$

where $u\in F_2^n$ is identified with an integer in $[0,2^n-1]$.

Since $H_n H_n = 2^n I_{2^n}$, post-multiplying both sides by H_n we get the inverse of Walsh transform,

$$(-1)^{f(u)} = \frac{1}{2^n} \sum_{w \in F_2^n} W_f(w)(-1)^{\langle u, w \rangle}. \tag{16}$$

Another commonly used tool in Boolean function analysis is the auto-correlation function. The auto-correlation function is an integer-valued map $C_f : \{0,1\}^n \to [-2^n, 2^n]$ defined by (see MacWilliams and Sloane, 1977, for a related concept called directional derivative)

$$C_f(u) = \sum_{w \in F_2^n} (-1)^{f(w) \oplus f(u \oplus w)}. \tag{17}$$

It is clear that $C_f(0) = 2^n$. The auto-correlation is not a transform in the sense that it does not uniquely determine the function.

For the weighted majority game $\widetilde{G} = (N; V; 1, 2, 3; 4)$ with minimum winning coalitions $\{1, 3\}$ and $\{2, 3\}$ the corresponding Boolean function f and Walsh transform W_f are given in Table 1. The variable x_i in the table represents player i.

The next result is called the Wiener–Khintchine Theorem in continuous analysis and has also been obtained for Boolean functions (see Carlet, 1992; Preneel, 1993; Zhang and Zheng, 1995).

Theorem 3. *Let f be an n-variable function. Then*

$$\big[C_f(0), \ldots, C_f(2^n - 1)\big] H_n = \big[W_f^2(0), \ldots, W_f^2(2^n - 1)\big]. \tag{18}$$

Applying the inverse transform gives $\sum_{u \in F_2^n} W_f^2(u) = 2^n C_f(0) = 2^{2n}$. This is a conservation law for the spectral values of f and is known as Parseval's Theorem (see, for example, Ding et al., 1978).

The next result states a useful property of Walsh transform (see Canteaut et al., 2000, Proposition 5). For a vector space E, we define $E^{\perp}$ to be the vector space which is orthogonal to E, i.e., $E^{\perp} = \{u \colon \langle u, v \rangle = 0, \forall v \in E\}$.

Table 1
The Walsh transform and autocorrelation

x_3	x_2	x_1	f	W_f	C_f
0	0	0	0	2	8
0	0	1	0	2	4
0	1	0	0	2	4
0	1	1	0	2	4
1	0	0	0	6	−4
1	0	1	1	−2	−4
1	1	0	1	−2	−4
1	1	1	1	−2	−4

Theorem 4. *Let f and g be n-variable functions and E be a subspace of F_2^n. Then*

$$\sum_{w\in E} W_f^2(w) = |E| \sum_{u\in E^\perp} C_f(u). \tag{19}$$

See (Sarkar and Maitra, 2002) for a discussion of the above results in a more general setting.

5.2. The results

In this section, we present the results mentioned at the beginning of Section 5 along with complete proofs. First we generalize the notion of swing. The notion of swing is quite general in the sense that we do not require monotonicity (condition (iii) in Definition 1) for swing to be defined.

Definition 12. Given a game $G = (N; V) \in \mathbf{F}^*$, and $i \in N$, number of negative swings of i is defined as

$$m_i^-(G) = \left|\left\{S \subseteq N - \{i\}\colon\ V(S) - V(S \cup \{i\}) = 1\right\}\right|.$$

For any $G = (N; V) \in \mathbf{F}^*$, we write

$$m^-(G) = \sum_{i\in N} m_i^-(G). \tag{20}$$

The following proposition, whose proof is very easy, states the relationship between $m_i^-(G)$ and $m_i(\overline{G})$.

Proposition 5. *Let $G = (N; V) \in \mathbf{F}^*$ be arbitrary. Then for any $i \in N$, $m_i^-(G) = m_i(\overline{G})$.*

Proposition 6. *Let $G = (N; V) \in \mathbf{F}^*$. Then $m(\overline{G}) = 0$ if and only if G satisfies monotonicity, that is, condition* (iii) *in Definition* 1.

Proof. The sufficiency part of the proof is easy to verify. We therefore establish the necessity. If $m(\overline{G}) = 0$, then $m_i(\overline{G}) = 0$ for all $i \in N$. Let S and T be two coalitions in G such that $V(S) = 1$ and $S \subseteq T$. Then we need to show that $V(T) = 1$. This is shown by induction on $r = |T| - |S|$. For $r = 0$, we have $T = S$ and the result follows trivially. Assume that the result is true for $r - 1$. Let T' be such that $S \subseteq T' \subseteq T$ and $|T'| = r - 1$. By induction hypothesis $V(T') = 1$. Let $j \in N$ be such that $T = T' \cup \{j\}$. If possible, let $V(T) = 0$. Then $\overline{V}(T') = 0$ and $\overline{V}(T) = 1$, which in turn implies that $m_j(\overline{G}) \neq 0$. This contradicts the assumption that $m_i(\overline{G}) = 0$ for all $i \in N$. Therefore G will fulfil monotonicity. □

Corollary 7. *Let $G = (N; V) \in \mathbf{F}^*$. Then $M(G) = \sum_{i\in N} M_i(G) = m(G)$ if and only if G is monotone.*

Corollary 8. *Let $G = (N; V) \in \mathbf{F}^*$. Then $B(G) + B(\overline{G}) = B(G)$, that is $B(\overline{G}) = 0$ if and only if G meets monotonicity.*

Remarks. (a) Propositions 5 and 6 show that a game does not have negative swing if and only if it is monotone.

(b) Since $m(G) = M(G) - m(\overline{G})$ and $m(\overline{G}) \geqslant 0$, $m(G)$ is maximized if and only if G satisfies monotonicity.

(c) It is evident that $M(G)$ can be regarded as a sensitivity index on the set $\mathbf{F}^*$.

Given $G = (N; V) \in \mathbf{F}^*$, we now express $m_i(G)$ in terms of the autocorrelation values of V. For $i \in N$, let ε_i be the n-vector, which has 1 in the ith position and 0 elsewhere.

Lemma 9. *For any n-player game $G = (N; V) \in \mathbf{F}^*$ and $i \in N$, we have*

$$M_i(G) = 2^{n-2} - \frac{1}{4} C_V(\varepsilon_i) \tag{21}$$

Proof. Let $\mu_i(V) = |\{S \subseteq N\colon V(S\Delta\{i\}) \oplus V(S) = 1\}|$, where for any two sets A and B, $A\Delta B = (A - B) \cup (B - A)$. Then it is easy to verify that

$$m_i(G) + m_i(\overline{G}) = \frac{1}{2}\mu_i(V).$$

We now compute

$$\begin{aligned} C_V(\varepsilon_i) &= \sum_{x \in F_2^n} (-1)^{V(x) \oplus V(x \oplus e_i)} \\ &= \left|\{x\colon V(x) = V(x \oplus \varepsilon_i)\}\right| - \left|\{x\colon V(x) \neq V(x \oplus \varepsilon_i)\}\right| \\ &= 2^n - 2\left|\{x\colon V(x) \neq V(x \oplus \varepsilon_i)\}\right| \\ &= 2^n - 2\mu_i(V) = 2^n - 4\big(m_i(G) + m_i(\overline{G})\big). \end{aligned}$$

This gives us the desired result. □

Corollary 10. *For any n-player game $G = (N; V) \in \mathbf{F}^*$, we have*

$$M(G) = n2^{n-2} - \frac{1}{4} \sum_{i=1}^{n} C_V(\varepsilon_i). \tag{22}$$

Thus the problem reduces to computing $\sum_{i=1}^{n} C_V(\varepsilon_i)$. We use algebraic techniques to tackle this problem. The first two steps are the following.

For two n-bit vectors u and v we denote $u \leqslant v$ if $u_i \leqslant v_i$ for each $i \in N$. Also by $\bar{u}$ we denote the bitwise complement of u.

Lemma 11. *For any n-player game $G = (N; V) \in \mathbf{F}^*$, we have*

$$\sum_{i=1}^{n} C_V(\varepsilon_i) = -n2^n + \frac{1}{2^{n-1}} \sum_{i=1}^{n} \sum_{u \leqslant \bar{\varepsilon}_i} W_V^2(u). \tag{23}$$

Proof. For $1 \leqslant i \leqslant n$, let E_i be the subspace of F_2^n defined by $E_i = \{u \in F_2^n\colon u \leqslant \bar{\varepsilon}_i\}$. Then $E_i^{\perp} = \{u \in F_2^n\colon u \leqslant \varepsilon_i\} = \{0, \varepsilon_i\}$. It is easy to see that $|E_i| = 2^{n-1}$. We now apply Theorem 4 to get

$$\sum_{u \leqslant \varepsilon_i} C_V(u) = \frac{1}{2^{n-1}} \sum_{u \leqslant \bar{\varepsilon}_i} W_V^2(u).$$

Note that

$$\sum_{u \leqslant \varepsilon_i} C_V(u) = C_V(0) + C_V(\varepsilon_i) = 2^n + C_V(\varepsilon_i).$$

Hence summing both the sides from 1 to n we obtain the desired result. □

The next task is to simplify the right hand side of Eq. (23).

Lemma 12. *For any n-player game $G = (N; V) \in \mathbf{F}^*$, we have*

$$\sum_{i=1}^{n} \sum_{u \leqslant \bar{\varepsilon}_i} W_V^2(u) = n 2^{2n} - \sum_{u \in F_2^n} wt(u) W_V^2(u). \tag{24}$$

Proof. Let $u \in F_2^n$ be arbitrary. The number of times $W_V^2(u)$ occurs in the left-hand side of Eq. (24) is $(n - wt(u))$. Hence the left-hand side is equal to

$$\sum_{u \in F_2^n} \big(n - wt(u)\big) W_V^2(u) = n \sum_{u \in F_2^n} W_V^2(u) - \sum_{u \in F_2^n} wt(u) W_V^2(u).$$

Using Parseval's Theorem, we have $\sum_{u \in F_2^n} W_V^2(u) = 2^{2n}$. This gives us the desired result. □

Let $(N; V)$ be a n-player game. For $0 \leqslant i \leqslant n$, we define

$$K_V(i) = \sum_{u \in F_2^n,\, wt(u) = i} \frac{W_V^2(u)}{2^{2n}}.$$

Note that using Parseval's Theorem, we have $\sum_{i=0}^{n} K_V(i) = 1$. We rewrite Lemma 12 in the following manner:

Lemma 13.

$$\sum_{i=1}^{n} \sum_{u \leqslant \bar{\varepsilon}_i} W_V^2(u) = n 2^{2n} - 2^{2n} \sum_{i=0}^{n} i K_V(i). \tag{25}$$

Combining Corollary 10, Lemma 11, and Lemma 13, we obtain the main result.

Theorem 14. *Let $G = (N; V) \in \mathbf{F}^*$ be an n-player game. Then*

$$M(G) = 2^{n-1} \sum_{i=0}^{n} i K_V(i). \tag{26}$$

Remark. We make some observations on the complexity of computing $M(G)$. Theorem 14 relates $M(G)$ to the Walsh transform of V. Using the fast Walsh transform algorithm, the Walsh transform of an n-variable function can be computed in time $O(n2^n)$ (see MacWilliams and Sloane, 1977). From this we obtain the K_i in $O(2^n)$ time. Hence the value of $M(G)$ can be computed in time $O(n2^n)$.

Recall that an n-player game $(N; V)$ is balanced if the number of winning coalitions (i.e., the weight) of V is 2^{n-1}.

Corollary 15. *Let $G = (N; V) \in \mathbf{F}^*$ be an n-player game. Assume further that G is balanced. Then $M(G) \geqslant 2^{n-1}$.*

Proof. If G is balanced, then $W_V(0) = 0$ and consequently $K_V(0) = 0$. Thus

$$K_V(1) + \cdots + K_V(n) = 1 \quad \text{and} \quad M(G) = 2^{n-1} \sum_{i=0}^{n} i K_V(i) \geqslant 2^{n-1}. \qquad \square$$

Corollary 16. *Let $G = (N; V) \in \mathbf{F}$ be an n-player game. Assume also that G is balanced and monotone. Then $m(G) \geqslant 2^{n-1}$. Further, equality is attained if there is a dictator.*

A class of Boolean functions called resilient functions has been extensively studied for cryptographic applications. These were introduced by Siegenthaler (1984) and were characterized in terms of Walsh transform (see Xiao and Massey, 1988). An n-variable Boolean function f is called k-resilient if $W_f(u) = 0$ for all $0 \leqslant wt(u) \leqslant k$. We can prove improved lower bound for games corresponding to resilient functions. The proof is similar to that of Corollary 15.

Corollary 17. *Let $G = (N; V) \in \mathbf{F}^*$ be an n-player game which is k-resilient. Then $M(G) \geqslant (k+1)2^{n-1}$.*

Let X be a random variable on $\{0, \ldots, n\}$ such that $P[X = i] = K_V(i)$. Then $\sum_{i=0}^{n} i K_V(i)$ is the expected value of X. Bounds on this expected value provide bounds on $m(G)$.

Theorem 18. *Let $G = (N; V) \in \mathbf{F}^*$ be an n-player game and $w = |\mathbf{W}_G|$ is the number of winning coalitions. Then*

$$\frac{w(2^n - w)}{2^{n-1}} \leqslant M(G) = m(G) + m(\overline{G}) \leqslant n \frac{w(2^n - w)}{2^{n-1}}. \tag{27}$$

Further, both the upper and lower bounds are attained.

Proof. We have

$$\sum_{i=1}^{n} K_V(i) \leqslant \sum_{i=0}^{n} i K_V(i) \leqslant n \sum_{i=1}^{n} K_V(i).$$

Using $\sum_{i=1}^{n} K_V(i) = 1 - K_V(0)$, we obtain

$$1 - K_V(0) \leqslant \sum_{i=0}^{n} i K_V(i) \leqslant n\bigl(1 - K_V(0)\bigr). \tag{28}$$

By definition,

$$K(0) = \frac{W_V^2(0)}{2^{2n}} = \frac{(2^n - 2w)^2}{2^{2n}}.$$

Putting this value of $K(0)$ in inequality (28) and using (26) we obtain the desired result. The lower bound is attained if any one player becomes the dictator. The upper bound is attained if G is the parity game, i.e., $V(x) \equiv wt(x) \bmod 2$ for all $x \in F_2^n$. (Note that for a parity game the number of swings of any player i in both G and $\overline{G}$ is 2^{n-2}. Therefore, for such a game $m(G) = m(\overline{G}) = n2^{n-2}$.) □

Corollary 19. *If $G = (N; V) \in \mathbf{F}^*$ is monotone, then*

$$\frac{w(2^n - w)}{2^{n-1}} \leqslant m(G) \leqslant n\frac{w(2^n - w)}{2^{n-1}}.$$

6. Conclusion

Dubey and Shapley (1979) argued that in a voting situation the sum of the number of ways in which each voter can affect a 'swing' in the outcome is a measure of the sensitivity of the situation. Following Felsenthal and Machover (1998), we consider a normalized value of this sum and refer to it as the Banzhaf (1965)–Coleman (1971)–Dubey–Shapley (1979) sensitivity index. This paper investigates some of its properties, the main topics being a characterization from a set of independent axioms and derivation of bounds for a very general class of games.

Acknowledgments

For comments and suggestions, we are grateful to two anonymous referees. We thank Antti Pajala for pointing out that in the 'historical sketch' part of their book, Felsenthal and Machover (1998) mention that the Banzhaf–Coleman–Dubey–Shapley index was invented originally by Lionel Penrose in 1946.

References

Banzhaf, J.F., 1965. Weighted voting doesn't work: a mathematical analysis. Rutgers Law Rev. 19, 317–343.

Barua, R., Chakravarty, S.R., Roy, S., 2002. Measuring power in weighted majority games. Mimeo. Indian Statistical Institute, Kolkata.

Burgin, M., Shapley, L.S., 2001. Enhanced Banzhaf power index and its mathematical properties. Working paper 797. Department of Mathematics, UCLA.

Canteaut, A., Carlet, C., Charpin, P., Fontaine, C., 2000. Propagarion characteristics and correlation immunity of highly non-linear Boolean functions. In: Advances in Cryptology—Eurocrypt 2000, Lecture Notes in Comput. Sci. Springer-Verlag, New York, pp. 507–522.

Carlet, C., 1992. Partially bent functions. In: Advances in Cryptology—CRYPTO'92, Lecture Notes in Comput. Sci. Springer-Verlag, New York, pp. 280–291.

Coleman, J.S., 1971. Control of collectives and the power of a collectivity to act. In: Lieberman, B. (Ed.), Social Choice. Gordon and Breach, New York, pp. 269–298.

Deegan, J., Packel, E.W., 1978. A new index of power for simple n-person games. Int. J. Game Theory 7, 113–123.

Ding, C., Xiao, G., Shan, W., 1978. The Stability Theory of Stream Ciphers. Lecture Notes in Comput. Sci., vol. 561. Springer-Verlag, New York.

Dubey, P., 1975. On the uniqueness of the Shapley value. Int. J. Game Theory 4, 131–140.

Dubey, P., Shapley, L.S., 1979. Mathematical properties of the Banzhaf power index. Math. Operations Res. 4, 99–131.

Felsenthal, D.S., Machover, M., 1995. Postulates and paradoxes of relative voting power—a critical reappraisal. Theory Dec. 38, 195–229.

Felsenthal, D., Machover, M., 1998. The Measurement of Voting Power. Edward Elgar, Cheltenham.

Hart, S., 1976. A note on edges of the n-cube. Discrete Math. 14, 157–163.

Johnston, R.J., 1978. On the measurement of power: some reactions to Laver. Environ. Planning A 10, 907–914.

Lehrer, E., 1988. An axiomatization of the Banzhaf value. Int. J. Game Theory 17, 89–99.

MacWilliams, F.J., Sloane, N.J.A., 1977. The Theory of Error Correcting Codes. North-Holland, Amsterdam.

Nowak, A.S., 1997. On an axiomatization of the Banzhaf value without the additivity axiom. Int. J. Game Theory 26, 137–141.

Owen, G., 1978. Characterization of the Banzhaf–Coleman index. SIAM J. Appl. Math. 35, 315–327.

Preneel, B., 1993. Analysis and design of cyptographic hash functions. Doctoral dissertation. K.U. Leuven.

Sarkar, P., Maitra, S., 2002. Cross-correlation analysis of cryptographically useful boolean functions and S-boxes. Theory Comput. Syst. 35, 39–57.

Shapley, L.S., 1953. A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (Eds.), Contributions to the Theory of Games II. Ann. of Math. Stud. Princeton Univ. Press, Princeton, pp. 307–317.

Shapley, L.S., Shubik, M.J., 1954. A method for evaluating the distribution of power in a committee system. Amer. Polit. Sci. Rev. 48, 787–792.

Siegenthaler, T., 1984. Correlation-immunity of nonlinear combining functions for cryptographic applications. IEEE Trans. Inform. Theory IT-30 (5), 776–780.

Straffin, P.D., 1977. Homogeneity, independence and power indices. Public Choice 30, 107–118.

Xiao, G., Massey, J.L., 1988. A spectral characterization of correlation-immune combining functions. IEEE Trans. Inform. Theory, 569–571.

Young, H.P., 1985. Monotonic solutions of cooperative games. Int. J. Game Theory 14, 65–72.

Zhang, X.M., Zheng, Y., 1995. GAC—the criterion for global avalanche characteristics of cryptographic functions. J. Universal Comp. Sci. 1, 316–333.