ON CONFIDENCE INTERVAL FOR TWO-MEANS PROBLEM BASED ON SEPARATE ESTIMATES OF VARIANCES AND TABULATED VALUES OF t-TABLE

By SAIBAL BANERJEE

Indian Statistical Institute

SUMMARY. Given k samples of n_i units from k normal populations $N_i(m_i, \sigma_i^2)$ $(i = 1, 2, \ldots, k)$ a confidence interval for any linear function $\sum_1^k c_i m_i$ (where c_i are known coefficients) with confidence coefficient not less than some pre-assigned probability α is possible in terms of sample estimates of population means and variances and tabulated values of Student's t-table. Some generalizations of the result including testing of hypothesis have been considered.

1. Introduction

Given two samples of n_i $(i = 1, 2)$ units from two normal populations with means m_1 and m_2 and variances σ_1^2 and σ_2^2 a confidence interval for any linear function $c_1m_1+c_2m_2$ (where c_1 and c_2 are known coefficients) in terms of sample estimates of population means and variances is of interest. It was shown by the author (Banerjee, 1960) that given k-samples of n_i units from k normal populations $N_i(m_i, \sigma_i^2)(i = 1, 2, \ldots, k)$ and some pre-assigned probability α, a confidence interval for $\sum_1^k c_i m_i$ with confidence coefficient not less than pre-assigned probability α, could be built up from the relation

$$\text{prob}\left[\left\{\sum_1^k c_i(\bar{x}_i - m_i)\right\}^2 \leqslant \sum_1^k \frac{t_i^2 c_i^2 s_i^2}{n_i}\right] \geqslant \alpha \qquad \ldots \quad (1.1)$$

where $\bar{x}_i$ and s_i^2 $(i = 1, 2, \ldots, k)$ are sample estimates of population means and variances, c_i $(i = 1, 2, \ldots, k)$ are known coefficients and t_i $(i = 1, 2, \ldots, k)$ are so chosen that

$$\frac{1}{\sqrt{\nu_i}} \frac{1}{B\left(\frac{\nu_i}{2}, \frac{1}{2}\right)} \int_{-t_i}^{t_i} \left(1+\frac{t^2}{\nu_i}\right)^{-\frac{\nu_i+1}{2}} dt = \alpha \quad (\nu_i = n_i - 1;\ i = 1, 2, \ldots, k).$$

The probability statement (1.1) or the confidence interval associated with the probability statement (1.1) is applicable to more general situations. Some generalizations are considered in this paper.

2. Notations

The following notations have been used throughout this paper:

$f(\chi_i^2)$ denotes the frequency function of a χ_i^2 variate which is distributed as a χ^2 variate with ν_i degrees of freedom.

$h(\chi^2)$ denotes the frequency function of a χ^2 variate which is distributed as a χ^2 variate with one degree of freedom.

$f(t/\nu_i)$ denotes the frequency function of a t-variate which is distributed as Student's t-variate with ν_i degrees of freedom.

The terminology that t_i $(i = 1, 2, \ldots, k)$ are t-values of Student's t-table of ν_i $(i = 1, 2, \ldots, k)$ d.f. corresponding to confidence coefficient α denotes t-values of Student's t-table so that

$$\int_{-t_i}^{t_i} f(t/\nu_i)dt = \alpha.$$

3. Theorems

Theorem 1 : *If u be a normal variate distributed about zero mean and unit variance and χ_i^2 $(i = 1, 2, \ldots, k)$ be χ^2 variates distributed mutually independently and also independently of u with ν_i $(i = 1, 2, \ldots, k)$ degrees of freedom and w_i $(i = 1, 2, \ldots, k)$ be a set of arbitrary weights satisfying the relation*

$$\sum_1^k w_i = 1; \quad w_i \geqslant 0; \quad (i = 1, 2, \ldots, k)$$

then

$$\text{prob}\left\{u^2 \leqslant \sum_1^k \frac{t_i^2}{\nu_i} w_i \chi_i^2\right\} \geqslant \alpha \quad \ldots \quad (3.1)$$

where t_i $(i = 1, 2, \ldots, k)$ are tabulated values of Student's t-table of ν_i $(i = 1, 2, \ldots, k)$ degrees of freedom corresponding to confidence coefficient α.

Proof : We have

$$\text{prob}\left[u^2 \leqslant \sum_1^k \frac{t_i^2}{\nu_i} w_i\chi_i^2\right] = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_1^k f(\chi_i^2)\left\{\int_0^R h(\chi^2)d\chi^2\right\} d\chi_1^2 \ldots d\chi_k^2. \quad \ldots \quad (3.2)$$

where

$$R = \sum_1^k \frac{t_i^2}{\nu_i} w_i\chi_i^2.$$

As $\int_0^z h(\chi^2)d\chi^2$ is an upward convex function of z,

$$\int_0^{\sum_1^k \frac{t_i^2}{\nu_i} w_i\chi_i^2} h(\chi^2) \geqslant \sum_1^k w_i \int^{\frac{t_i^2}{\nu_i}\chi_i^2} h(\chi^2)d\chi^2. \quad \ldots \quad (3.3)$$

It can be shown

$$\int_0^\infty f(\chi_i^2)\left\{\int^{\frac{t_i^2}{\nu_i}\chi_i^2} h(\chi^2)d\chi^2\right\} d\chi_i^2 = \alpha; \quad (i = 1, 2, \ldots, k). \quad \ldots \quad (3.4)$$

As $\sum_1^k w_i = 1$, from (3.2), (3.3) and (3.4),

$$\text{prob}\left\{u^2 < \sum_1^k \frac{t_i^2}{\nu_i} w_i\chi_i^2\right\} \geq \alpha. \qquad \ldots \quad (3.5)$$

Theorem 2 : *Let Y be a normal variate distributed about mean M with variance $\sum_1^k \lambda_i\sigma_i^2 + \sum_1^l \theta_j\sigma_j^2$, where λ_i and θ_j $(i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, l)$ are known positive constants. If σ_j^2 $(j = 1, 2, \ldots, l)$ be known and s_i^2 be estimates of σ_i^2 $(i = 1, 2, \ldots, k)$ where $\nu_i s_i^2/\sigma_i^2$ $(i = 1, 2, \ldots, k)$ are mutually independently (and also independently of Y) distributed as χ^2 with ν_i $(i = 1, 2, \ldots, k)$ degrees of freedom, then*

$$\text{prob}\left[(Y-M)^2 \leq \sum_1^k t_i^2\lambda_i s_i^2 + d^2 \sum_1^l \theta_j\sigma_j^2\right] \geq \alpha \qquad \ldots \quad (3.6)$$

where t_i $(i = 1, 2, \ldots, k)$ and d are respectively tabulated values of Student's t-table of ν_i $(i = 1, 2, \ldots, k)$ d.f. and the normal probability table corresponding to confidence coefficient α.

Proof: We have prob $[(Y-M)^2 \leq \sum_1^k t_i^2\lambda_i s_i^2 + d^2 \sum_1^l \theta_j\sigma_j^2]$

$$= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_1^k f(\chi_i^2) \left\{\int_0^R h(\chi^2)d\chi^2\right\} d\chi_1^2 \ldots d\chi_k^2 \qquad \ldots \quad (3.7)$$

where
$$\chi^2 = (Y-M)^2/\left(\sum_1^k \lambda_i\sigma_i^2 + \sum_1^l \theta_j\sigma_j^2\right)$$

$$\chi_i^2 = \frac{\nu_i s_i^2}{\sigma_i^2} \qquad (i=1, 2, \ldots, k)$$

$$w_i = \frac{\lambda_i\sigma_i^2}{\Sigma\lambda_i\sigma_i^2+\Sigma\theta_j\sigma_j^2} \quad (i = 1, 2, \ldots, k)$$

$$w_j' = \frac{\theta_j\sigma_j^2}{\Sigma\lambda_i\sigma_i^2+\Sigma\theta_j\sigma_j^2} \quad (j = 1, 2, \ldots, l)$$

$$R = \sum_1^k \frac{t_i^2}{\nu_i} w_i\chi_i^2 + d^2 \sum_1^l w_j'$$

and $f(\chi_i^2)$ denotes the frequency function of a χ_i^2 variate which is distributed as a χ^2 variate with ν_i degrees of freedom and $h(\chi^2)$ denotes the frequency function of a χ^2 variate with one degree of freedom.

As $\int_0^z h(\chi^2)d\chi^2$ is an upward convex function of z,

$$\int_0^R h(\chi^2)d\chi^2 > \sum_1^k w_i \int_0^{\frac{t_i^2}{\nu_i}\chi_i^2} h(\chi^2)d\chi^2 + \sum_1^l w_j' \int_0^{d^2} h(\chi^2)d\chi^2. \quad \ldots \quad (3.8)$$

It can be shown that $$\int_0^\infty f(\chi_i^2)\left\{\int_0^{\frac{t_i^2}{\nu_i}\chi_i^2} h(\chi^2)d\chi^2\right\} d\chi_i^2 = \alpha \qquad (i = 1, 2, \ldots, k)$$

and $$\int_0^{d^2} h(\chi^2)d\chi^2 = \alpha. \quad \ldots \quad (3.9)$$

As $\sum_1^k w_i + \sum_1^l w_j' = 1$, from (3.7), (3.8) and (3.9)

$$\text{prob}\left[(Y-M)^2 \leqslant \sum_1^k t_i^2\lambda_i s_i^2 + d^2 \sum_1^l \theta_j \sigma_j^2\right] \geqslant \alpha. \quad \ldots \quad (3.10)$$

Result 1 : Let three samples of N_1, N_2 and N_3 units be drawn from three normal populations with means m_1, m_2 and m_3 and variances not necessarily equal. Let $(\bar{x}_i, s_i^2)$ $(i = 1, 2, 3)$ be respectively estimates for the population means and variances of the three populations. Suppose, based on previous experiments, we have estimates (i) $\bar{x}_{11}, s_{11}^2$; (ii) $\bar{x}_{21}$ and (iii) s_{31}^2 respectively for (i) mean and variance of the first population, (ii) mean of the second population and (iii) variance of the third population based on n_1, n_2 and n_3 units. Then a confidence interval for any linear function $\sum_1^3 c_i m_i$ of population means, with confidence coefficient not less than some preassigned probability α, may be built up from the relation :

$$\text{prob}\left[\left\{\sum_1^3 c_i(\bar{x}_{i0}-m_i)\right\}^2 \leqslant \sum_1^3 \frac{t_i^2 c_i^2 s_{i0}^2}{N_{i0}}\right] \geqslant \alpha \quad \ldots \quad (3.11)$$

where $$N_{10} = N_1+n_1; \quad \bar{x}_{10} = \frac{N_1\bar{x}_1+n_1\bar{x}_{11}}{N_{10}}$$

$$N_{20} = N_2+n_2; \quad \bar{x}_{20} = \frac{N_2\bar{x}_2+n_2\bar{x}_2}{N_{20}}$$

$$N_{30} = N_3; \qquad \bar{x}_{30} = \bar{x}_3$$

$$s_{10}^2 = \frac{(N_1-1)s_1^2 + (n_1-1)s_{11}}{N_1+n_1-2}$$

$$s_{20}^2 = s_2^2$$

$$s_{30}^2 = \frac{(N_3-1)s_3^2 + (n_3-1)s_{31}^2}{N_3+n_3-2}$$

and t_i $(i = 1, 2, 3)$ are tabulated values of Student's t-table of ν_i d.f. $(\nu_1 = N_1+n_2-2$, $\nu_2 = N_2-1$ and $\nu_3 = N_3+n_3-2)$ corresponding to confidence coefficient α.

Relation (3.11) follows directly from Theorem 2.

Result 2 : From two samples $(x_{ij}, y_{ij}\ ;\ i = 1, 2;\ j = 1, 2, \ldots, n_i)$ of n_i $(i = 1, 2)$ units from two bivariate normal populations regression equations of y on x are estimated as:

$$\text{Population I} \ : \quad Y_1 = \bar{y}_1+b_1(X_1-\bar{x}_1)$$

$$\text{Population II} : \quad Y_2 = \bar{y}_2+b_2(X_2-\bar{x}_2).$$

If β_1 and β_2 denote respectively population values of regression coefficients of the two populations a confidence interval for $\beta_1-\beta_2$ may be built up from the result :

$$\text{prob} \quad \left[\{b_1-b_2-(\beta_1-\beta_2)\}^2 \leqslant \frac{t_1^2 s_1^2}{S_1^2} + \frac{t_2^2 s_2^2}{S_2^2} \right] \geqslant \alpha \qquad \ldots \quad (3.12)$$

where
$$S_i^2 = \sum_1^{n_i} (x_{ij}-\bar{x}_i)^2, \qquad (i = 1, 2)$$

$$s_i^2 = \frac{\sum_1^{n_i} (y_{ij}-Y_{ij})^2}{n_i-2}\ ; \qquad (i = 1, 2).$$

$$Y_{ij} = \bar{y}_i+b_i(x_{ij}-\bar{x}_i); \qquad (i = 1, 2)$$

and t_i $(i = 1, 2)$ are tabulated values of Student's t-table of n_i-2 $(i = 1, 2)$ d.f. corresponding to confidence coefficient α.

As b_1 and b_2 respectively are distributed normally about means β_1 and β_2 with variances say $\frac{\sigma_1^2}{S_1^2}$ and $\frac{\sigma_2^2}{S_2^2}$ and s_1^2 and s_2^2 are estimates of σ_1^2 and σ_2^2 with n_1-2 and n_2-2 d.f., the result follows from Theorem 2.

4. Some results on the two-means problem

In this section four results on the two-means problem would be presented as under :

(1) proof that restricting to class of functions of the form $A_1s_1^2+A_2s_2^2$ the only function which with minimum values of A_1 and A_2 would satisfy the relation

$$\{ \sum_1^2 c_i(\bar{x}_i-m_i)\}^2 \leq A_1s_1^2+A_2s_2^2$$

with probability not less that any pre-assigned probability α is

$$\frac{t_1^2c_1^2s_1^2}{n_1} + \frac{t_2^2c_2^2s_2^2}{n_2}$$

(2) numerical values of upper bounds of probability of the inequality

$$\left\{ \sum_{1}^{2} c_i(\bar{x}_i - m_i) \right\}^2 \leqslant \frac{t_1^2 c_1^2 s_1^2}{n_1} + \frac{t_2^2 c_2^2 s_2^2}{n_2}$$

(3) proof that the confidence interval

$$\sum_{1}^{2} c_i \bar{x}_i \pm \sqrt{\sum_{1}^{2} t_i^2 c_i^2 s_i^2 / n_i} \quad \text{is unbiassed}$$

(4) non-central confidence intervals.

Throughout this section and the following sections, unless otherwise stated, $\bar{x}_1$ and s_1^2 denote sample estimates of population mean m_1 and variance σ_1^2 based on a sample of size n_1 drawn from a normal population $N(m_1, \sigma_1^2)$. Also $\bar{x}_2$ and s_2^2 denote sample estimates of population mean m_2 and variance σ_2^2 based on a sample of size n_2 drawn from a normal population $N(m_2, \sigma_2^2)$. Also $\nu_1 (= n_1 - 1)$ and ν_2 $(= n_2 - 1)$ denote degrees of freedom of s_1^2 and s_2^2. Also c_1 and c_2 denote known constants, positive or negative, and α a pre-assigned probability level (between 0 and 1) and $t_i (i = 1, 2)$ are tabulated values of Student's t-table of ν_i $(i = 1, 2)$ d.f. corresponding to confidence coefficient α.

Result 1: Let $P\{A_1, A_2\}$ denote the probability of the inequality

$$\left\{ \sum_{1}^{2} c_i(\bar{x}_i - m_i) \right\}^2 \leqslant A_1 s_1^2 + A_2 s_2^2. \qquad \ldots \quad (4.1)$$

It can be easily shown that $P\{A_1, A_2\}$ is equal to

$$\frac{1}{\Gamma(p_1)} \frac{1}{\Gamma(p_2)} \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \int_0^\infty e^{-y_1 - y_2} (y_1)^{p_1 - 1} (y_2)^{p_2 - 1} \left\{ \int^{a_1 w_1 y_1 + a_2(1 - w_1) y_2} e^{-z} z^{-\frac{1}{2}} \, dz \right\} dy_1 dy_2 \qquad \ldots \quad (4.2)$$

where

$$p_i = \nu_i / 2 \; ; \; a_i = \frac{A_i / \nu_i}{c_i^2 / n_i} \; ; \; (i = 1, 2)$$

and

$$w_1 = \frac{c_1^2 \sigma_1^2 / n_1}{c_1^2 \sigma_1^2 / n_1 + c_2^2 \sigma_2^2 / n_2}.$$

Further, it can be shown that $P\{A_1, A_2\}$ as defined in (4.2) is continuous in w_1 for $0 \leqslant w_1 \leqslant 1$.

Let numerical values of A_1 and A_2 be so taken that

$$\left. \begin{aligned} & A_1 < \frac{t_1^2 c_1^2}{n_1} \\ \text{and} \quad & A_2 \geqslant \frac{t_2^2 c_2^2}{n_2} \end{aligned} \right\} \qquad \ldots \quad (4.3)$$

With numerical values of A_1 and A_2 so determined by (4.3) as w_1 tends to unity $P\{A_1, A_2\}$ tends to α^1 where α^1 is defined as

$$\frac{1}{\Gamma(p_1)} \cdot \frac{1}{\Gamma(\frac{1}{2})} \cdot \int_0^\infty e^{-y_1} (y_1)^{p_1-1} \left\{ \int_0^{\frac{A_1/\nu_1}{t_1^2/n_1} y_1} e^{-z} z^{-\frac{1}{2}} \, dz \right\} dy_1 \qquad \dots \quad (4.4)$$

which is less than α.

Since $P\{A_1, A_2\}$ is continuous in w_1 it is possible to choose a value of w_1 near $w_1 = 1$ such that $P\{A_1, A_2\} < \alpha$. This means that even if A_2 is made arbitrarily large,

$$\text{prob}\left[\left\{\sum_1^2 c_i(\bar{x}_i - m_i)\right\}^2 \leqslant A_1 s_1^2 + A_2 s_2^2\right]$$

where $A_1 < \dfrac{t_1^2 c_1^2}{n_1}$ depending upon the value of w_1, would be less than the pre-assigned probability level for some of the populations. Hence the only function of the form $A_1 s_1^2 + A_2 s_2^2$ which, with minimum values of A_1 and A_2, would satisfy the relation

$$\left\{\sum_1^2 c_i(\bar{x}_i - m_i)\right\}^2 \leqslant A_1 s_1^2 + A_2 s_2^2$$

with probability not less than α is

$$\frac{t_1^2 c_1^2 s_1^2}{n_1} + \frac{t_2^2 c_2^2 s_2^2}{n_2}$$

Result 2: From (4.2), $P\left\{\dfrac{t_1^2 c_1^2}{n_1}, \dfrac{t_2^2 c_2^2}{n_2}\right\}$ standing for the probability of the inequality

$$\left\{\sum_1^2 c_i(\bar{x}_i - m_i)\right\}^2 \leqslant \frac{t_1^2 c_1^2 s_1^2}{n_1} + \frac{t_2^2 c_2^2 s_2^2}{n_2}$$

is equal to

$$\frac{1}{\Gamma(p_1)} \cdot \frac{1}{\Gamma(p_2)} \cdot \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \int_0^\infty e^{-y_1 - y_2} (y_1)^{p_1 - 1} (y_2)^{p_2 - 1} \left\{ \int_0^{a_1 w_1 y_1 + a_2 w_2 y_2} e^{-z} . z^{-\frac{1}{2}} \, dz \right\} dy_1 dy_2 \qquad \dots \quad (4.5)$$

where $\quad p_i = \nu_i/2\,; \quad a_i = t_i^2/\nu_i\,; \quad (i = 1, 2)$

$$w_1 = \frac{c_1^2 \sigma_1^2/n_1}{c_1^2\sigma_1^2/n_1 + c_2^2\sigma_2^2/n_2} \text{ and } w_2 = 1 - w_1.$$

$P\left\{\dfrac{t_1^2 c_1^2}{n_1}, \dfrac{t_2^2 c_2^2}{n_2}\right\}$ is essentially a function of w_1 and it tends to α when w_1 tends to zero or unity. For values of w_1 between zero and unity $P\left\{\dfrac{t_1^2 c_1^2}{n_1}, \dfrac{t_2^2 c_2^2}{n_2}\right\}$ is numerically greater than α. Differentiating $P\left\{\frac{t_1^2c_1^2}{n_1}, \frac{t_2^2c_2^2}{n_2}\right\}$ with respect to w_1 within the sign of integration (such differentiation is permissible for $\nu_1, \nu_2 \geqslant 2$ and for $\nu_1, \nu_2 = 1$ if $0 < \epsilon \leqslant w_1 \leqslant 1-\epsilon$) it can be shown that

$$\frac{d}{dw_1} P\left\{\frac{t_1^2c_1^2}{n_1}, \frac{t_2^2c_2^2}{n_2}\right\} = I_1 - I_2 \qquad \ldots \quad (4.6)$$

where for (i) $a_2w_2 \geqslant a_1w_1$,

$$I_1 = K.\ \frac{a_1p_1}{1+a_1w_1} \cdot c_2^{-\frac{1}{2}}.\ F\left\{\tfrac{1}{2}, p_1+1, p_1+p_2+1;\ \frac{c_2-c_1}{c_2}\right\} \qquad \ldots \quad (4.7)$$

$$I_2 = K.\ \frac{a_2p_2}{1+a_2w_2} \cdot c_2^{-\frac{1}{2}}.\ F\left\{\tfrac{1}{2}, p_1, p_1+p_2+1;\ \frac{c_2-c_1}{c_2}\right\} \qquad \ldots \quad (4.8)$$

and for (ii) $a_1w_1 \geqslant a_2w_2$,

$$I_1 = K\cdot\ \frac{a_1p_1}{1+a_1w_1} \cdot c_1^{-\frac{1}{2}}.\ F\left\{\tfrac{1}{2}, p_2, p_1+p_2+1;\ \frac{c_1-c_2}{c_1}\right\} \qquad \ldots \quad (4.9)$$

$$I_2 = K.\ \frac{a_2p_2}{1+a_2w_2} \cdot c_1^{-\frac{1}{2}}.\ F\left\{\tfrac{1}{2}, p_2+1, p_1+p_2+1;\ \frac{c_1-c_2}{c_1}\right\} \qquad \ldots \quad (4.10)$$

where $c_1 = \dfrac{a_1w_1}{1+a_1w_1}$; $c_2 = \dfrac{a_2w_2}{1+a_2w_2}$; and K is a function of p_i, a_i and w_i $(i = 1, 2)$.

If $\nu_1 = \nu_2 = \nu$, so that $a_1 = a_2 = a_0$, from (4.7) and (4.8) for the case $a_0w_2 \geqslant a_0w_1$ (i.e. $w_1 \leqslant \frac{1}{2}$)

$$I_1 = K'.\ \frac{1}{1+a_0w_1} \cdot F\left\{\tfrac{1}{2}, \frac{\nu}{2}+1, \nu+1;\ \frac{w_2-w_1}{w_2(1+a_0w_1)}\right\} \qquad \ldots \quad (4.11)$$

$$I_2 = K'.\ \frac{1}{1+a_0w_2} \cdot F\left\{\tfrac{1}{2}, \frac{\nu}{2}, \nu+1;\ \frac{w_2-w_1}{w_2(1+a_0w_1)}\right\}. \qquad \ldots \quad (4.12)$$

As

$$F\left\{\tfrac{1}{2}, \frac{\nu}{2}+1, \nu+1; z\right\} > F\left\{\tfrac{1}{2}, \frac{\nu}{2}, \nu+1; z\right\}, \ (z > 0)$$

from (4.11) and (4.12) it follows for $w_1 < \frac{1}{2}$,

$$I_1 > I_2 \qquad \ldots \quad (4.13)$$

which means $P\left\{\frac{t_1^2c_1^2}{n_1}, \frac{t_0^2c_2^2}{n_2}\right\}$ increases as w_1 increases. Further from (4.11) and (4.12) it is evident that at $w_1 = \frac{1}{2}$

$$I_1 = I_2. \qquad \ldots \quad (4.14)$$

Also from (4.9) and (4.10) for the case $a_0w_1 > a_0w_2$ (i.e. $w_1 > \frac{1}{2}$) it can be shown that I_2 is greater than I_1. $P\left\{\frac{t_0^2c_1^2}{n}, \frac{t_0^2c_2^2}{n}\right\}$ for variation in w_1 takes a maximum value at $w_1 = \frac{1}{2}$. In Table 1 below maximum value of $P\left\{\frac{t_0^2c_1^2}{n}, \frac{t_0^2c_2^2}{n}\right\}$ for suitable values of $\nu\ (= n-1)$ and α (i) 0.90; (ii) 0.95 and (iii) 0.98 are presented. The values have been worked out from Table 9 (Probability Integral $P(t/\nu)$ of the t-distribution) of *Biometrika Tables*, Volume I, using the relation :

$$P\left\{\frac{t_0^2c_1^2}{n}, \frac{t_0^2c_2^2}{n}\right\}_{w_1=\frac{1}{2}} = \int_{-t_0}^{t_0} f(t/2\nu)dt$$

where t_0 satisfies the relation

$$\int_{-t_0}^{t_0} f(t/\nu)dt = \alpha.$$

Also, occasionally, Incomplete Beta Function tables were used.

TABLE 1. MAXIMUM VALUES OF $P\left\{\frac{t_0^2c_1^2}{n}, \frac{t_0^2c_2^2}{n}\right\}$

FOR VARIATION IN w_1

$\nu = n-1$	α—value		
	0.90	0.95	0.98
(0)	(1)	(2)	(3)
1	.9758	.9939	.9990
2	.9567	.9873	.9978
3	.9430	.9809	.9961
4	.9342	.9759	.9943
5	.9283	.9721	.9928
6	.9239	.9691	.9916
8	.9183	.9651	.9895
10	.9149	.9623	.9880
12	.9125	.9605	.9869
14	.9108	.9591	.9860
16	.9090	.9579	.9854
18	.9080	.9572	.9848
20	.9077	.9565	.9844
22	.9062	.9559	.9840
24	.9055	.9554	.9836
30	.9051	.9542	.9830

Result 3: The confidence interval

$$\Sigma c_i\bar{x}_i \pm \sqrt{\frac{t_1^2c_1^2s_1^2}{n_1} + \frac{t_2^2c_2^2s_2^2}{n_2}} \qquad \ldots \quad (4.15)$$

is unbiased.

Let

$$\left.\begin{aligned} P_0 &\equiv \text{prob}\left[\left\{\sum_1^2 c_i\bar{x}_i - \sum_1^2 c_i m_i\right\}^2 < \sum_1^2 \frac{t_i^2 c_i^2 s_i^2}{n_i}\right] \\ \text{and} \quad P_1 &\equiv \text{prob}\left[\left\{\sum_1^2 c_i\bar{x}_i - M'\right\}^2 < \sum_1^2 \frac{t_i^2 c_i^2 s_i^2}{n_i}\right] \end{aligned}\right\} \quad \ldots \quad (4.16)$$

Since for fixed (s_1, s_2), $\sum_1^2 c_i\bar{x}_i$ is distributed normally about mean $\sum_1^2 c_i m_i$ with variance, say θ^2,

$$P_0 = \int_0^\infty \int_0^\infty g(s_1)g(s_2) \left\{ \int_{-T/\theta}^{T/\theta} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \right\} ds_1\, ds_2$$

and

$$P_1 = \int_0^\infty \int_0^\infty g(s_1)\, g(s_2) \left\{ \int_{-T/\theta}^{T/\theta} \frac{1}{\sqrt{2\pi}} e^{-(u-M'')^2/2} du \right\} ds_1\, ds_2$$

where

$g(s_1)$ = frequency function of s_1

$g(s_2)$ = " " of s_2

$$T = \sqrt{\frac{\Sigma\, t_i^2 c_i^2 s_i^2}{n_i}} \text{ and } M'' = \sum_1^2 c_i m_i - M'.$$

As

$$\int_{-d}^{d} e^{-u^2/2}\, du > \int_{-d}^{d} e^{-(u-M'')/2}\, du,$$

unbiassedness follows.

Result 4: The confidence interval so far considered for the two-means problem is central by nature. Non-central confidence interval with confidence coefficient not less than any pre-assigned probability level is, however, possible. Consider the confidence interval :

$$\left\{ -T_1(s_1^2, s_2^2) \leqslant \sum_1^2 c_i(\bar{x}_i - m_i) \leqslant T_2(s_1^2, s_2^2) \right\} \quad \ldots \quad (4.17)$$

where

$$T_1(s_1^2, s_2^2) \equiv \sqrt{\frac{t_{11}^2 c_1^2 s_1^2}{n_1} + \frac{t_{21}^2 c_2^2 s_2^2}{n_2}}$$

$$T_2(s_1^2, s_2^2) \equiv \sqrt{\frac{t_{12}^2 c_1^2 s_1^2}{n_1} + \frac{t_{22}^2 c_2^2 s_2^2}{n_2}}$$

where t_{ij} ($i = 1, 2,; j = 1, 2$) have been so determined that

$$\left.\begin{array}{l} \int_{-t_{11}}^{0} f(t/\nu_1)dt = \alpha_{11};\ \int_{0}^{t_{12}} f(t/\nu_1)dt = \alpha_{12}; \\ \int_{-t_{21}}^{0} f(t/\nu_2)dt = \alpha_{21};\ \int_{0}^{t_{22}} f(t/\nu_2)dt = \alpha_{22}; \end{array}\right\} \quad \ldots (4.18)$$

where $\quad t_{ij} \geqslant 0 (i, j = 1, 2)$

and $\quad \alpha_{11}+\alpha_{12} = \alpha_{21}+\alpha_{22}=\alpha.$

Now $\quad$ prob $\{-T_1 \leqslant \Sigma\, c_i(\bar{x}_i-m_i) \leqslant T_2\}$

$$= \int_0^\infty \int_0^\infty g(s_1)\, g(s_2) \left\{ \int_{-T_1/\theta}^{T_2/\theta} \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2} \right\} ds_1\, ds_2$$

$$= \tfrac{1}{2} \int_0^\infty \int_0^\infty g(s_1)\, g(s_2) \left\{ \int_0^{T_1^2/\theta^2} h(\chi^2)d\chi^2 + \int_0^{T_2^2/\theta^2} h(\chi^2)d\chi^2 \right\} ds_1\, ds_2 \quad \ldots (4.19)$$

where

$g(s_1)$ = frequency function of s_1

$g(s_2)$ = ,, ,, of s_2

$h(\chi^2)$ = ,, ,, of χ^2 variate with 1.d.f.

and

$$\theta^2 = \frac{c_1^2\sigma_1^2}{n_1} + \frac{c_2^2\sigma_2^2}{n_2}.$$

Now

$$\int_0^{T_i^2/\theta^2} h(\chi^2)d\chi^2 \geqslant w_1 \int_0^{t_{i1}^2 s_1^2/\sigma_1^2} h(\chi^2)d\chi^2 + (1-w_1) \int_0^{t_{2i}^2 s_2^2/\sigma_2^2} h(\chi^2)d\chi^2 \quad \ldots (4.20)$$

where

$$w_1 = \frac{c_1^2\sigma_1^2/n_1}{c_1^2\sigma_1^2/n_1+c_2^2\sigma_2^2/n_2}, \quad (i = 1, 2).$$

From (4.19) and (4.20)

$$\text{prob}\,[-T_1 \leqslant \Sigma\, c_i(x_i-m_i) \leqslant T_2] \geqslant \tfrac{1}{2}\{w_1(\alpha_{11}+\alpha_{12})+(1-w_1)(\alpha_{21}+\alpha_{22})\} = \alpha.$$

5. Test of hypothesis

The problem of the two-means has so far been considered from the approach of estimation by confidence interval corresponding to the region R

$$\left\{ \sum_1^2 c_i(\bar{x}_i-m_i) \right\}^2 \leqslant \frac{t_1^2c_1^2s_1^2}{n_1} + \frac{t_2^2c_2^2s_2^2}{n_2}.$$

The region R', complementary to the region of acceptance R, may be used as a critical region for testing hypothesis H regarding population means. For example, if it is required to test the hypothesis H that $m_1 = m_2$ the corresponding statistic T may be defined as

$$T = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{t_1^2 s_1^2}{n_1} + \frac{t_2^2 s_2^2}{n_2}}},$$

t_1 and t_2 being suitably chosen. If such tests are applied, the first kind of error, i.e., the probability of rejection of the hypothesis when true, will not be exactly equal to an assigned value $1-\alpha$ but would depend upon w_1 or the variances of the two populations. For w_1 tending to zero or unity, the error of the first kind would tend to the assigned value $1-\alpha$. Hence, one can go in for such tests, if one is prepared to accept tests whose first kind of error would not be exactly equal to a given value $1-\alpha$ but would depend upon the values of the free parameters (here σ_1^2, σ_2^2) in such a way that the first kind of error would always be less than or equal to $1-\alpha$. Such tests, however, would be unbiassed as the complementary region R, the region of acceptance, as proved earlier is unbiassed

Two examples have been considered below. Example 1 has been taken from *Biometrika Tables*, Volume I, 1954 and Example 2 from *Statistical Tables* by Fisher and Yates.

Example 1 : Two samples of sizes $n_1 = 10$ and $n_2 = 15$ furnish the following estimates :

population	mean	variance
I	73.4	51
II	47.1	141

To test the hypothesis about the equality of population means with maximum value of error of the first kind fixed at (i) 0.10, (ii) 0.05 and (iii) 0.02 three statistics (i) T_1, (ii) T_2 and (ii) T_3 respectively may be computed as under :

$$T_j = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{t_{1j}^2 \frac{s_1^2}{n_1} + t_{2j}^2 \frac{s_2^2}{n_2}}} \quad (j = 1, 2, 3); \qquad T_2 = \frac{73.4-47.1}{\sqrt{69.34}} = \frac{26.3}{8.33} = 3.16;$$

$$T_1 = \frac{73.4-47.1}{\sqrt{46.29}} = \frac{26.3}{6.80} = 3.87; \qquad T_3 = \frac{73.4-47.1}{\sqrt{105.31}} = \frac{26.3}{10.26} = 2.56;$$

where t_{ij} are $100.\alpha_j$ percentile points of Student's t-table of ν_i d.f. with $\nu_1 = 9$, $\nu_2 = 14$ and $\alpha_1 = 0.10$, $\alpha_2 = 0.05$ and $\alpha_3 = 0.02$. All the statistics are numerically greater than unity. It is seen that with maximum value of the error of the first kind fixed even at 0.02 the hypothesis cannot be accepted.

Apart from the question of testing of hypothesis, confidence intervals I_1, I_2 and I_3 for the difference of population means with confidence coefficient respectively not less than (i) 0.90, (ii) 0.95 and (iii) 0.98 may be built up as

$$\text{(i)} \quad I_1 \rightarrow 26.3 \pm 6.80 \rightarrow (19.50 \rightarrow 33.10)$$

$$\text{(ii)} \quad I_2 \rightarrow 26.3 \pm 8.33 \rightarrow (17.97 \rightarrow 34.63)$$

$$\text{(iii)} \quad I_3 \rightarrow 26.3 \pm 10.26 \rightarrow (16.04 \rightarrow 36.56)$$

90 per cent confidence limits, according to Welch's solution for this case is 19.8 to 32.8 which corresponds to interval I_1 having a range 19.5 to 33.1.

Example 2 : A physical constant evaluated by a new method gives a mean of twelve determinations,

$$x_1 = 4.77383$$

and that the sum of squares of the deviations of these values from their mean is

$$\sum_j (x_{1j} - x_1)^2 = 0.11580 \times 10^{-1}$$

so that from 11 degrees of freedom the variance of the mean is estimated to be

$$s_1^2(x_1) = 0.8773 \times 10^{-4}$$

and the estimated standard deviation of the mean is

$$s_1(\bar{x}_1) = .9366 \times 10^{-2}.$$

Numerous previous determinations, using different methods, have given the value,

$$x_2 = 4.744$$

where $\bar{x}_2$ has a standard error based on a large number of degrees of freedom with $s_2(x_2) = .00382$.

To test the hypothesis about the equality of population means with maximum value of error of the first kind fixed at (i) 0.10, (ii) 0.05 and (iii) 0.02 three statistics (i) T_1, (ii) T_2 and (iii) T_3 respectively may be computed as under :

$$T_j = \frac{\bar{x}_1 - x_2}{\sqrt{t_{1j}^2 \; s_1^2(\bar{x}_1) + t_{2j} \; s_2^2(x_2)}}; \qquad T_2 = \frac{\cdot 02983}{10^{-2} \times \sqrt{4 \cdot 8105}} = \frac{2 \cdot 983}{2 \cdot 19} = 1 \cdot 36;$$

$$T_1 = \frac{\cdot 02983}{10^{-2} . \sqrt{3 \cdot 2246}} = \frac{2 \cdot 983}{1 \cdot 80} = 1 \cdot 66; \quad T_3 = \frac{\cdot 02893}{10^{-2} \times \sqrt{7 \cdot 2704}} = \frac{2 \cdot 983}{2 \cdot 70} = 1 \cdot 11;$$

where t_{ij} are $100 \cdot \alpha_j$ percentile points of Student's t-table of ν_i d.f. with $\nu_1 = 11$ and $\nu_2 = \infty$ and $\alpha_1 = 0 \cdot 10$, $\alpha_2 = 0 \cdot 05$ and $\alpha_3 = 0 \cdot 02$. All the statistics are numerically greater than unity. It is seen that with maximum value of the error of the first kind fixed even at 0.02 the hypothesis cannot be accepted.

6. Behrens-Fisher and Welch's solution

Fisher (1935) indicated that by means of fiducial argument in statistical inference given two samples of $n_i (i = 1, 2)$ units from two normal populations with $\bar{x}_i$ and s_i^2 $(i = 1, 2)$ as sample estimates of population means and variances, the relation

$$d = \frac{(\bar{x}_1 - m_1) - (\bar{x}_2 - m_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = u_1 \sin\theta - u_2 \cos\theta \tag{6.1}$$

where $$u_i = \frac{\bar{x}_i - m_i}{s_i/\sqrt{n_i}} \ (i = 1, 2) \text{ and } \tan\theta = \frac{s_1/\sqrt{n_1}}{s_2/\sqrt{n_2}}$$

originally due to Behrens (1929), could be used to test the hypothesis that $d(= m_1 - m_2)$ has the value zero. Sukhatme in 1938 published critical values of Behrens-Fisher test as defined in (6.1) for 5 per cent level of significance for $\nu_1, \nu_2 = 6, 8, 12, 24$ and ∞ for $\theta = 0°, 15°, 30°, 45°, 60°, 75°, 90°$ where $\tan\theta$ is $\frac{s_1/\sqrt{n_1}}{s_2/\sqrt{n_2}}$. Further critical values for 1 per cent level of significance for $\nu_1, \nu_2 = 6, 8, 12, 24$ and ∞ were published later. To calculate critical values of (6.1) Sukhatme assumed that for given value of θ, u_1 and u_2 were independently distributed as Student's t-variate with $n_i - 1$ $(i = 1, 2,)$ d.f. Critical values of Behrens-Fisher test for small odd degrees of freedom $\nu_1, \nu_2 = 1, 3, 5$ and 7 were published by Fisher and Healy in 1956.

Critical values of Behrens-Fisher test have been tabulated for different values of θ where $\tan\theta = \frac{s_1/\sqrt{n_1}}{s_2/\sqrt{n_2}}$. For $\theta = 0°$, critical values of Behrens-Fisher test is equal to t-values of Student's t-table with ν_2 d.f. Also for $\theta = 90°$, critical values of Behrens-Fisher test is exactly equal to t-values of Student's t-table with ν_1 d.f. For intermediate values of θ, critical values of the Behrens-Fisher test for $\nu_1 = \nu_2 = \nu$ and $\nu = 6, 8, 12,$ and 24 is numerically less than tabulated critical values for $\theta = 0°$ (or $90°$, which are numerically equal in such cases) for 5 per cent and 1 per cent level of significance. For $\nu_1 \neq \nu_2$ and $\nu_1, \nu_2 = 6, 8, 12, 24$ and ∞ critical values of Behrens-Fisher test for intermediate values of θ usually lies in between tabulated critical values for $\theta = 0°$ and $\theta = 90°$ and is occasionally less than both of them. For $\nu_1 = \nu_2 = 1$ critical values of the test for intermediate values θ are, however, numerically higher than corresponding critical values for $\theta = 0°$(or $90°$) for 10 per cent, 5 per cent, 2 per cent and 1 per cent level of significance.

Welch (1947) considered the problem of finding a function which is such that

$$\text{prob}\left[\frac{|y-\eta|}{\sqrt{\sum_1^k \lambda_i s_i^2}} \leqslant V(s_1^2, s_2^2, \ldots, s_k^2)\right] = \alpha \qquad \ldots \ (6.2)$$

for all values of $\sigma_1^2, \sigma_2^2, \ldots \sigma_k^2$, where y is a normal variate distributed about mean η with variance $\sum_1^k \lambda_i \sigma_i^2$ and s_i^2 are independent estimates of σ_i^2 and λ_i are known positive constants ($i = 1, 2, \ldots, k$). For $y \equiv \bar{x}_1 - \bar{x}_2$ Welch's problem is to find $V(s_1^2, s_2^2)$ which is such that

$$\text{prob}\left[\frac{|\bar{x}_1-\bar{x}_2-(m_1-m_2)|}{\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}} \leqslant V(s_1^2, s_2^2)\right] = \alpha. \quad \ldots (6.3)$$

Wilks (1940) stated that an exact solution of the form (6.3) is not possible; no proof has, however, been published. Welch has put forward a series for the case (6.2) (which includes the case (6.3) as well) as under:

$$V(s_1^2, s_2^2, \ldots, s_k^2) = \xi + \frac{\xi(1+\xi^2)}{4}\left(\sum_1^k c_i^2/f_i\right) - \frac{\xi(1+\xi^2)}{2}\left(\sum_1^k c_i^2/f_i^2\right)$$

$$+ \frac{\xi(3+5\xi^2+\xi^4)}{3}\left(\sum_1^k c_i^3/f_i^2\right) \text{ etc.} \quad \ldots (6.4)$$

where $$c_i = \lambda_i s_i^2 / \sum_1^k \lambda_i s_i^2 ;$$

$$f_i = \text{d.f. of } s_i^2 ; \quad (i = 1, 2, \ldots, k)$$

and ξ is tabulated value of normal probability table so that

$$\frac{1}{\sqrt{2\pi}} \int_{-\xi}^{\xi} e^{-\frac{y^2}{2}} dy = \alpha.$$

In the words of Bartlett (1956) "there is a permissible criticism of Welch's solution, namely, that the existence of an exact solution in his sense has never been rigorously established." According to Wallace (1958) "it is still not known whether a non-randomized similar level α test exists."

Critical values of Welch's solution for the two sample case only have been calculated by Aspin and are given in *Biometrika Tables*, Volume I (Table No. 11, 1954 print). The values given in *Biometrika Tables* cover the range (i) $\nu_1, \nu_2 = 6, 8, 10, 15, 20$ and ∞ for $\alpha = 0.90$ and (ii) $\nu_1\ \nu_2 = 10, 12, 15, 20, 30$ and ∞ for $\alpha = 0.98$. Also further critical values for $\alpha = 0.95$ and 0.99 are given in Welch, Trickett and James (1956). Critical values of Welch's solution for very small degrees of freedom ($\nu_1, \nu_2 \leqslant 5$) have not been published.

Critical values $V(c, \nu_1, \nu_2, \alpha)$ of Welch's solution have been tabulated for given ν_1, ν_2 and α for the ratio $c = \frac{\lambda_1 s_1^2}{\lambda_1 s_1^2 + \lambda_2 s_2^2}$. This, of course, does not mean that critical values of Welch's solution refer to sub-sets having observed variance ratios. Both Welch's solution and the present solution refer to unrestricted variation of $\bar{x}_1$,

$\bar{x}_2$, s_1^2 and s_2^2. For $c = 0$ critical values of Welch's solution are exactly equal to t-values of the Students t-values with ν_2 degrees of freedom. Also for $c = 1$ critical values of Welch's solution are exactly equal to t-values of the Student's t-values with ν_1 degrees of freedom. For intermediate values of c, $V(c, \nu_1, \nu_2, \alpha)$ numerically lies in between $V(0, \nu_1, \nu_2, \alpha)$ and $V(1, \nu_1, \nu_2, \alpha)$.

To compare the solution derived in this paper with Welch's solution, the solution derived in this paper can be written as under :

$$\text{prob}\left[\frac{(y-\eta)^2}{\lambda_1 s_1^2+\lambda_2 s_2^2} \leqslant \frac{t_1^2\lambda_1 s_1^2+t_2^2\lambda_2 s_2^2}{\lambda_1 s_1^2+\lambda_2 s_2^2}\right] \geqslant \alpha. \qquad \ldots \quad (6.5)$$

Using relation (6.5), critical values of (i) Welch's solution and (ii) the present solution for the cases (i) $\alpha = 0{\cdot}95$; $\nu_2 = 8$; $\nu_1 = 8$, 12 and ∞ and (ii) $\alpha = 0{\cdot}99$; $\nu_2 = 12$; $\nu_1 = 12$ and ∞ have been given in Tables 2 and 3. Critical values of the present solution as given in column (4) of Tables 2 and 3 have been worked out from the relation:

$$\sqrt{t_1^2 c+t_2^2(1-c)}$$

where
$$c = \frac{\lambda_1 s_1^2}{\lambda_1 s_1^2+\lambda_2 s_2^2}$$

for different values of c as indicated in column (1) of Table 2.

It is seen from Tables 2 and 3 that for $c = 0$ or 1, there is no difference between the critical values of the two solutions. For intermediate values of c there are differences in the values but the magnitude of the differences falls off as the number of degrees of freedom increases. Maximum magnitude of the differences expressed as percentage critical values of the Welch's solution are of the order of (i) 11 per cent (ii) 7 per cent and (iii) 4 per cent respectively for the cases $\nu_1 =$ (i) 8, (ii) 12 and (iii) ∞ with $\nu_2 = 8$ and $\alpha = 0.95$. Difference in the mathematical expectation of the lengths of the two confidence intervals, which would be a weighted average of the difference in the lengths of the two intervals at different c values, properly weighted by the frequency of occurrence of c, would, however, be smaller than the maximum magnitude of the difference.

A comparison of the critical values of the present solution with the critical values of the Behrens-test as given in *Statistical Tables* (by Fisher and Yates) may be of interest. This comparison, however, would not be strictly valid because the present solution is based on the theory of confidence interval whereas the tabulated values as given in statistical tables are based on the theory of fiducial inference. Further, presumably the Behrens-Fisher test calculates critical values restricting to sub-sets

TABLE 2. CRITICAL VALUES OF (i) WELCH (ii) BEHRENS-FISHER AND (iii) PRESENT SOLUTION FOR $\nu_2 = 8$; $\nu_1 = 8$, 12 AND ∞ FOR 5 PER CENT LEVEL OF SIGNIFICANCE

	Welch's solution	Behrens-Fisher test*	present solution	difference col. (4)-col. (2)	difference col. (4)-col. (3)
(1)	(2)	(3)	(4)	(5)	(6)
			$\nu_2 = 8$; $\nu_1 = 8$		
0.0	2.31	2.31	2.31	0.00	0.00
0.1	2.25	2.30	2.31	0.06	0.01
0.2	2.20	2.30	2.31	0.11	0.01
0.3	2.14	2.29	2.31	0.17	0.02
0.4	2.10	2.29	2.31	0.21	0.02
0.5	2.08	2.29	2.31	0.23	0.02
0.6	2.10	2.29	2.31	0.21	0.02
0.7	2.14	2.29	2.31	0.17	0.02
0.8	2.20	2.30	2.31	0.11	0.01
0.9	2.25	2.30	2.31	0.06	0.01
1.0	2.31	2.31	2.31	0.00	0.00
			$\nu_2 = 8$; $\nu_1 = 12$		
0.0	2.31	2.31	2.31	0.00	0.00
0.1	2.25	2.29	2.29	0.04	0.00
0.2	2.20	2.27	2.28	0.08	0.01
0.3	2.15	2.26	2.27	0.12	0.01
0.4	2.10	2.24	2.26	0.16	0.02
0.5	2.07	2.22	2.24	0.17	0.01
0.6	2.07	2.22	2.23	0.16	0.01
0.7	2.08	2.21	2.22	0.14	0.01
0.8	2.11	2.20	2.20	0.09	0.00
0.9	2.14	2.19	2.19	0.05	0.00
1.0	2.18	2.18	2.18	0.00	0.00
			$\nu_2 = 8$; $\nu_1 = \infty$		
0.0	2.31	2.31	2.31	0.00	0.00
0.1	2.25	2.27	2.27	0.02	0.00
0.2	2.20	2.23	2.24	0.04	0.01
0.3	2.14	2.20	2.21	0.07	0.01
0.4	2.09	2.16	2.17	0.08	0.01
0.5	2.05	2.13	2.14	0.09	0.01
0.6	2.01	2.09	2.10	0.09	0.01
0.7	1.99	2.06	2.07	0.08	0.01
0.8	1.97	2.03	2.03	0.06	0.00
0.9	1.96	1.99	2.00	0.04	0.01
1.0	1.96	1.96	1.96	0.00	0.00

* Values have been worked out from tabulated values as given in Table VI, *Statistical Tables* by Fisher and Yates, by interpolation.

TABLE 3. CRITICAL VALUES OF (i) WELCH (ii) BEHRENS-FISHER AND (iii) PRESENT SOLUTION FOR $\nu_2 = 12$; $\nu_1 = 12$ AND ∞ FOR 1 PER CENT LEVEL OF SIGNIFICANCE

c	Welch's solution	Behrens-Fisher test*	present solution	difference col. (4)-col. (2)	difference col. (4)-col. (3)
(1)	(2)	(3)	(4)	(5)	(6)
			$\nu_2 = 12$; $\nu_1 = 12$		
0.0	3.05	3.06	3.06	0.01	0.00
0.1	2.98	3.02	3.06	0.08	0.04
0.2	2.91	2.99	3.06	0.15	0.07
0.3	2.84	2.97	3.06	0.22	0.09
0.4	2.78	2.96	3.06	0.28	0.10
0.5	2.76	2.95	3.06	0.30	0.11
0.6	2.78	2.96	3.06	0.28	0.10
0.7	2.84	2.97	3.06	0.22	0.09
0.8	2.91	2.99	3.06	0.15	0.07
0.9	2.98	3.02	3.06	0.08	0.04
1.0	3.05	3.06	3.06	0.01	0.00
			$\nu_2 = 12$; $\nu_1 = \infty$		
0.0	3.05	3.06	3.06	0.01	0.00
0.1	2.98	3.00	3.01	0.03	0.01
0.2	2.91	2.94	2.96	0.05	0.02
0.3	2.84	2.88	2.92	0.08	0.04
0.4	2.77	2.83	2.87	0.10	0.04
0.5	2.71	2.78	2.83	0.12	0.05
0.6	2.65	2.73	2.78	0.13	0.05
0.7	2.62	2.68	2.73	0.11	0.05
0.8	2.59	2.64	2.68	0.09	0.04
0.9	2.58	2.61	2.63	0.05	0.02
1.0	2.58	2.58	2.58	0.00	0.00

* Values have been worked out from tabulated values as given in Table VI, *Statistical Tables* by Fisher and Yates, by interpolation.

having observed values $\frac{s_1/\sqrt{n_1}}{s_2/\sqrt{n_2}}$, whereas in the present case critical values refer to unrestricted variation of the four sample estimates $\bar{x}_1$, $\bar{x}_2$, s_1^2 and s_2^2.

Bearing in mind the broad limitations of comparing the critical values of the present solution with the critical values of Behrens-Fisher test, in column (6) of Tables 2 and 3 differences in critical values of the two solutions have been shown. It is seen that the differences in the critical values are small.

Further comparison of critical values of the present solution with the critical values of the Behrens-Fisher test for small d.f. ν_1, $\nu_2 = 1$, 3 and 5 has been done in

Table 4. [Welch's solution has not been considered because critical values of Welch's' solution for ν_1, $\nu_2 < 5$ have not been published.] It is seen from columns (4) and (7) that excepting the cases (i) $\nu_1 = \nu_2 = 1$; (ii) $\nu_1 = 1$; $\nu_2 = 3$ differences in the critical values are usually small.

TABLE 4. CRITICAL VALUES OF (i) BEHRENS-FISHER TEST AND (ii) PRESENT SOLUTION FOR SMALL ODD DEGREES OF FREEDOM FOR 5 PER CENT LEVEL OF SIGNIFICANCE

θ	Behrens-Fisher test*	present solution	difference col.(3)-col. (2)	Behrens-Fisher test*	present solution	difference col.(6)-col. (5)
(1)	(2)	(3)	(4)	(5)	(6)	(7)
	$\nu_1 = 1$; $\nu_2 = 1$			$\nu_1 = 1$; $\nu_2 = 3$		
0°	12.706	12.706	0.000	3.182	3.182	.000
15°	15.562	12.706	—2.856	4.960	4.501	—.459
30°	17.357	12.706	—4.651	7.123	6.925	—.198
45°	17.969	12.706	—5.263	9.303	9.262	—.041
60°	17.357	12.706	—4.651	11.112	11.118	.006
75°	15.562	12.706	—2.856	12.294	12.300	.006
90°	12.706	12.706	0.000	12.706	12.706	.000
	$\nu_1 = 1$; $\nu_2 = 5$			$\nu_1 = 3$; $\nu_2 = 3$		
0°	2.571	2.571	.000	3.182	3.182	.000
15°	4.218	4.121	—.097	3.191	3.182	—.009
30°	6.636	6.732	.096	3.225	3.182	—.043
45°	9.090	9.166	.076	3.244	3.182	—.062
60°	11.043	11.077	.034	3.225	3.182	—.043
75°	12.282	12.292	.010	3.191	3.182	—.009
90°	12.706	12.706	.000	3.182	3.182	.000
	$\nu_1 = 3$; $\nu_2 = 5$			$\nu_1 = 5$; $\nu_2 = 5$		
0°	2.571	.571	.000	2.571	2.571	.000
15°	2.626	2.617	—.009	2.564	2.571	.007
30°	2.756	2.737	—.019	2.562	2.571	.009
45°	2.897	2.893	—.004	2.565	2.571	.006
60°	3.026	3.041	.015	2.562	2.572	.009
75°	3.134	3.145	.011	2.564	2.571	.007
90°	3.182	3..82	.000	2.571	2.571	.000

* Values taken from Table VI, *Statistical Tables* (1957 edition) by Fisher and Yates.

7. Conclusion

Given two samples of n_i $(i = 1, 2)$ units from two normal populations, having variances not necessarily equal, a confidence interval for any linear function of population means in terms of sample estimates of population means and variances and tabulated values of Student's t-tables is possible. If the population variances are unknown the only function of the form $A_1s_1^2+A_2s_2^2$ which with minimum values of A_1 and A_2 would satisfy the relation

$$\left\{ \sum_1^2 c_i(x_i-m_i)\right\}^2 \leq A_1s_1^2+A_2s_2^2$$

with probability not less an α is $\frac{t_1^2c_1^2s_1^2}{n_1}+\frac{t_2^2c_2^2s_2^2}{n_2}$. Further with maximum value of the error of the first kind (probability of rejection of hypothesis when true) fixed at any given value any hypothesis regarding the equality of population means (or any linear function of population means) of the two populations can be tested. Such tests are unbiassed.

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