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STRONG BERTRAND EQUILIBRIA

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Abstract: In this paper we examine the strong Nash equilibria (Aumann(1959)) of a Bertrand-Chamberlin model of price competition where firms supply all demand. We provide a necessary and sufficient condition for the existence of such an equilibrium. We find that whenever a strong Nash equilibrium exists, it is unique, symmetric and quasi-competitive.

Key words: Bertrand-Chamberlin oligopoly, strong Nash equilibrium.

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1. INTRODUCTION

This paper examines the strong Nash equilibria of a Bertrand-Chamberlin model of price competition where firms supply all demand. Under the Bertrand-Chamberlin approach[1] a firm supplies the entire demand that it faces at any given price.[2] Such an assumption is justified by invoking reputational reasons, as well as government regulations. Vives (1999) argues that in regulated industries like electricity or telephone, such governmental regulations are, in fact, often in force. For example, under the "common carrier" regulation firms are required to supply all demand at the given prices. If the supply of the commodity is exhausted, then the consumers can take a "rain-check", a coupon to purchase the good at the posted price at a later date (Spulber (1989)). In certain sealed bid auctions also, the winning firm(s) must supply the entire demand corresponding to their winning bid(s).

Dastidar (1995) has demonstrated the existence of pure strategy Nash equilibria in such a framework. In general, however, multiple equilibria exist. Novshek and Roy Chowdhury (2001) show that even when the number of firms goes to infinity, the set

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[1] This approach was pioneered by Chamberlin (1933). Some of the other authors to adopt this framework include Vives (1990), and Bulow, Geanakoplos and Klemperer (1985).

[2] In fact the existence of such costs is routinely assumed in fields like operations research. (See Dixon (1990), as well as Taha (1987)).

of Nash equilibria is a non-degenerate interval. Both these papers, however, analyze the set of Nash equilibria, i.e. equilibria where only unilateral deviations by firms are allowed. In this paper, we interested in equilibrium solutions that are immune to group deviations. Formally, the solution concept used in this paper is the notion of *strong Nash equilibrium* developed in Aumann (1959). We call our solution concept the *Strong Bertrand Equilibrium*, or SBE for short.

Thus in this paper we solve for the SBE of a Bertrand-Chamberlin model of price competition where firms supply all demand. We provide a necessary and sufficient condition for the existence of an SBE. We show that an SBE, if it exists, must be unique and symmetric. We also show that this price is decreasing in the number of firms. (Okuguchi (1973) calls this property *quasi-competitiveness*.)

2. THE MODEL

We examine a game of Bertrand-Chamberlin price competition where the firms must supply all demand. There are $n \geq 2$ firms producing a homogenous commodity. We assume that the demand function $F(p)$ satisfies the following assumption.

ASSUMPTION 1. (a) $F : [0, \infty) \to [0, \infty)$. *Moreover, $F(p)$ is twice differentiable for all p such that $F(p) > 0$.*

(b) *$F(p)$ is strictly decreasing for all p such that $F(p) > 0$.*

(c) *Moreover, $F(p)$ intersects the price axis.*

All firms are identical with cost functions $c(q)$, where $c(q)$ is increasing and convex.

ASSUMPTION 2. (a) $c : [0, \infty) \to [0, \infty)$. *Moreover, $c(q)$ is twice differentiable.*

(b) *Costs are increasing and strictly convex, i.e. $c'(q) > 0$ and $c''(q) > 0$, $\forall q > 0$. Moreover, $c(0) = 0$.*[3]

We then introduce some notations.

For any integer m, $1 \leq m \leq n$, define

$$\pi^m(p) = \frac{pF(p)}{m} - c\left(\frac{F(p)}{m}\right).$$

Thus $\pi^m(p)$ represents the profit of a firm charging the price p when m firms in the market charge p and share the market equally.

ASSUMPTION 3. *For all p such that $F(p) > 0$, $\pi^m(p)$ is strictly concave in p. Moreover, $\pi^m(p) > 0$ for some p.*

Given Assumptions 2 and 3, there is a unique price p^m which maximizes $\pi^m(p)$. Moreover, $\pi^m(p^m) > 0$. The following observation will be used later.

OBSERVATION 1. *p^m is strictly decreasing in m.*

The proof is in the appendix.

[3] All our results generalize to the case where $c(0) > 0$.

Next let $\mathbf{p} = (p_1, \cdots, p_n)$, denote the price vector when firm i charges a price of p_i. (Throughout we will use boldface letters to denote a price vector.) If the n firms choose the price vector $\mathbf{p}$, then the demand curve facing firm i is

$$D_i(p_1, \cdots, p_i, \cdots, p_n) = \begin{cases} 0, & \text{if } p_i > p_j, \quad \text{for some } j, \\ \dfrac{F(p_i)}{m}, & \text{if } p_i \le p_j, \ \forall j, \quad \text{and} \quad \#(l : p_l = p_i) = m. \end{cases}$$

The corresponding profit of the i-th firm is

$$\pi_i(p_1, \cdots, p_n) = \begin{cases} 0, \quad \text{if } p_i > p_j, \quad \text{for some } j, \\ (p_i - AC(D_i(p_1, \cdots, p_n)))D_i(p_1, \cdots, p_n), \text{if } p_i \le p_j, \forall j, \end{cases}$$

where $AC(q)$ denotes the average cost of producing q.

Given a price vector $\mathbf{p}$ and a coalition $T \subset N$, we let $\mathbf{p_T}$ denote the price vector corresponding to the coalition T.

Finally, given $\mathbf{p'_T}$ and $\mathbf{p_S}$, $T \subset S$, $(\mathbf{p'_T}, \mathbf{p_{S/T}})$ denotes the s-vector where the prices correspond to $\mathbf{p'_T}$ for firms in T, and to $\mathbf{p_S}$ for firms in S/T.

In this paper we restrict attention to pure strategies.

We now provide some definitions, leading upto the definition of a strong Bertrand equilibrium.

DEFINITION 1. For any coalition $T \subseteq N$, the price vector $\mathbf{p'_T}$ constitutes a *profitable deviation* from $\mathbf{p}$ if

$$\pi_i(\mathbf{p'_T}, \mathbf{p_{N/T}}) > \pi_i(\mathbf{p}), \ \forall i \in T. \tag{1}$$

Clearly any profitable deviation must be symmetric.

We then follow the ideas of Aumann (1959) and introduce an equilibrium concept which we call *Strong Bertrand Equilibrium*, or SBE for short.

DEFINITION 2. A vector of prices $\mathbf{p}^*$ is said to be a *Strong Bertrand Equilibrium* (SBE) if no coalition T, $T \subseteq N$, has a profitable deviation from $\mathbf{p}^*$.

Thus SBE formalizes a solution concept where not only individual, but even coalitions of firms have no incentive to deviate. Note that any strong Bertrand equilibrium must be Pareto optimal.

We begin by establishing that any SBE, if it exists, must be symmetric and unique. We also characterize the unique SBE price.

PROPOSITION 1. (i) *In any SBE all firms charge the same price.*

(ii) *An SBE, if it exists, is unique and involves all firms charging the price p^n.*

Proof. (i) Suppose to the contrary there exist i, j such that $p_i > p_j$ and firm j is one of the k firms serving the market. Note that the profit of firm i is zero while that of firm j is given by $x[p_j - (c(x)/x)]$, where $x = F(p_j)/k$. Since in equilibrium, profit must be nonnegative (any firm can charge a large enough price and ensure that it gets no consumers), we must have $p_j \ge c(x)/x$. Now if firm i deviates to p_j, its profit will be $x'[p_j - (c(x')/x')]$, where $x' = F(p_j)/(k+1)$. Since $x > x'$, from the convexity of the cost function we have that $c(x)/x > c(x')/x'$. Hence, $p_j > c(x')/x'$. Thus if firm i charges p_j, its profit will be strictly positive, so that it has a profitable deviation from p_j.

(ii) Note that from proposition 1(i) any SBE must be symmetric. Moreover, recall that any SBE must be Pareto optimal. Thus given that $\pi^n(p)$ is maximized at p^n, any SBE must involve all firms charging p^n. ■

We then introduce a further piece of notation that we require for our next proposition. For every $m \leq n$, let $p(m,n)$ solve the equation $\pi^m(p) = \pi^n(p)$ with $\pi^n(p) > 0$. We can use Lemma 1 below to prove that such a $p(m,n)$ exists and is unique.

The following lemma helps us to identify the existence of profitable group deviations. The proof is in the appendix.

LEMMA 1. *Fix a coalition T, with $|T| = t$ $(< n)$. Then, there exists a price $p(t,n)$ with $0 < p(t,n) < p^t$, such that the following hold.*

(i) *For all $p < p(t,n)$, T has no profitable deviation from $\mathbf{p} = (p, \cdots, p)$.*

(ii) *The coalition T has a profitable deviation from $\mathbf{p}$ whenever $p > p(t,n)$.*

(iii) *If a coalition T has no profitable deviation from $\mathbf{p}$, then no subset of T has a profitable deviation from $\mathbf{p}$.*

Proposition 2 below provides a necessary and sufficient condition for the existence of an SBE.

PROPOSITION 2. *An SBE exists if and only if $p^n \leq p(n-1,n)$.*

Proof. From Proposition 1 recall that the only possible SBE involves all firms charging the price p^n.

Necessity: Suppose that $p(n-1,n) < p^n$.

From Lemma 1(ii), any coalition consisting of $n-1$ firms has a profitable deviation from p^n. Thus p^n cannot be an SBE.

Sufficiency: Suppose that $p^n \leq p(n-1,n)$.

Since p^n is the profit maximizing price for the grand coalition, clearly the grand coalition does not have a profitable deviation. Since $p^n \leq p(n-1,n)$, by Lemma 1(i), no coalition consisting of $n-1$ firms has a profitable deviation. Moreover, from Lemma 1(iii), no other coalition of size less than $n-1$ can have a profitable deviation. ■

The existence result is interesting as a strong Nash equilibrium fails to exist in many games. Peleg (1984), however, demonstrates the existence of strong Nash equilibria in some classes of voting games.

The following example shows that depending on parameter values, the condition for existence may or may not be satisfied.

EXAMPLE 1. Let the demand function be $q = a - p$ and let the cost function be cq^2. We also assume that $n = 2$. It is easy to see that $p(1,2) = \frac{3ac}{2+3c}$ and $p^2 = \frac{a(1+c)}{2+c}$. Thus $p(1,2) < p^2$ if and only if $c < 2$.

We then ask if p^n satisfies *quasi-competitiveness* (see Okuguchi (1973)), i.e. the property that the market price is decreasing in the degree of competitiveness. From Observation 1 we find that it does.

3. CONCLUSION

In this paper we consider a model of Bertrand-Chamberlin price competition where firms supply all demand and solve for the strong Bertrand equilibrium of this game. We provide a necessary and sufficient condition for the existence of such an equilibrium. We find that whenever an SBE exists, it is unique, symmetric and quasi-competitive.

4. APPENDIX

Proof of Observation 1. Note that p^m solves

$$Y(p,m) = \frac{pF'(p) + F(p)}{m} - c'\left(\frac{F(p)}{m}\right)\frac{F'(p)}{m} = 0 .$$

From the concavity of the profit function it follows that $Y_p(p,m) < 0$. We can then use the fact that $Y(p^m, m) = 0$ to show that

$$Y_m(p^m, m) = \frac{F(p^m)F'(p^m)c''(F(p^m)/m)}{m^3} < 0 .$$

Hence, $dp^m/dm = -Y_m(p^m, m)/Y_p(p^m, m) < 0$. ■

Proof of Lemma 1. (i) and (ii). Note that since $n > t$, $\pi^n(0) = -c(\frac{F(p)}{n}) > -c(\frac{F(p)}{t}) = \pi^t(0)$. We then define $q_n = \frac{F(p^t)}{n}$ and let $\bar{p}$ satisfy $F(\bar{p}) = tq_n$. Clearly, $\bar{p} > p^t$. Since $F(\bar{p}) = \frac{tF(p^t)}{n}$ and $\bar{p} > p^t$, straightforward calculations yield that $\pi^t(\bar{p}) > \pi^n(p^t)$. Now p^t maximizes $\pi^t(p)$, and hence $\pi^t(p^t) \geq \pi^t(\bar{p}) > \pi^n(p^t)$. Thus there exists a $p(t,n)$ such that $\pi^t(p(t,n)) = \pi^n(p(t,n))$.

Next define $Y(p) = \pi^n(p) - \pi^t(p)$. Observe that

$$\begin{aligned}\frac{dY(p)}{dp} &= \frac{1}{n}\left[F(p) + pF'(p) - c'\left(\frac{F(p)}{n}\right)F'(p)\right] \\ &\quad - \frac{1}{t}\left[F(p) + pF'(p) - c'\left(\frac{F(p)}{t}\right)F'(p)\right].\end{aligned} \tag{2}$$

Since $c'(\frac{F(p)}{n}) < c'(\frac{F(p)}{t})$, it follows that

$$\frac{dY(p)}{dp} < \left[F(p) + pF'(p) - c'\left(\frac{F(p)}{t}\right)F'(p)\right]\left(\frac{1}{n} - \frac{1}{t}\right). \tag{3}$$

Finally, since $p \leq p^t$, one can use the concavity of the profit function to show that $[F(p) + pF'(p) - c'(\frac{F(p)}{t})F'(p)] > 0$. Hence $\frac{dY(p)}{dp} < 0$, thus proving the first two parts of the Lemma.

(iii) It is sufficient to prove that $p(t,n)$ is decreasing in t. Note that $p(t,n)$ solves the following equation in p

$$p = \frac{c\left(\frac{F(p)}{t}\right) - c\left(\frac{F(p)}{n}\right)}{\frac{F(p)}{t} - \frac{F(p)}{n}}. \tag{4}$$

From Lemma 1(i) and 1(ii), in fact, it follows that $p(t, n)$ is the minimum p satisfying the above equation. Let $Z(p, t, n)$ denote the right hand side of the above equation. Clearly, $Z(0, t, n) > 0$. Next observe that

$$\frac{\partial Z(p, t, n)}{\partial t} = \frac{n^2}{F(p)(n - t)^2}\left[\left\{c\left(\frac{F(p)}{t}\right) - c\left(\frac{F(p)}{n}\right)\right\} - c'\left(\frac{F(p)}{t}\right)\left(\frac{F(p)}{t} - \frac{F(p)}{n}\right)\right]. \tag{5}$$

From the convexity of the cost function it follows that $Z(p, t, n)$ is strictly decreasing in t. Hence $p(t, n)$ is decreasing in t. ■

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