On the Lipschitz Continuity of the Solution Map in Semidefinite Linear Complementarity Problems

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In this paper, we investigate the Lipschitz continuity of the solution map in semidefinite linear complementarity problems. For a monotone linear transformation defined on the space of real symmetric $n \times n$ matrices, we show that the Lipschitz continuity of the solution map implies the globally uniquely solvable (GUS)-property. For Lyapunov transformations with the Q-property, we prove that the Lipschitz continuity of the solution map is equivalent to the strong monotonicity property. For the double-sided multiplicative transformations, we show that the Lipschitz continuity of the solution map implies the GUS-property.

Key words: semidefinite linear complementarity problem (SDLCP); Lipschitz continuity; P-property; Q-property; GUS-property.
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1. Introduction. Let $\mathcal{S}^n$ be the vector space of all real symmetric $n \times n$ matrices and $\mathcal{S}^n_+$ the set of positive semidefinite matrices in $\mathcal{S}^n$. Given a linear transformation L: $\mathcal{S}^n \to \mathcal{S}^n$ and $Q \in \mathcal{S}^n$, the semidefinite linear complementarity problem SDLCP(L, Q) is to find a matrix $X \in \mathcal{S}^n$ such that

$$X \in \mathcal{S}^n_+, \quad Y = L(X) + Q \in \mathcal{S}^n_+, \quad \langle X, Y \rangle = \mathrm{tr}(XY) = 0 \ (\Leftrightarrow XY = 0),$$

where tr denotes the trace. We shall refer to $XY = 0$ as the complementarity condition.

This problem was originally introduced by Kojima et al. [9] in a different form. The SDLCP can be considered as a generalization of the linear complementarity problem (LCP); see Cottle et al. [2]. Motivated by the significance of P-matrices in linear complementarity theory, Gowda and Song [6] introduced the P-property and its variants, namely, the globally uniquely solvable (GUS) property and the P_2-property for the SDLCP. The relationship between these properties was established in Gowda and Song [6] and in Parthasarathy et al. [12].

One of the important problems in the LCP is to characterize the Lipschitz continuity of its solution map $q \to S_M(q)$, where $S_M(q)$ is the set of all solutions to the linear complementarity problem LCP(M, q). Mangasarian and Shiau [10] showed that if M is a P-matrix, then the multivalued map $q \to S_M(q)$ is Lipschitz continuous. Pang conjectured that if S_M is Lipschitz continuous and $S_M(q) \neq \phi$ for all $q \in R^n$, then M must be a P-matrix; see Gowda [4]. This result was established in Murthy et al. [11].

These results motivate us to find the interconnections between the Lipschitz continuity of the solution map ϕ_L of the SDLCP and the P-property and its variants. Because the cone $\mathcal{S}^n_+$ is nonpolyhedral and the matrix multiplication is not commutative in general, the problem becomes more difficult than in the case of the LCP. We have shown that for a monotone linear transformation L, the Lipschitz continuity of ϕ_L implies the P_2-property. By specializing to Lyapunov transformations L_A, we show that if L_A has the Q-property, then ϕ_{L_A} is Lipschitz continuous if and only if L_A has the strong monotonicity property. Using this result, we give an example to illustrate that the GUS-property of L need not imply the Lipschitz continuity of ϕ_L. For the transformation $M_A(X) = AXA^T$, we prove that if ϕ_{M_A} is Lipschitz continuous, then M_A has the GUS-property. If we make the additional assumption that A is symmetric, then for the transformation M_A, we show that ϕ_{M_A} is Lipschitz continuous if and only if M_A has the strong monotonicity property.

1.1. Notation and Preliminaries.

(i) Let $X \in \mathcal{S}^n$. We write $X \succeq 0$ ($\succ 0$) if X is positive semidefinite (definite).

(ii) Given a linear transformation $L: \mathcal{S}^n \to \mathcal{S}^n$, let $\phi_L(Q)$ denote the set of all solutions to SDLCP(L, Q).

(iii) We use I to denote the identity matrix.

(iv) For $A \in \mathcal{S}^n$, let $\|A\|$ denote the Frobenius norm.

We list below some well-known results on matrices; see Zhang [13].

(i) $X \succeq 0 \Rightarrow PXP^T \succeq 0$ for any nonsingular matrix P.

(ii) $X \succeq 0$, $Y \succeq 0$, $\langle X, Y \rangle = 0 \Rightarrow XY = YX = 0$.

(iii) Let $A \in \mathcal{S}^n$. Then, there exists a real invertible matrix Q such that $QAQ^T = \operatorname{diag}[I_k, -I_r, 0]$, where k is the number of positive eigenvalues of A and r is the number of negative eigenvalues of A.

Definition 1.1. For $A \in R^{n\times n}$, we recall the following definitions.

(i) A is positive stable if every eigenvalue of A has a positive real part.

(ii) A is Schur stable if all the eigenvalues of A lie in the open unit disk of the complex plane.

(iii) A is a signature matrix if A is a diagonal matrix and its diagonal entries are either 1 or -1.

Definition 1.2. For a matrix $A \in R^{n\times n}$, we define the corresponding Lyapunov transformation $L_A: \mathcal{S}^n \to \mathcal{S}^n$ by

$$L_A(X) = AX + XA^T.$$

Definition 1.3. For a linear transformation $L: \mathcal{S}^n \to \mathcal{S}^n$, we say that

(i) L has the Q-property if SDLCP(L, Q) has a solution for all $Q \in \mathcal{S}^n$.

(ii) L has the P-property if X and $L(X)$ commute, $XL(X) \preceq 0 \Rightarrow X = 0$.

(iii) L has the GUS-property, if for all $Q \in \mathcal{S}^n$, SDLCP(L, Q) has a unique solution.

(iv) L has the monotonicity property if $\langle L(X), X \rangle \geq 0$ for any $X \in \mathcal{S}^n$.

(v) L has the strong monotonicity property if $\langle L(X), X \rangle > 0$ for any nonzero $X \in \mathcal{S}^n$.

(vi) L has the R_0-property if SDLCP($L, 0$) has a unique solution.

(vii) L has the P_2-property if $X \succeq 0$, $Y \succeq 0$, $(X - Y)L(X - Y)(X + Y) \preceq 0 \Rightarrow X = Y$.

Definition 1.4. Let $L: \mathcal{S}^n \to \mathcal{S}^n$ be a linear transformation. Corresponding to any $\alpha \subseteq \{1, \ldots, n\}$, we define a linear transformation $L_{\alpha\alpha}: \mathcal{S}^{|\alpha|} \to \mathcal{S}^{|\alpha|}$ by

$$L_{\alpha\alpha}(Z) = [L(X)]_{\alpha\alpha} \quad (Z \in \mathcal{S}^{|\alpha|}),$$

where, corresponding to any $Z \in \mathcal{S}^{|\alpha|}$, $X \in \mathcal{S}^n$ is the unique matrix such that $X_{\alpha\alpha} = Z$ and $x_{ij} = 0$ for all $(i, j) \notin \alpha \times \alpha$. We call $L_{\alpha\alpha}$ the principal subtransformation of L corresponding to α.

Definition 1.5. Let $L: \mathcal{S}^n \to \mathcal{S}^n$ be a linear transformation and $\phi_L(Q)$ be the set of all solutions to SDLCP(L, Q). Then, the multivalued map $\phi_L: \mathcal{S}^n \to \mathcal{S}^n_+$ is Lipschitzian, if there exists $C > 0$ such that

$$\phi_L(Q) \subseteq \phi_L(Q') + C\|Q - Q'\|\mathbf{B}$$

for all $Q, Q' \in \mathcal{S}^n$ satisfying $\phi_L(Q) \neq \phi$ and $\phi_L(Q') \neq \phi$. Here, $\|\cdot\|$ denotes the Frobenius norm in $\mathcal{S}^n$ and $\mathbf{B}$ denotes the closed unit ball in $\mathcal{S}^n$.

THEOREM 1.1 (KARAMARDIAN [8]). *If* SDLCP$(L, 0)$ *and* SDLCP(L, I) *have unique solutions, then* L *has the* Q*-property.*

THEOREM 1.2 (GOWDA AND SONG [6]). *For a matrix* $A \in R^{n\times n}$, *consider the Lyapunov transformation* L_A. *Then, the following statements are equivalent*:

(i) A *is positive stable and positive semidefinite.*

(ii) L_A *has the GUS-property.*

THEOREM 1.3 (GOWDA AND SONG [6]). *If a linear transformation* $L: \mathcal{S}^n \to \mathcal{S}^n$ *has the* P_2*-property, then* L *has the GUS-property.*

2. Main results. We first give a sufficient condition for the Lipschitz continuity of ϕ_L. The result is known in a more general setting than the SDLCP. For a proof we refer to Proposition 2.3.11 in Facchinei and Pang [3]. Specializing this result to the SDLCP we get the following theorem.

THEOREM 2.1. *Let* $L: \mathcal{S}^n \to \mathcal{S}^n$ *be a linear transformation. If* L *has the strong monotonicity property, then* ϕ_L *is Lipschitzian.*

We now give an example to show that the monotonicity property is not sufficient to conclude that ϕ_L is Lipschitzian.

EXAMPLE 2.1. Let $L: \mathcal{S}^2 \to \mathcal{S}^2$ be defined as

$$L\left(\begin{bmatrix} x & y \\ y & z \end{bmatrix}\right) := \begin{bmatrix} 2x & y \\ y & 0 \end{bmatrix}.$$

Then, L has the monotonicity property. It is simple to verify that $\phi_L(I) = \{0\}$ and

$$\left\{\begin{bmatrix} 0 & 0 \\ 0 & n \end{bmatrix} : n = 1, 2, \ldots\right\} \subseteq \phi_L(0).$$

If ϕ_L is Lipschitzian, then $\phi_L(0)$ will be a compact set, which is clearly a contradiction.

The next example shows that ϕ_{-L} is Lipschitzian, but ϕ_L is not Lipschitzian.

EXAMPLE 2.2. Let $L(X) := -X$. We claim that ϕ_L is not Lipschitzian. Assume the contrary. Let $C > 0$ be the Lipschitzian constant. Without any loss of generality, assume that $C > 1/\sqrt{2}$. Let

$$Q_C := \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \quad \text{and} \quad Q'_C := \begin{bmatrix} C & 1/2 \\ 1/2 & C \end{bmatrix}.$$

It is not hard to show that

$$\phi_L(Q'_C) = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} C & 1/2 \\ 1/2 & C \end{bmatrix}, \begin{bmatrix} \frac{C-1/2}{2} & \frac{1/2-C}{2} \\ \frac{1/2-C}{2} & \frac{C-1/2}{2} \end{bmatrix}, \begin{bmatrix} \frac{C+1/2}{2} & \frac{C+1/2}{2} \\ \frac{C+1/2}{2} & \frac{C+1/2}{2} \end{bmatrix}\right\}.$$

Choose

$$X_C = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \in \phi_L(Q_C).$$

Then, $\|X_C - X'_C\| > C\|Q_C - Q'_C\|$ for every $X'_C \in \phi_L(Q'_C)$, which contradicts that ϕ_L is Lipschitzian.

We now show that if L has the monotonicity property and ϕ_L is Lipschitzian, then SDLCP(L, Q) has a unique solution for all $Q \in \mathcal{S}^n$. The proof of this result is based on the following lemmas.

LEMMA 2.1. *Suppose that ϕ_L is Lipschitzian and $\phi_L(I) = \{0\}$. Then, $\phi_L(Q) = \{0\}$ for every $Q \succeq 0$.*

PROOF. Let $\mathcal{S}^n_{++}$ be the set of all positive definite matrices in $\mathcal{S}^n$. Let $\Omega = \{Q \in \mathcal{S}^n_{++} : \phi_L(Q) = \{0\}\}$. Then, Ω is nonempty by our assumption. We claim that Ω is an open set in $\mathcal{S}^n_{++}$. Fix $P \in \Omega$. Because ϕ_L is Lipschitzian, for all $U \in \mathbf{B}$, we have $\phi_L(kP + U) \subseteq \phi_L(kP) + C\|U\|\mathbf{B}$ for all $k = 1, 2, 3, \ldots$. From $\phi_L(P) = \{0\}$, we see that $\phi_L(kP) = \{0\}$. Therefore, the sets $\phi_L(kP + U)$ are uniformly bounded. For any $X \in \phi_L(kP + U)$, we have

$$X \succeq 0, \quad Y := L(X) + kP + U \succeq 0, \quad YX = 0.$$

It follows that for all large k, $Y \succ 0$. Hence, $\phi_L(kP + U) = \{0\}$ for all large k. Because $kP + U \succ 0$, for all large k, $kP + U \in \Omega$. Hence, $P + U/k \in \Omega$ for large k. Thus, P is an interior point and Ω is an open set in $\mathcal{S}^n_{++}$. Now we claim that Ω is closed in $\mathcal{S}^n_{++}$. Let $P \in \mathcal{S}^n_{++}$ be in the closure of Ω. Because every neighbourhood of P contains a point U of Ω, and for any such point $\phi_L(U) = \{0\}$, it follows from the Lipschitzian property of ϕ_L that $\phi_L(P)$ must be contained in an arbitrary small ball around zero vector. Hence, $\phi_L(P) = \{0\}$, so that $P \in \Omega$. Because $\mathcal{S}^n_{++}$ is connected, $\Omega = \mathcal{S}^n_{++}$. Let $Q' \succeq 0$. Then, $Q' + \epsilon I \succ 0$ for all $\epsilon > 0$. Because ϕ_L is Lipschitzian, $\phi_L(Q') \subseteq \phi_L(Q' + \epsilon I) + C\sqrt{n}\epsilon\mathbf{B}$. Hence, if $X \in \phi_L(Q')$, then $\|X\| \leq C\sqrt{n}\epsilon$ for all $\epsilon > 0$. Thus, $X = 0$. □

The proof of the next lemma is straightforward and thus omitted.

LEMMA 2.2. *Let L: $\mathcal{S}^n \to \mathcal{S}^n$ be linear. For an invertible matrix $Q \in R^{n\times n}$, define $\hat{L}(X) := QL(Q^TXQ)Q^T$. Then, ϕ_L is Lipschitzian if and only if $\phi_{\hat{L}}$ is Lipschitzian.*

LEMMA 2.3. *Suppose that ϕ_L is Lipschitzian, and $\phi_L(I) = \{0\}$. Let $L_{\alpha\alpha}$ be a principal subtransformation of L. Then, $\phi_{L_{\alpha\alpha}}$ is Lipschitzian.*

PROOF. Because $\phi_L(I) = \{0\}$, by Lemma 2.1, $\phi_L(0) = \{0\}$. Thus, L has the Q-property by Theorem 1.1 and hence $\phi_L(Q) \neq \phi$ for all $Q \in \mathcal{S}^n$. Without any loss of generality, assume that $\alpha = \{1, \ldots, k\}$, where $k \leq n$. Let $P, Q \in \mathcal{S}^{|\alpha|}$ and $Z \in \phi_{L_{\alpha\alpha}}(P)$. Put

$$Z' = \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$L(Z') = \begin{bmatrix} M & N \\ N^T & R \end{bmatrix}.$$

From the block form of Z', it follows that $L_{\alpha\alpha}(Z) = M$. Define

$$P'_m = \begin{bmatrix} P & -N \\ -N^T & mI \end{bmatrix} \quad \forall m = 1, 2, 3, \ldots.$$

We now claim that $Z' \in \phi_L(P'_m)$ for all large m. Because $Z \succeq 0$, it follows that $Z' \succeq 0$. Now

$$L(Z') + P'_m = \begin{bmatrix} M + P & 0 \\ 0 & R + mI \end{bmatrix}.$$

Because $M + P = L_{\alpha\alpha}(Z) + P$, we see that $M + P \succeq 0$. Now choose large m such that $R + mI \succ 0$. Thus, $L(Z') + P'_m \succeq 0$ for large m. The verification of the complementarity condition is straightforward and thus $Z' \in \phi_L(P'_m)$ for large m. Now for $Q \in \mathcal{S}^{|\alpha|}$, define

$$Q'_m = \begin{bmatrix} Q & -N \\ -N^T & mI \end{bmatrix} \quad \forall m = 1, 2, 3, \ldots.$$

Let $X_m \in \phi_L(Q'_m)$. We claim that the sequence $\{X_m\}$ is bounded. Put

$$Q''_m = \begin{bmatrix} 0 & 0 \\ 0 & mI \end{bmatrix} \quad \forall\, m = 1, 2, 3, \ldots.$$

Because $Q''_m \succeq 0$, by Lemma 2.1, $\phi_L(Q''_m) = \{0\}$. By the Lipschitz continuity of ϕ_L, we have $X_m \in C\|Q'_m - Q''_m\|\mathbf{B}$. Thus, $\{X_m\}$ is bounded. Now there exists a subsequence of $\{X_m\}$ which converges. Without any loss of generality, assume that $\{X_m\}$ itself converges. Partition X_m and $L(X_m)$ conformally as before:

$$X_m = \begin{bmatrix} Y_m & Z_m \\ Z_m^T & W_m \end{bmatrix} \quad \text{and} \quad L(X_m) = \begin{bmatrix} A_m & B_m \\ B_m^T & C_m \end{bmatrix}.$$

Because $X_m(L(X_m) + Q'_m) = 0$, we have $Z_m^T(B_m - N) + mW_m + W_m C_m = 0$. Taking limits, we see that $W_m \to 0$. Because $X_m \succeq 0$, Z_m converges to 0. Now let $Y_m \to Y^*$. We claim that $Y^* \in \phi_{L_{\alpha\alpha}}(Q)$. Clearly, $Y^* \succeq 0$. Because L is linear, we see that A_m converges to $L_{\alpha\alpha}(Y^*)$. Now applying the limits to $A_m + Q \succeq 0$, we get $L_{\alpha\alpha}(Y^*) + Q \succeq 0$. The verification of the complementarity condition is direct, because $\{Z_m\}$ and $\{W_m\}$ converge to zero. Because ϕ_L is Lipschitzian, for all large m, we have

$$\begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} Y_m & Z_m \\ Z_m^T & W_m \end{bmatrix} + C\|P'_m - Q'_m\|\mathbf{B}.$$

Taking limits, we see that $\|Z - Y^*\| \le C\|P - Q\|$ and hence $\phi_{L_{\alpha\alpha}}$ is Lipschitzian. □

Lemma 2.4. *If ϕ_L is Lipschitzian and L has the Q-property, then L is* 1-1.

Proof. Let L^T denote the adjoint of L. If L is not 1-1, then there exists a nonzero $X \in \mathcal{S}^n$ such that $L^T(X) = 0$. Put $Q = -L(I)$. Clearly, $I \in \phi_L(Q)$. Now we can find a sequence $\{Q_m\}$ such that $Q_m \to Q$, and $\langle Q_m, X\rangle \neq 0$ for all m. Let $X_m \in \phi_L(Q_m)$. We claim that each X_m is singular. Suppose that X_k is nonsingular for some k. Then, the complementarity condition $X_k(L(X_k) + Q_k) = 0$ implies that $L(X_k) + Q_k = 0$. Taking the inner product with X, we see that $\langle X, L(X_k) + Q_k\rangle = 0$. Because $L^T(X) = 0$, we get $\langle X, Q_k\rangle = 0$. This contradicts our assumption on $\{Q_m\}$. Now by the Lipschitz continuity of ϕ_L, we have $I \in X_m + C\|Q - Q_m\|\mathbf{B}$. Because Q_m converges to Q, we see that $X_m \to I$. This is clearly a contradiction, because each X_m is singular. Hence, L must be 1-1. □

We now prove our main result.

Theorem 2.2. *Let $L\colon \mathcal{S}^n \to \mathcal{S}^n$ be a linear transformation. Suppose that L has the monotonicity property and ϕ_L is Lipschitzian. Then, L has the P_2-property. In particular, L has the GUS-property.*

Proof. Assume that there exists $X \succeq 0$, $Y \succeq 0$ satisfying $(X - Y)L(X - Y)(X + Y) = 0$. We claim that $X = Y$, and this will imply that L has the P_2-property; see Parthasarathy et al. [12, Proposition 1]. If possible, let $X \neq Y$. Because $X + Y \succeq 0$ and nonzero, there exists a real invertible matrix Q such that

$$X + Y = Q\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^T,$$

where I_r is the identity matrix of size $r \times r$ and $1 \le r \le n$. Let $A := Q^{-1}X(Q^{-1})^T$ and $B := Q^{-1}Y(Q^{-1})^T$. Then, A and B are symmetric positive semidefinite matrices with

$$A + B = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that

$$A = \begin{bmatrix} A_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_r & 0 \\ 0 & 0 \end{bmatrix},$$

where A_r and B_r are $r \times r$ matrices. Now $(X-Y)[L(X-Y)](X+Y) = 0$ gives

$$Q^{-1}(X-Y)(Q^{-1})^T Q^T [L(QAQ^T - QBQ^T)] QQ^{-1}(X+Y)(Q^{-1})^T = 0.$$

This gives $(A-B)[\hat{L}(A-B)](A+B) = 0$, where $\hat{L}(Z) := Q^T L(QZQ^T)Q$. Writing

$$\hat{L}(A-B) = \begin{bmatrix} P & N \\ N^T & R \end{bmatrix},$$

we get

$$\begin{bmatrix} A_r - B_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P & N \\ N^T & R \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (A_r - B_r)P & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

From the block form of $A - B$, we note that $P = \hat{L}_{\alpha\alpha}(A_r - B_r)$, where $\alpha = \{1, \ldots, r\}$. Thus,

$$(A_r - B_r)(\hat{L}_{\alpha\alpha})(A_r - B_r) = 0.$$

By Lemma 2.2, $\phi_{\hat{L}}$ is Lipschitzian. From the monotonicity property of L, it follows that $\hat{L}$ has the monotonicity property. Hence, $\phi_{\hat{L}}(I) = \{0\}$. Thus, $\phi_{\hat{L}_{\alpha\alpha}}$ is Lipschitzian, by Lemma 2.3. Now the monotonicity property of $\hat{L}$ implies the monotonicity property of $\hat{L}_{\alpha\alpha}$. Hence, $\phi_{\hat{L}_{\alpha\alpha}}(I_{\alpha\alpha}) = \{0\}$, where $I_{\alpha\alpha}$ denotes the identity matrix in $\mathcal{S}^{|\alpha|}$. Thus, from Lemma 2.1, $\phi_{\hat{L}_{\alpha\alpha}}(0) = \{0\}$. From Theorem 1.1, we see that $\hat{L}_{\alpha\alpha}$ has the Q-property. Hence, $\hat{L}_{\alpha\alpha}$ is 1-1, by Lemma 2.4. If $A_r - B_r$ is invertible, then $A_r = B_r$, proving $A = B$ and $X = Y$. This is a contradiction. Suppose that $A_r - B_r$ is not invertible. Let

$$A_r - B_r = U \begin{bmatrix} D_s & 0 \\ 0 & 0 \end{bmatrix} U^T.$$

Now define $T(X) = U\hat{L}_{\alpha\alpha}(U^T XU)U^T$. Then, $D_s T_{\beta\beta}(D_s) = 0$, where $T_{\beta\beta}$ is the principal subtransformation of T corresponding to $\beta = \{1, \ldots, s\}$. By repeating the same argument as before, we see that $T_{\beta\beta}$ has the Q-property and $\phi_{T_{\beta\beta}}$ is Lipschitzian. Hence, by Lemma 2.4, $T_{\beta\beta}$ is 1-1. This shows that $D_s = 0$, proving $A_r = B_r$ and thus $X = Y$. This ends the proof. □

3. Some special linear transformations. In this section, we specialize to some specific linear transformations which are prominently studied in the SDLCP literature; see Gowda and Song [6], Gowda and Parthasarathy [5], and Gowda et al. [7].

3.1. Stein transformations. For a matrix $A \in R^{n\times n}$, we define the corresponding Stein transformation $S_A\colon \mathcal{S}^n \to \mathcal{S}^n$ by

$$S_A(X) = X - AXA^T.$$

We first prove the following theorem which is needed in the sequel.

Theorem 3.1. *Suppose that $L\colon \mathcal{S}^n \to \mathcal{S}^n$ is linear. Let $i \in \{1, \ldots, n\}$. If ϕ_L is Lipschitzian and $\phi_L(I) = \{0\}$, then the (i, i)-entry of $L(E_{ii})$ is positive, where E_{ii} is the matrix in $\mathcal{S}^n$ with one in the (i, i)-entry and zeros elsewhere.*

Proof. Let $\alpha = \{i\}$. Now consider the principal subtransformation $L_{\alpha\alpha}$ of L. If the (i,i)-entry of $L(E_{ii})$ is β (say), then it is easy to verify that $L_{\alpha\alpha}(x) = \beta x$. By Lemma 2.1, $\phi_L(Q) = \{0\}$ for all $Q \succeq 0$. Now using the same technique as in Gowda and Song [6, proof of Theorem 8], we see that β is nonnegative. Now by Lemma 2.3, $\phi_{L_{\alpha\alpha}}$ is Lipschitzian and thus β cannot be zero. Hence, $\beta > 0$. □

The proof of the next lemma is similar to Lemma 1 in Gowda and Song [6].

Lemma 3.1. *Let $A \in R^{n\times n}$. Then, A is positive definite if and only if every diagonal entry of UAU^T is positive, for any orthogonal matrix U.*

We now derive the following result for Stein transformations.

Theorem 3.2. *Let $A \in R^{n\times n}$. Then, for the Stein transformation S_A, consider the following statements*:

(i) *ϕ_{S_A} is Lipschitzian and S_A has the Q-property.*
(ii) *$I - A$ is positive definite.*

Then, (i) $\Rightarrow$ (ii).

Proof. Assume (i). Because S_A has the Q-property, by Gowda and Parthasarathy [5, Theorem 11], S_A has the P-property. It is now easy to verify that $\phi_{S_A}(I) = \{0\}$. By Theorem 3.1, the (i,i)-entry of $S_A(E_{ii})$ is positive for each index $i \in \{1, \dots, n\}$. It follows that if α is a diagonal entry of A, then $1 - \alpha^2 > 0$. Thus, every diagonal entry of $I - A$ is positive. Now by Lemma 2.2, $\phi_{S_{UAU^T}}$ is Lipschitzian for any U orthogonal. Because S_A has the Q-property, S_{UAU^T} has the Q-property for all orthogonal U. Thus, by repeating the same argument as before, we see that for any orthogonal U, the diagonal of $U(I-A)U^T$ is positive. Hence, by Lemma 3.1, $I - A$ is positive definite. This gives (ii). □

The following example shows that the P-property need not imply that ϕ_L is Lipschitz continuous.

Example 3.1. For

$$A = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix},$$

consider the corresponding Stein transformation S_A. Because $I - A$ is not positive definite, from Theorem 3.2, it follows that ϕ_{S_A} is not Lipschitzian. But A is Schur stable. Hence, S_A has the P-property, by Gowda and Parthasarathy [5, Theorem 11].

3.2. Lyapunov transformations. For $A \in R^{n\times n}$, we consider the corresponding Lyapunov transformation L_A. We now show that if ϕ_{L_A} is Lipschitzian and L_A has the Q-property, then L_A has the strong monotonicity property.

Theorem 3.3. *The following statements are equivalent for a Lyapunov transformation L_A*:

(i) *A is positive definite.*
(ii) *L_A has the strong monotonicity property.*
(iii) *L_A has the P_2-property.*
(iv) *ϕ_{L_A} is Lipschitzian and L_A has the Q-property.*

Proof. The equivalence of (i), (ii), and (iii) follows from Parthasarathy et al. [12, Theorem 5]. The implication (ii) $\Rightarrow$ (iv) follows from Theorem 2.1 and Theorem 1.1. We now show that (iv) $\Rightarrow$ (i). For L_A, the Q-property implies the P-property in (Gowda and Song [6, Theorem 5]). Hence, $\phi_{L_A}(I) = \{0\}$. Now by Theorem 3.1, the (i,i)-entry of $L_A(E_{ii})$ is positive for each index $i \in \{1, \dots, n\}$. If γ is the (i,i)-entry of A, then it is easy to see that the (i,i)-entry of $L_A(E_{ii})$ is 2γ. Thus, $\gamma > 0$. Hence, the diagonal of A is positive. Because ϕ_{L_A} is Lipschitzian, by Lemma 2.2, $\phi_{L_{UAU^T}}$ is Lipschitzian for any orthogonal matrix U. Now the Q-property of L_A implies the Q-property of L_{UAU^T} for all orthogonal U. By repeating the same argument as before, we see that the diagonal of UAU^T is positive for all orthogonal U. Hence, by Lemma 3.1, A is positive definite. □

The following example shows that ϕ_L is not Lipschitzian, but L has the GUS-property.

EXAMPLE 3.2. Let

$$A = \begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix}.$$

Then, A is positive stable and positive semidefinite. Hence, by Theorem 1.2, L_A has the GUS-property. Because A is not a positive definite matrix, ϕ_{L_A} is not Lipschitzian by the above theorem.

3.3. Multiplicative transformations. For $A \in R^{n\times n}$, we now consider the transformation $M_A\colon \mathcal{S}^n \to \mathcal{S}^n$ defined by $M_A(X) := AXA^T$. We first prove the following lemma.

LEMMA 3.2. *Let $A \in R^{n\times n}$. Then, the following are equivalent*:

(i) *A is positive definite or negative definite.*

(ii) *If U is an orthogonal matrix, then every diagonal entry of UAU^T is different from zero.*

PROOF. The implication (i) $\Rightarrow$ (ii) is obvious. We now show that (ii) $\Rightarrow$ (i). Suppose that A is neither positive definite nor negative definite. Then, there exists a matrix $X \in \mathcal{S}^n$ of rank 1 such that $XAX = 0$. Let $UXU^T = D$, where D is diagonal. Put $C = UAU^T$. Then, we see that $DCD = 0$. Because D is of rank 1, $D = \text{diag}[0, ..d_i, .., 0]$. From the equation $DCD = 0$, we see that the (i, i)-entry of C is zero which contradicts (ii). □

THEOREM 3.4. *For the transformation M_A, consider the following statements*:

(i) *ϕ_{M_A} is Lipschitzian.*

(ii) *A is positive definite or negative definite.*

(iii) *M_A has the P_2-property.*

(iv) *M_A has the GUS-property.*

Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

PROOF. (i) $\Rightarrow$ (ii): Let $X \in \phi_{M_A}(I)$. Then, $Y = M_A(X) + I \succ 0$. Hence, the complementarity condition $XY = 0$ implies that $X = 0$. Thus, by Theorem 3.1, the (i, i)-entry of $M_A(E_{ii})$ is positive for all i. If a_i is the (i, i)-entry of A, then it is easy to see that the (i, i)-entry of $M_A(E_{ii})$ is a_i^2. Hence, we can conclude that every diagonal entry of A is different from zero. If U is an orthogonal matrix, then $\phi_{M_{UAU^T}}$ is Lipschitzian (by Lemma 2.2). Thus, item (ii) in Lemma 3.2 holds. Hence, A is either positive definite or negative definite. Now (ii) $\Rightarrow$ (iii) follows from Gowda et al. [7, Corollary 6] and (iii) $\Rightarrow$ (iv) follows from Theorem 1.3. □

When A is symmetric, we prove a stronger result for M_A.

THEOREM 3.5. *If A is symmetric, the following are equivalent for the transformation $M_A(X) := AXA$.*

(i) *M_A has the Q-property.*

(ii) *M_A has the strong monotonicity property.*

(iii) *ϕ_{M_A} is Lipschitzian.*

(iv) *M_A has the P-property.*

PROOF. The implication (ii) $\Rightarrow$ (iii) follows from Theorem 2.1. We now show that (iii) $\Rightarrow$ (i). It is simple to verify that $\phi_{M_A}(I) = \{0\}$ and hence $\phi_{M_A}(0) = \{0\}$ by Lemma 2.1. Now using Theorem 1, we see that M_A has the Q-property. Thus, we get (iii) $\Rightarrow$ (i).

We now show that (i) $\Rightarrow$ (ii). Let S be a signature matrix of order n. We claim that if M_S has the Q-property, then $S = \pm I$. If $S \neq \pm I$, without any loss of generality, assume that

$$S = \begin{bmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{bmatrix},$$

where I_k denotes the identity matrix in $\mathcal{S}^k$, $k < n$. Define $Q \in \mathcal{S}^n$ as follows:

$$Q(i,j) = \begin{cases} -1 & \text{if } (i,j) = (k,k+1) \text{ or } (i,j) = (k+1,k), \\ 0 & \text{otherwise.} \end{cases}$$

If possible, let $X \in \phi_{M_S}(Q)$. Now expanding $X(SXS+Q)$, and summing its $(k,k+1)$ and $(k+1,k)$-entries, we see that $-x_{kk} - x_{k+1k+1} = 0$. Because $X \succeq 0$, we get $x_{kk} = x_{k+1k+1} = 0$. Hence, its principal submatrix

$$\begin{bmatrix} x_{kk} & x_{kk+1} \\ x_{k+1k} & x_{k+1k+1} \end{bmatrix} = 0.$$

Because $SXS + Q \succeq 0$, its principal submatrix

$$\begin{bmatrix} x_{kk} & x_{kk+1} - 1 \\ x_{k+1k} - 1 & x_{k+1k+1} \end{bmatrix} \succeq 0.$$

This is clearly a contradiction and it follows that M_S cannot have the Q-property. Hence, $S = I$ or $S = -I$. Now let $X \in \phi_{M_A}(-I)$. Then, $AXA \succeq I$. This shows that A cannot be singular. Hence, there exists a real invertible matrix B such that $BAB^T = S$, where S is a signature matrix. Now the Q-property of M_A implies the Q-property of M_{BAB^T}. Hence, $S = \pm I$. This implies that A is either positive definite or negative definite. Now using Parthasarathy et al. [12, Theorem 6], we see that M_A has the strong monotonicity property. Thus, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

It is obvious to see that the strong monotonicity property implies the P-property. Hence, we get the implication (ii) $\Rightarrow$ (iv). Finally, (iv) $\Rightarrow$ (i) follows from Gowda and Song [6, Theorem 4]. □

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