

On \mathbf{Q} and \mathbf{R}_0 properties of a quadratic representation in linear complementarity problems over the second-order cone

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Received 19 May 2004; accepted 28 September 2004

Submitted by H. Schneider

Abstract

This paper studies the linear complementarity problem $\text{LCP}(M, q)$ over the second-order (Lorentz or ice-cream) cone denoted by A_+^n , where M is a $n \times n$ real square matrix and $q \in \mathbb{R}^n$. This problem is denoted as $\text{SOLCP}(M, q)$. The study of second-order cone programming problems and hence an independent study of SOLCP is motivated by a number of applications. Though the second-order cone is a special case of the cone of squares (symmetric cone) in a Euclidean Jordan algebra, the geometry of its faces is much simpler and hence an independent study of LCP over A_+^n may yield interesting results. In this paper we characterize the \mathbf{R}_0 -property ($x \in A_+^n$, $M(x) \in A_+^n$ and $\langle x, M(x) \rangle = 0 \Rightarrow x = 0$) of a quadratic representation $P_a(x) := 2a \circ (a \circ x) - a^2 \circ x$ of A^n for $a, x \in A^n$ where ' \circ ' is a Jordan product and show that the \mathbf{R}_0 -property of P_a is equivalent to stating that $\text{SOLCP}(P_a, q)$ has a solution for all $q \in A^n$.

AMS classification: 90C33; 15A48

Keywords: Second-order cone; Face; \mathbf{R}_0 -property; \mathbf{Q} -property

1. Introduction

In a n -dimensional real vector space R^n with the usual inner product, the *second-order cone* denoted by A_+^n is defined as $A_+^n := \{x \in R^n : \|\bar{x}\| \leq x_0\}$ where $x = (x_0, \bar{x})^T$ is indexed from zero. Given a $n \times n$ real square matrix M and a vector q in R^n the *second-order linear complementarity problem* (SOLCP(M, q)) is to find a vector x in A_+^n such that $Mx + q$ is in A_+^n and $\langle x, Mx + q \rangle = x^T(Mx + q) = 0$. Note that the second-order cone is a closed convex cone which is self-dual in nature, that is, $A_+^n = \{x \in R^n : \langle x, y \rangle \geq 0 \forall y \in A_+^n\}$. It is well known that the second-order cone is the cone of squares (see Section 2) of its associated Euclidean Jordan algebra, see [4]. In this regard the complementarity problem SOLCP is a special case of the more general linear complementarity problem studied in the setting of a Euclidean Jordan algebra, see [6,9]. However, the important feature which makes the SOLCP interesting and draws a special attention is the nature of the faces of the second-order cone. Unlike the cone of symmetric positive semidefinite matrices, which is also studied in the setting of a Euclidean Jordan algebra, the only nontrivial faces of A_+^n are its extreme rays and its only nonpolyhedral face is the cone A_+^n itself, see [5].

Second-order cone programming and complementarity problems have been subjects of some recent studies. Pang et al. have studied the stability of solutions to semidefinite and second-order cone complementarity problems in [11]. Study of smoothing functions for second-order cone complementarity problems to develop noninterior continuation methods has been made by Fukushima, Luo, and Tseng in [5]. One can see the same paper and the references therein for various applications of second-order cone complementarity problems. For a comprehensive exposition to various applications and algorithmic aspects of second-order cone programming problems, the reader is advised to refer to Alizadeh and Golfarb, [1].

Loewy and Schneider [8] have studied the closed convex cone of matrices which leave A_+^n invariant, denoted by $\Pi(A_+^n)$, and characterized the extreme rays of $\Pi(A_+^n)$. Our focus in this article is on the SOLCP(M, q) where the matrix $M \in \Pi(A_+^n)$. Our study is motivated by a result proved by Murty [10] in the context of a LCP over R_+^n with a nonnegative square matrix. It states that LCP(M, q) is solvable for all $q \in R^n$ if and only if the diagonal entries of M are positive, where M is a $n \times n$ nonnegative matrix, that is, $M(R_+^n) \subseteq R_+^n$. Though we do not have a complete generalization of Murty's result to a second-order cone, in this article we shall show that for a *quadratic representation* P_a of A^n for $a \in A^n$, defined as $P_a(x) := 2a \circ (a \circ x) - a^2 \circ x$, see [4], SOLCP(P_a, q) has a solution for all $q \in A^n$ if and only if a or $-a$ lies in the interior of A_+^n . An important feature that is useful in showing the above equivalence is the property of the faces of the second-order cone. Note that the quadratic representation $P_a \in \Pi(A_+^n)$ for all $a \in R^n$ and $P_a(A_+^n) = A_+^n$ when a is invertible. The quadratic representation plays a fundamental role in the study of Euclidean Jordan algebras. On the space of real symmetric matrices the quadratic representation is seen to be the map $X \rightarrow AXA$ where A is a real symmetric matrix. The solvability of semidefinite linear complementarity problem SDLCP(L, Q) with

$L(X) = AXA$, where A is real symmetric, has been characterized in terms of A being positive or negative definite in [12,13].

We shall begin with a brief survey of some Jordan algebraic properties of the second-order cone in Section 2. In Section 3 we present our main results and discuss some open problems.

2. Second-order cone and its Jordan algebra

A Euclidean Jordan algebra V is a finite dimensional vector space over the field of real numbers R equipped with an inner product $\langle x, y \rangle$ and a bilinear map $(x, y) \rightarrow x \circ y$, which satisfies the following conditions:

- (i) $x \circ y = y \circ x$,
- (ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ where $x^2 = x \circ x$, and
- (iii) $\langle x \circ y, z \rangle = \langle y, x \circ z \rangle$,

for all $x, y, z \in V$. The product $x \circ y$ is called a Jordan product. A Euclidean Jordan algebra V has an identity element, if there exists a (unique) element $e \in V$ such that $x \circ e = e \circ x = x$ for all $x \in V$. The cone of squares in V is defined as $K := \{x \circ x : x \in V\}$, which is a symmetric cone [4], that is, K is a self-dual closed convex cone such that for any two elements $x, y \in \text{int } K$, there exists an invertible linear transformation $\Theta : V \rightarrow V$ such that $\Theta(K) = K$ and $\Theta(x) = y$.

For $x \in V$ let d be the smallest positive integer such that the set $\{e, x, x^2, \dots, x^d\}$ is linearly dependent. Then d is called the *degree* of x . The *rank* of V is defined as the largest degree of any $x \in V$. An element $c \in V$ is an *idempotent* if $c^2 = c$. It is *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. A finite set $\{f_1, f_2, \dots, f_n\}$ of primitive idempotents in V is a *Jordan frame* if

$$f_i \circ f_j = 0, \quad i \neq j \quad \text{and} \quad \sum_{i=1}^n f_i = e.$$

Theorem 1 (Spectral theorem). *Let V be a Euclidean Jordan algebra with rank n . Then for every $x \in V$, there exists a Jordan frame $\{f_1, f_2, \dots, f_n\}$ and real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ such that*

$$x = \lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_n f_n.$$

The numbers λ_i (with their multiplicities) are uniquely determined by x and are called the eigenvalues of x .

With the above decomposition we shall define *determinant* of $x \in A^n$ as follows:

$$\det(x) := \lambda_1 \lambda_2 \cdots \lambda_n.$$

x is said to be invertible if $\det(x) \neq 0$, in which case the inverse of x is defined as $x^{-1} := \lambda_1^{-1} f_1 + \cdots + \lambda_n^{-1} f_n$.

For a complete treatment on Euclidean Jordan algebra, one can refer to the book [4] by Faraut and Koranyi. Brief summaries can also be found in [14,15].

In this paper we shall confine our attention to the case when the space is R^n whose elements $x = (x_0, \bar{x})^T$ are indexed from zero, equipped with the usual inner product and the Jordan product defined as

$$x \circ y = ((x, y), x_0 \bar{y} + y_0 \bar{x})^T.$$

Then R^n is a Euclidean Jordan algebra, denoted by A^n , with the cone of squares as second-order cone which is seen to be $A_+^n := \{x \in R^n : \|\bar{x}\| \leq x_0\}$. The interior of A_+^n is the cone given by $\text{int } A_+^n = \{x \in R^n : \|\bar{x}\| < x_0\}$, see [1]. The identity element in this algebra is given by $e = (1, 0, \dots, 0)^T$. Also the spectral decomposition of any x with $\bar{x} \neq 0$ is given by $x = \lambda_1 f_1 + \lambda_2 f_2$ with

$$\begin{aligned} \lambda_1 &:= x_0 + \|\bar{x}\|, \quad \lambda_2 := x_0 - \|\bar{x}\|, \\ f_1 &:= \frac{1}{2}(1, \bar{x}/\|\bar{x}\|)^T, \quad \text{and} \quad f_2 := \frac{1}{2}(1, -\bar{x}/\|\bar{x}\|)^T, \end{aligned}$$

where $\{f_1, f_2\}$ constitutes a Jordan frame. From the above decomposition $\det(x) = x_0^2 - \|\bar{x}\|^2$. The rank of A^n is always 2 and it can be shown that all Jordan frames are of the above form. Also $x \in A_+^n$ ($\text{int } A_+^n$) if and only if both λ_1 and λ_2 are nonnegative (positive), see [1,5].

A linear transformation $L_x : A^n \rightarrow A^n$ for $x \in A^n$ is defined as $L_x(y) := x \circ y$ for all $y \in A^n$. We say that two elements x, y of a Euclidean Jordan algebra V operator commute if $x \circ (y \circ z) = y \circ (x \circ z)$ for all $z \in V$, which is equivalent to stating that $L_x L_y = L_y L_x$. Specializing Lemma X.2.2 in [4] to the space A^n , we have the following characterization of operator commutativity. Also see [1] where an independent proof is provided.

Proposition 1. *Two elements x, y of A^n operator commute iff there is a Jordan frame $\{f_1, f_2\}$ such that $x = \lambda_1 f_1 + \lambda_2 f_2$ and $y = \beta_1 f_1 + \beta_2 f_2$ for some real numbers $\lambda_1, \lambda_2, \beta_1$, and β_2 .*

In view of the above proposition it is easy to see that vectors x and y in A^n operator commute iff either \bar{y} is a multiple of \bar{x} or \bar{x} is a multiple of \bar{y} .

The quadratic representation of A^n , denoted by P_a for $a \in A^n$, is the matrix

$$P_a := 2L_a^2 - L_{a^2} = 2aa^T - \det(a)\mathcal{J}_n,$$

where \mathcal{J}_n is the $n \times n$ matrix defined as

$$\mathcal{J}_n := \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}.$$

Observation 1. *For $a \in A^n$, $P_a \in \Pi(A_+^n)$.*

Proof. Case 1: When $\det(a) \neq 0$, $P_a(A_+^n) = A_+^n$ and $P_a(\text{int } A_+^n) = \text{int } A_+^n$, see [1].

Case 2: When $\det(a) = 0$, $a_0^2 = \|\bar{a}\|^2$ and $P_a(x) = 2a^T x a$ for $x \in A_+^n$. If $a_0 = \|\bar{a}\|$, then $a \in A_+^n$ and $2a^T x a_0 - 2|a^T x|a_0 = 0$, because $a^T x \geq 0$. Again if $a_0 = -\|\bar{a}\|$, then $2a^T x a_0 + 2|a^T x|a_0 = 0$, because $a^T x \leq 0$. \square

Below we shall state some of the important properties of a quadratic representation.

Proposition 2 [1]. *Let α be a real number and $x \in A^n$. For $a \in A^n$ with the spectral decomposition $a = \lambda_1 f_1 + \lambda_2 f_2$ with $\lambda_1 \neq \lambda_2$, we have the following properties:*

- (i) $\lambda_1 = a_0 + \|\bar{a}\|$ and $\lambda_2 = a_0 - \|\bar{a}\|$ are the eigenvalues of L_a , each with multiplicity one and corresponding eigenvectors f_1 and f_2 , respectively. Also a_0 is an eigenvalue of L_a with multiplicity $n - 2$.
- (ii) $\lambda_1^2 = (a_0 + \|\bar{a}\|)^2$ and $\lambda_2^2 = (a_0 - \|\bar{a}\|)^2$ are eigenvalues of P_a , each with multiplicity one and corresponding eigenvectors f_1 and f_2 , respectively. Also $\det(a) = a_0^2 - \|\bar{a}\|^2$ is an eigenvalue of P_a with multiplicity $n - 2$.
- (iii) P_a is an invertible matrix iff a is invertible.
- (iv) $P_{\alpha a} = \alpha^2 P_a$.
- (v) $P_{P_a(x)} = P_a P_x P_a$.
- (vi) $\det P_a(x) = \det^2(a) \det(x)$.
- (vii) If a is invertible, then $P_a(a^{-1}) = a$ and $P_{a^{-1}} = P_a^{-1}$.

Proposition 3 [5,6]. *For $x, y \in A^n$, the following conditions are equivalent:*

- (i) $x \in A_+^n, y \in A_+^n$, and $\langle x, y \rangle = 0$.
- (ii) $x \in A_+^n, y \in A_+^n$, and $x \circ y = 0$.

In each case, elements x and y operator commute.

Definition 1 [2].

(a) A subset F of a closed convex cone C in A^n is called a *face* of C , denoted by $F \triangleleft C$, if F is a convex cone and

$$x \in C, y - x \in C \text{ and } y \in F \implies x \in F.$$

(b) The *smallest face* of a closed convex cone C containing a point $x \in C$ is defined as

$$\Phi(x) := \bigcap \{F : F \triangleleft C, x \in F\}.$$

(c) For a self-dual closed convex cone C the *complementary face* of the face F of C is defined as

$$F^\Delta := \{x \in C : \langle x, y \rangle = 0 \forall y \in F\}.$$

Theorem 2 [2]. Let $F \triangleleft C$ and $x \in C$. Then F is the smallest face of C containing x if and only if x lies in the relative interior of F .

In view of the above discussions the second-order cone A_+^n can also be represented by $A_+^n := \{x \in A^n : x^T \mathcal{J}_n x \geq 0, x_0 \geq 0\}$ and any $x \in A^n$ has $\det(x) = x^T \mathcal{J}_n x$. The elements on the boundary of A_+^n are exactly those for which $x_0 = \|\bar{x}\|$. Below we shall state some relations on the boundary structure of the cone A_+^n , which will be useful in this paper.

$$\begin{aligned} x \in A_+^n \cup (-A_+^n) &\Rightarrow \mathcal{J}_n x \in A_+^n \cup (-A_+^n), \\ x^T \mathcal{J}_n x > 0 &\Rightarrow x \in \text{int } A_+^n \cup \text{int } (-A_+^n), \\ x^T \mathcal{J}_n x \geq 0 &\Rightarrow x \in A_+^n \cup (-A_+^n), \\ x^T \mathcal{J}_n x = 0 &\Rightarrow x \in \text{bd } A_+^n \cup \text{bd } (-A_+^n), \\ x^T \mathcal{J}_n x < 0 &\Rightarrow x \notin A_+^n \cup (-A_+^n). \end{aligned}$$

Definition 2 [4]. A linear transformation $\Psi : A^n \rightarrow A^n$ is said to be an *automorphism* of A^n if Ψ is invertible and $\Psi(x \circ y) = \Psi(x) \circ \Psi(y)$ for all $x, y \in A^n$. The set of all automorphisms of A^n is denoted by $\text{Aut}(A^n)$.

One of the important properties of automorphisms of A^n , which we shall make use of in this paper, is that for any two Jordan frames $\{e_1, e_2\}$ and $\{f_1, f_2\}$ in A^n there exists an automorphism Ψ such that $\Psi f_1 = e_1$ and $\Psi f_2 = e_2$, see [4]. Also any automorphism Ψ of A^n can be written as

$$\Psi = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix},$$

where U is an $(n-1) \times (n-1)$ orthogonal matrix, see [8].

Definition 3

(a) A matrix $M : A^n \rightarrow A^n$ has the \mathbf{R}_0 -property if

$$x \in A_+^n, M(x) \in A_+^n, \langle x, M(x) \rangle = 0 \Rightarrow x = 0.$$

(b) M is said to have the \mathbf{Q} -property if $\text{SOLCP}(M, q)$ has a solution for all $q \in A^n$.

There is a nice geometric interpretation of \mathbf{Q} -property in terms of the complementary cones associated with a matrix M which are defined as follows. Given a matrix $M : A^n \rightarrow A^n$ a *complementary cone* of M corresponding to the face F of A_+^n is defined as

$$K_F := \{y - L(x) : x \in F, y \in F^\Delta\}.$$

The $\text{SOLCP}(M, q)$ has a solution if and only if $q \in K_F$ for some face F , see [9]. Thus M has the \mathbf{Q} -property if and only if the union of all complementary cones is the whole space A^n . The above interpretation is motivated by the study of comple-

mentary cones in LCP over R_+^n , see [3]. But due to the nonpolyhedral nature of the second-order cone, complementary cones in SOLCP need not be closed.

Example 1. Let $M : A^3 \rightarrow A^3$ be a matrix defined as

$$M(x) = \begin{pmatrix} x_0 + x_1 \\ 0 \\ x_2 \end{pmatrix}.$$

Note that

$$M \begin{pmatrix} \frac{1}{2}(\epsilon + 1/\epsilon) \\ \frac{1}{2}(\epsilon - 1/\epsilon) \\ -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \text{ as } \epsilon \rightarrow 0.$$

However, there exists no $x \in A_+^3$ such that

$$M(x) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Thus the complementary cone of M corresponding to the face A_+^3 is not closed.

However, it can be shown (in a more general setting) that the \mathbf{R}_0 -property of a matrix M provides a sufficient condition for the closedness of complementary cones, see [9].

Fix a canonical Jordan frame $\{e_1, e_2\}$ where

$$e_1 := (1/2, 1/2, 0)^T \text{ and } e_2 := (1/2, -1/2, 0)^T.$$

In case of a second-order cone with a fixed Jordan frame $\{e_1, e_2\}$ consider the following subspaces of A^n . (Similar subspaces have been considered in Theorem IV 2.1 in [4].)

$$\begin{aligned} V_{11} &= \{x \in A^n : x \circ e_1 = x\} = \{\lambda e_1 : \lambda \in R\}, \\ V_{22} &= \{x \in A^n : x \circ e_2 = x\} = \{\beta e_2 : \beta \in R\}, \text{ and} \\ V_{12} &= \left\{x \in A^n : x \circ e_1 = \frac{1}{2}x = x \circ e_2\right\} = \{x \in A^n : x_0 = x_1 = 0\}. \end{aligned}$$

Thus given an $x \in A^n$ we can write

$$x = (x_0 + x_1)e_1 + (x_0 - x_1)e_2 + (0, 0, x_2, \dots, x_{n-1})^T.$$

We shall designate $(x_0 + x_1)$ and $(x_0 - x_1)$ as the diagonal entries of a vector x with respect to the Jordan frame $\{e_1, e_2\}$.

Observation 2. For $a \in A_+^n$ we have the following relationship among its entries. $a \in A_+^n$ (int A_+^n) if and only if $(a_0 + a_1) \geq 0$ (> 0), $(a_0 - a_1) \geq 0$ (> 0), and $(a_0 + a_1)(a_0 - a_1) - \|\tilde{a}\|^2 \geq 0$ (> 0) where $\tilde{a} = (a_2, \dots, a_{n-1})^T$.

Similar to the notion of a diagonal matrix we introduce the notion of a diagonal vector $d \in A^n$ with respect to a canonical Jordan frame $\{e_1, e_2\}$ as

$$d = \lambda_1 e_1 + \lambda_2 e_2, \quad \lambda_1, \lambda_2 \in R.$$

3. Equivalence of Q and R_0 -property of a quadratic representation

Proposition 4. *Let $M \in \Pi(A_+^n)$. Then M has the R_0 -property if and only if $\langle x, M(x) \rangle > 0$ for all $0 \neq x \in A_+^n$.*

The proof of the above proposition follows easily from the definitions.

Remark 1. For a linear complementarity problem $\text{LCP}(M, q)$ over R_+^n with $M(R_+^n) \subseteq R_+^n$, we have $\text{LCP}(M, q)$ is solvable for all $q \in R^n$ iff $\text{LCP}(M, 0)$ has a unique solution zero, which is also equivalent to stating that M is strictly copositive, see [3].

Theorem 3. *Let $M \in \Pi(A_+^2)$. Then M has the Q-property if and only if M has the R_0 -property.*

Proof. Observing the fact that $\{e_1, e_2\}$ is the only Jordan frame of A_+^2 unique up to permutation, R_0 -property of the above matrix M is equivalent to the property that $\langle e_1, M(e_1) \rangle > 0$ and $\langle e_2, M(e_2) \rangle > 0$. Now suppose without loss of generality that $\langle e_1, M(e_1) \rangle = 0$. We shall show that M does not have the Q-property. Take $q = e_2 - e_1 = (0, -1, 0)^T$. Let $x \in A_+^2$ be a solution to $\text{SOLCP}(M, q)$. Then we have

$$x = \lambda_1 e_1 + \lambda_2 e_2 \quad \text{and} \quad M(x) + q = \beta_1 e_1 + \beta_2 e_2$$

such that $\lambda_1 \beta_1 = 0$ and $\lambda_2 \beta_2 = 0$. Since $M(x) \in A_+^2$ and $x \neq 0$, $\lambda_2 = \beta_1 = 0$. Thus $\langle e_1, M(x) + q \rangle = \langle e_1, \lambda_1 M(e_1) + e_2 - e_1 \rangle = -\langle e_1, e_1 \rangle < 0$, which contradicts the fact that $M(x) + q \in A_+^2$. The ‘if part’ is apparent from Karamardian’s theorem [7] on the solvability of the complementarity problem. \square

Loewy and Schneider [8] proved the following result which characterizes the extreme matrices of the closed convex cone of square matrices which leave the cone A_+^n invariant.

Theorem 4 [8]. *Let M be an $n \times n$ real matrix, with $n \geq 3$. Then M is an extreme matrix of $\Pi(A_+^n)$ (generates a 1-dimensional face of $\Pi(A_+^n)$) if and only if either $M(A_+^n) = A_+^n$ or $M = uv^T$ for $u, v \in \text{bd } A_+^n$.*

Remark 2. Any $M : A^n \rightarrow A^n$ satisfying $M(A_+^n) = A_+^n$ can be written as $M = P_a \Psi$, where $a \in \text{int } A_+^n$ and $\Psi \in \text{Aut}(A^n)$, see page 56 of [4].

Proposition 5. Let $M \in \Pi(A_+^n)$. Then M has the **Q**-property if and only if $\Theta M \Theta^T$ has the **Q**-property for all $\Theta : A^n \rightarrow A^n$ such that $\Theta(A_+^n) = A_+^n$.

Proof. Take an arbitrary $q \in A^n$ and define $\tilde{q} = \Theta^{-1}q$. Then $\text{SOLCP}(M, \tilde{q})$ has a solution \tilde{x} such that

$$\tilde{x} \in A_+^n, \quad M(\tilde{x}) + \tilde{q} \in A_+^n \quad \text{and} \quad \langle \tilde{x}, M(\tilde{x}) + \tilde{q} \rangle = 0.$$

Define $x = (\Theta^{-1})^T \tilde{x}$. Since both $\Theta^{-1}, \Theta^T \in \Pi(A_+^n)$, we have

$$x \in A_+^n, \quad M(\Theta^T x) + \Theta^{-1}q \in A_+^n \quad \text{and} \quad \langle \Theta^T x, M\Theta^T x + \Theta^{-1}q \rangle = 0,$$

equivalently,

$$x \in A_+^n, \quad \Theta M \Theta^T x + q \in A_+^n \quad \text{and} \quad \langle x, \Theta M \Theta^T x + q \rangle = 0,$$

which implies that $\Theta M \Theta^T$ has the **Q**-property. \square

Theorem 5. Let $M = uv^T$ for $u, v \in \text{bd } A_+^n$. Then M does not have the **Q**-property.

Proof. First we shall show that if $M = e_i e_j^T$, where $i, j \in \{1, 2\}$, then M does not have the **Q** property. When $M = e_1 e_2^T$ or $M = e_1 e_1^T$ we can take $q = (0, 1, 0)^T$ and can show easily that $\text{SOLCP}(M, q)$ does not have a solution. Again when $M = e_2 e_1^T$ or $M = e_2 e_2^T$ we can take $q = (0, -1, 0)^T$ and can easily prove that $\text{SOLCP}(M, q)$ is not solvable. In fact, in both the above cases $\text{SOLCP}(M, q)$ is not even feasible. Now we consider the two cases.

Case 1: Suppose that u and v are linearly dependent. Then there exists an automorphism Ψ of A^n such that $\Psi u = e_1$. By Proposition 5, uv^T has the **Q**-property implies that $e_1 e_1^T$ has the **Q**-property. Since we know that $e_1 e_1^T$ is not **Q**, we have proved our claim.

Case 2: Suppose that u and v are linearly independent. Then by Lemma 3.7 in [8], there exists a $\Theta \in \Pi(A_+^n)$ with $\Theta(A_+^n) = A_+^n$ such that $\Theta u = e_1$ and $\Theta v = e_2$. Again by Proposition 5, uv^T has the **Q**-property implies that $e_1 e_2^T$ has the **Q**-property. Since we know that $e_1 e_2^T$ is not **Q**, uv^T does not have the **Q**-property. \square

Theorem 6. For a quadratic representation P_a of A^n , $n \geq 3$, we have the following equivalence:

- (i) $a \in \text{int } A_+^n$ or $-a \in \text{int } A_+^n$.
- (ii) P_a is positive definite.
- (iii) $\text{SOLCP}(P_a, q)$ has a unique solution for all $q \in A^n$.
- (iv) P_a has the **R**₀-property.

Proof. The proof of (i) \Rightarrow (ii) follows from Theorem 3 in [1], since all the eigenvalues of P_a are positive. From (ii), P_a is positive definite implies that $\text{SOLCP}(P_a, q)$ has atmost one solution for all $q \in A^n$. Also from Karamardian’s theorem [7], P_a

has the **Q**-property. Thus (iii) follows. The proof of (iii) \Rightarrow (iv) is obvious. To prove (iv) \Rightarrow (i), let us suppose that neither $a \in \text{int } A_+^n$ nor $-a \in \text{int } A_+^n$. To complete the proof, it is sufficient to show that there exists some $z \in \text{bd } A_+^n$ such that $\langle a, z \rangle = 0$. We consider the following two cases.

Case 1. When $a \in \text{bd } A_+^n$ or $-a \in \text{bd } A_+^n$ we can find a nonzero $z \in \text{bd } A_+^n$ or $-z \in \text{bd } A_+^n$ on a face complementary to which a or $-a$ lies.

Case 2. When $a \notin \text{bd } A_+^n$ and $-a \notin \text{bd } A_+^n$, there exists an $x \in \text{bd } A_+^n$ and $y \in \text{bd } A_+^n$ such that $\langle a, x \rangle < 0$ and $\langle a, y \rangle > 0$. Since the inner product $\langle a, \cdot \rangle$ is continuous on the line segment joining x and y denoted as $[x, y]$, there exists a $\tilde{z} = \alpha x + (1 - \alpha)y$ for $0 < \alpha < 1$ such that $\langle a, \tilde{z} \rangle = 0$. Since A_+^n has faces of dimension 0, 1 and n only, either \tilde{z} lies on a one dimensional face of A_+^n in which case we are done otherwise $\tilde{z} \in \text{int } A_+^n$. It means that $a^\perp \cap \text{int } A_+^n$ is nonempty where a^\perp is the orthogonal complement of the span of a in A^n . Note that a^\perp is a $(n - 1)$ dimensional subspace of A^n intersecting a full dimensional cone A_+^n in its interior where $n \geq 3$. Hence from [5, Lemma 3.6] there exists an invertible linear transformation Γ on A^n such that $\Gamma(A_+^n) = A_+^n$ and $\Gamma(a^\perp) = \{x \in A^n : x_{n-1} = 0\}$. Thus there exists a nonzero $z \in a^\perp$ such that $\Gamma z = e_1$, which by [5, Lemma 3.3] lies on the boundary of A_+^n .

Now for the above chosen z we have $\langle z, P_a(z) \rangle = \langle z, (2aa^T - \det(a)\mathcal{J}_n)z \rangle = 0$, which contradicts that P_a has the **R**₀-property. \square

Lemma 1. *If the quadratic representation P_a has the **Q**-property then a is invertible.*

Proof. Taking $-q \in \text{int } A_+^n$ let $x \in A_+^n$ be a solution to $\text{SOLCP}(P_a, q)$. Since $P_a(x) + q \in A_+^n$ and $-q \in \text{int } A_+^n$, $P_a(x) \in \text{int } A_+^n$. From Proposition 2, $\det P_a(x) = \det^2(a) \det(x)$ and since $x \in A_+^n$, we have a is invertible. \square

The following lemma is analogous to Lemma 4.3.2 in [13].

Lemma 2. *Let $d = d_1e_1 + d_2e_2$ be a diagonal vector with nonzero entries. Let $|d| = |d_1|e_1 + |d_2|e_2$. Write $d = |d| \circ s$ where $s = \pm e_1 \pm e_2$. The coefficients of e_1 and e_2 in the expression for s are determined by the signs of d_1 and d_2 , respectively. If P_d has the **Q**-property then P_s has the **Q**-property.*

Proof. Fix an arbitrary $q \in A^n$ and define $\tilde{q} := P_{\sqrt{|d|}}(q)$ where $\sqrt{|d|} = \sqrt{|d_1|}e_1 + \sqrt{|d_2|}e_2$. Let $x \in A_+^n$ be a solution to $\text{SOLCP}(P_d, \tilde{q})$. Then $z := P_d(x) + \tilde{q} \in A_+^n$ is such that $x \circ z = 0$. We have $z = P_{|d| \circ s}(x) + \tilde{q} = P_{|d|}P_s(x) + P_{\sqrt{|d|}}(q) = P_{\sqrt{|d|}}P_{\sqrt{|d|}}P_s(x) + P_{\sqrt{|d|}}(q)$, where the last equality follows from Proposition 2(v). Thus by Proposition 2(vii), $P_{\sqrt{|d|}^{-1}}(z) = P_{\sqrt{|d|}}P_s(x) + q = P_sP_{\sqrt{|d|}}(x) + q$. Define $y = P_{\sqrt{|d|}}(x)$ and $w = P_{\sqrt{|d|}^{-1}}(z)$. Then $y \in A_+^n$, $w = P_s(y) + q \in A_+^n$ and $\langle y, w \rangle = \langle x, P_{\sqrt{|d|}}P_{\sqrt{|d|}^{-1}}(z) \rangle = 0$. Hence $y \in A_+^n$ solves $\text{SOLCP}(P_s, q)$. \square

Lemma 3. *Let s be a diagonal vector of the form $s = \pm e_1 \pm e_2$ and $s \neq \pm e$. Then P_s does not have the **Q**-property.*

Proof. Without loss of generality take $s = e_1 - e_2 = (0, 1, 0)^T$. Then the matrix P_s will be

$$P_s = 2ss^T - \det(s)J_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -I \end{pmatrix}.$$

Consider a vector $q = (0, 0, -1, 0, \dots, 0)$. If $y \in A^n$ is the solution of the SOLCP (P_s, q) then

$$y \in A_+^n, P_s(y) + q \in A_+^n \quad \text{and} \quad \langle y, P_s(y) + q \rangle = 0.$$

The complementary condition $\langle y, P_s(y) + q \rangle = 0$ gives us $y_0^2 + y_1^2 - y_2 = \sum_{i=2}^{n-1} y_i^2$. Also from $y \in A_+^n$ and $P_s(y) + q \in A_+^n$ we have

$$y_0^2 \geq y_1^2 + \dots + y_{n-1}^2 \quad \text{and} \quad y_0^2 \geq y_1^2 + (1 + y_2)^2 + y_3^2 + \dots + y_{n-1}^2.$$

Substituting the value of $\sum_{i=2}^{n-1} y_i^2$ in the above two inequalities we get $2y_1^2 \leq y_2$ and $2y_1^2 + 1 + y_2 \leq 0$, which are inconsistent. \square

Theorem 7. *Let P_a be a quadratic representation of A^n , $n \geq 3$. Then P_a has the **Q**-property if and only if P_a has the **R**₀-property.*

Proof. Since P_a has the **Q**-property we have by Lemma 1, a is invertible. By the spectral decomposition and Theorem IV.2.5 in [4] we can write $a = d_1 \Psi e_1 + d_2 \Psi e_2$, where $d_1, d_2 \neq 0$ and $\Psi \in \text{Aut}(A^n)$. Since $\Psi P_a \Psi^T = P_{\Psi a}$, by Proposition 5, P_a has the **Q**-property implies that $P_{\Psi a}$ has the **Q**-property, which is equivalent to the statement that P_d has **Q**-property where $d := d_1 e_1 + d_2 e_2$. By Lemma 2, P_s has the **Q**-property where s is the vector corresponding to d as defined above. Since P_s has the **Q**-property if and only if $s = \pm e$, either $d_1 > 0$ and $d_2 > 0$ or $d_1 < 0$ and $d_2 < 0$. It means that either $a \in \text{int } A_+^n$ or $-a \in \text{int } A_+^n$, which by Theorem 6 proves that P_a has the **R**₀-property. Conversely, **R**₀ implies **Q** is evident from Karamardian’s theorem [7]. \square

The above result can also be extended to the linear complementarity problem over the direct product of second-order cones, which is defined as follows. Given a matrix M on R^n and $q \in R^n$ the linear complementarity problem over the cone K^n is the problem of finding an $x \in K^n$ such that $M(x) + q$ is in K^n and $\langle x, M(x) + q \rangle = 0$, where the cone K^n is defined as

$$K^n := A_+^{n_1} \times \dots \times A_+^{n_m},$$

with $n = n_1 + \dots + n_m$.

It should be noted that R^n is a Euclidean Jordan algebra with the usual inner product and the Jordan product defined as the componentwise Jordan product of the elements of A^{n_i} , $i = 1 \dots m$. Formally, for $x := (x_1, \dots, x_m)^T$ and $y := (y_1, \dots, y_m)^T$, where each component vector x_i is written in a row vector form, the Jordan product is defined as

$$x \circ y := (x_1 \circ y_1, \dots, x_m \circ y_m)^T.$$

The cone of squares with respect to the above Jordan product is the cone K^n . Also the quadratic representation P_a for $a \in R^n$ is a block diagonal matrix of the form

$$P_a = \begin{pmatrix} P_{a_1} & 0 & \dots & 0 \\ 0 & P_{a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & P_{a_m} \end{pmatrix},$$

where $a = (a_1, \dots, a_m)^T$. For further information one can see the Refs. [1,4].

Theorem 8. *In the space R^n with the cone of squares K^n and the Jordan product defined above the following statements are equivalent:*

- (i) For $a \in R^n$, P_a has the \mathbf{R}_0 -property, that is, $x \in K^n$, $P_a(x) \in K^n$ and $\langle x, P_a(x) \rangle = 0$ implies $x = 0$.
- (ii) The linear complementarity problem $\text{LCP}(P_a, q)$ over the cone K^n has the solution for all $q \in R^n$.

Proof. The proof of the above theorem is apparent once we notice that P_a has the \mathbf{R}_0 -property if and only if P_{a_i} has the \mathbf{R}_0 -property for each component vector $a_i \in A^i$ for $i = 1, \dots, m$. \square

An open problem

In this paper we have shown that when the matrix $M \in R^{n \times n}$ is a quadratic representation, $\text{SOLCP}(M, q)$ has a solution for all $q \in R^n$ if and only if $\text{SOLCP}(M, 0)$ has a unique solution zero. However, it is not known whether the above equivalence holds for all matrices $M \in \Pi(A_+^n)$ for $n \geq 3$. This question remains open.

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