Cylindrical and spherical ion acoustic waves in a plasma with nonthermal electrons and warm ions

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Using the reductive perturbation technique, nonlinear cylindrical and spherical Korteweg–de Vries (KdV) and modified KdV equations are derived for ion acoustic waves in an unmagnetized plasma consisting of warm adiabatic ions and nonthermal electrons. The effects of nonthermally distributed electrons on cylindrical and spherical ion acoustic waves are investigated. It is found that the nonthermality has a very significant effect on the nature of ion acoustic waves.

I. INTRODUCTION

Ion acoustic waves in unmagnetized plasma have been studied by a number of authors both experimentally and theoretically. Existence of nonlinear waves, viz., solitons has been studied in detail and the solitary waves have been observed theoretically and in laboratory.[1–6] Recently these studies have been extended to dusty plasma.[7–11] Generally the solitons are stable but it has been found that other factors like finite ion temperature play an important role[12–15] in determining the stability of ion acoustic waves. However most of the studies done so far in ion acoustic waves were confined to the unbounded planer geometry, though recently some works have been published in which cylindrical and spherical ion acoustic and dust-ion acoustic waves have been discussed.[16–20] Sometime ago Maxon and Viecelli[21] discussed this problem. It may be mentioned that cylindrical and spherical symmetric solitons have been observed in plasma.[22] However in these studies electrons were assumed to be isothermal. But nonisothermality plays an important role in determining the nature of solitary waves. Following Cairns *et al.*[23] we take the electrons to be nonisothermally distributed. The motivation for this came from the observations of solitary structures with density depletions made by the Freja and Viking satellites.[24,25] Mamun[26] and later Tang and Xue[27] also considered the nonthermal electrons and warm ion effects on ion acoustic waves. In fact it has been shown here that for certain values of β, the nonisothermal parameter, cylindrical and spherical Korteweg–de Vries (KdV) equations are not valid and one has to consider modified KdV (MKdV) equation. In this paper both cylindrical and spherical KdV equations are derived for ion acoustic waves in a nonmagnetized plasma consisting of warm ions and nonthermal electrons. For the special case when the coefficient of nonlinearity vanishes, modified cylindrical and spherical KdV equations are derived. Some exact solutions are obtained for cylindrical KdV and MKdV equations. Numerical solutions for both type of equations are obtained using initial KdV and MKdV profiles. Effect of σ, the ion temperature, and β, the nonthermal electron parameter, are studied numerically. The paper is organized as follows. In Sec. II the basic equations are given and cylindrical and spherical KdV and MKdV equations are derived. In Sec. III numerical results are discussed. Section IV is kept for discussion and conclusion.

II. BASIC EQUATIONS AND DERIVATION OF CYLINDRICAL AND SPHERICAL KdV EQUATIONS

We consider an unmagnetized plasma consisting of warm ions and nonthermal electrons. The normalized fluid equations in cylindrical and spherical geometry are

$$\frac{\partial n_i}{\partial t} + \frac{1}{r^\nu}\frac{\partial}{\partial r}(r^\nu n_i u_i) = 0, \tag{1}$$

$$\frac{\partial u_i}{\partial t} + u_i\frac{\partial u_i}{\partial r} + \frac{\sigma}{n_i}\frac{\partial p_i}{\partial r} = -\frac{\partial \phi}{\partial r}, \tag{2}$$

$$\frac{\partial p_i}{\partial t} + p_i\frac{\partial p_i}{\partial r} + 3p_i\frac{1}{r^\nu}\frac{\partial}{\partial r}(r^\nu u_i) = 0, \tag{3}$$

$$\frac{1}{r^\nu}\frac{\partial}{\partial r}\left(r^\nu\frac{\partial \phi}{\partial r}\right) = n_e - n_i, \tag{4}$$

where $\nu=0$ for one-dimensional geometry and $\nu=1,2$ for cylindrical and spherical geometries, respectively. In the above equations, the subscripts i and e refer to ion and electron, respectively. $\sigma=T_i/T_e$ is the ion temperature ratio. n_i is the ion number density normalized to its unperturbed equilibrium plasma density n_{i0}, u_i is the ion fluid speed normalized to the ion acoustic velocity $v_s=(T_e/m_i)^{1/2}$, and ϕ is the electrostatic wave potential normalized to T_e/e. The time and space variables are in units of the ion plasma period $\omega_{\mathrm{Pi}}^{-1}=(m_i/4\pi n_{i0}e^2)^{1/2}$ and the Debye radius $\lambda_{Dm}=(T_e/4\pi n_{i0}e^2)^{1/2}$, respectively.

The electrons are assumed to be nonthermally distributed and their distribution function is taken as

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$$f(E)=n_{i0}(1+\alpha E^2)\exp(-E)/(1+3\alpha), \quad (5)$$

where E is the nonthermalized electron energy. Consequently the electron number density is given by

$$n_e=(1-\beta\phi+\beta\phi^2)\exp(\phi), \quad (6)$$

where

$$\beta=\frac{4\alpha}{1+3\alpha}. \quad (7)$$

It is clear that Eq. (6) expresses the isothermally distributed electrons when $\beta=0$ (i.e., $\alpha=0$). The parameter α represents the nonthermality of electrons distribution, i.e., determining the fast particles present in our plasma model. Also it is assumed that $u_{\mathrm{thi}} \ll v_s \ll v_{\mathrm{the}}$, so that Landau damping can be neglected, where u_{thi} is the ion thermal velocity and v_{the} is the electron thermal velocity.

To derive the cylindrical and spherical KdV equations we use the stretched coordinates

$$\xi=\epsilon^{1/2}(r-v_0t), \quad \tau=\epsilon^{3/2}t. \quad (8)$$

The dependent variables are expanded as follows:

$$n_i=1+\epsilon n_i^{(1)}+\epsilon^2 n_i^{(2)}+\cdots, \quad (9)$$

$$u_i=\epsilon u_i^{(1)}+\epsilon^2 u_i^{(2)}+\cdots, \quad (10)$$

$$p_i=1+\epsilon p_i^{(1)}+\epsilon^2 p_i^{(2)}+\cdots, \quad (11)$$

$$\phi=\epsilon\phi^{(1)}+\epsilon^2\phi^{(2)}+\cdots. \quad (12)$$

Substituting Eqs. (9)–(12) into Eqs. (1)–(4), we obtain from the lowest order in ϵ, $n_i^{(1)}=[1/(v_0^2-3\sigma)]\phi^{(1)}$, $u_i^{(1)}=[v_0/(v_0^2-3\sigma)]\phi^{(1)}$, $p_i^{(1)}=[3/(v_0^2-3\sigma)]\phi^{(1)}$, $v_0=\sqrt{[3\sigma(1-\beta)+1]/(1-\beta)}$.

To next higher order in ϵ, we obtain a set of equations

$$\frac{\partial n_i^{(1)}}{\partial\tau}-v_0\frac{\partial n_i^{(2)}}{\partial\xi}+\frac{\partial}{\partial\xi}(n_i^{(1)}u_i^{(1)})+\frac{\partial u_i^{(2)}}{\partial\xi}+\frac{\nu u_i^{(1)}}{v_0\tau}=0, \quad (13)$$

$$\frac{\partial u_i^{(1)}}{\partial\tau}-v_0\frac{\partial u_i^{(2)}}{\partial\xi}+u_i^{(1)}\frac{\partial u_i^{(1)}}{\partial\xi}-\sigma n_i^{(1)}\frac{\partial p_i^{(1)}}{\partial\xi}+\sigma\frac{\partial p_i^{(2)}}{\partial\xi}=-\frac{\partial\phi^{(2)}}{\partial\xi}, \quad (14)$$

$$\frac{\partial p_i^{(1)}}{\partial r}-v_0\frac{\partial p_i^{(2)}}{\partial\xi}+u_i^{(1)}\frac{\partial p_i^{(1)}}{\partial\xi}+3\frac{\partial u_i^{(2)}}{\partial\xi}+3p_i^{(1)}\frac{\partial u_i^{(1)}}{\partial\xi}+\frac{3\nu u_i^{(1)}}{v_0\tau}=0, \quad (15)$$

$$\frac{\partial^2\phi^{(1)}}{\partial\xi^2}=\phi^{(2)}+\frac{1}{2}\phi^{(1)^2}-\beta\phi^{(2)}-n_i^{(2)}. \quad (16)$$

Combining Eqs. (13)–(16), we get a modified Korteweg–de Vries equation

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\frac{\nu}{2\tau}\phi^{(1)}+A\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}+B\frac{\partial^3\phi^{(1)}}{\partial\xi^3}=0, \quad (17)$$

where

$$A=\frac{12\sigma(1-\beta)^3+3(1-\beta)^2-1}{2\sqrt{3\sigma(1-\beta)+1}(1-\beta)^{(3/2)}},$$
$$B=\frac{1}{2\sqrt{3\sigma(1-\beta)+1}(1-\beta)^{(3/2)}}. \quad (18)$$

It can be shown[28] that a suitable coordinate transformation reduces the cylindrical KdV equation into the ordinary KdV equation which can be solved analytically. In this way the exact solution of Eq. (17) can be written as

$$\phi^{(1)}=\frac{1}{\tau}\left[\frac{\xi}{2A}+\frac{3V}{A}\operatorname{sech}^2\left(\sqrt{\frac{V}{4B\tau}}(\xi+2V)\right)\right], \quad (19)$$

where V is the solitary wave velocity and the solution is valid for $\tau>0$.

Another analytical solution of the cylindrical and spherical KdV equations can be found by the group analysis method.[20,29] By this method the solution of cylindrical KdV equation is given by

$$\phi^{(1)}=\frac{\xi}{2A\tau}+\frac{1}{\tau}[a_0+a_2\tanh^2(\xi\tau^{-1/2})], \quad (20)$$

where

$$a_0=\frac{8B}{A}, \quad a_2=-\frac{12B}{A}. \quad (21)$$

The solution of spherical KdV equation is given by

$$\phi^{(1)}=\frac{\xi}{A\tau\ln\tau}+\frac{c}{\tau\ln\tau}. \quad (22)$$

For detailed discussion see Ref. 20.

It is seen that for $\sigma=[1-3(1-\beta)^2]/12(1-\beta)^3$, $A=0$, whence the coefficient of nonlinearity vanishes. Then one has to consider the modified KdV (MKdV) equation. For example if $\sigma=1/6$ and $\beta=1/2$ (i.e., $\alpha=1/5$), then $A=0$.

For this we introduce the stretched coordinates $\xi=\epsilon(r-v_0t)$, $\tau=\epsilon^3 t$ and expand n_i, u_i, p_i, and ϕ in a power series of ϵ as given by Eqs. (9)–(12) and develop equations in various powers of ϵ. To lowest order in ϵ, Eqs. (1)–(4) give $n_i^{(1)}=[1/(v_0^2-3\sigma)]\phi^{(1)}$, $u_i^{(1)}=[v_0/(v_0^2-3\sigma)]\phi^{(1)}$, $p_i^{(1)}=[3/(v_0^2-3\sigma)]\phi^{(1)}$, $v_0=\sqrt{[3\sigma(1-\beta)+1]/(1-\beta)}$. To next higher order in ϵ, we obtain a set of equations, from which we get a equation which is an identity if $v_0=\sqrt{[3\sigma(1-\beta)+1]/(1-\beta)}$ and $\sigma=[1-3(1-\beta)^2]/[12(1-\beta)^3]$.

To next higher order in ϵ, we obtain a set of equations

$$\frac{\partial n_i^{(1)}}{\partial\tau}-v_0\frac{\partial n_i^{(3)}}{\partial\xi}+\frac{\partial}{\partial\xi}(u_i^{(3)}+n_i^{(1)}+u_i^{(2)}+n_i^{(2)}u_i^{(1)})+\frac{\nu u_i^{(1)}}{v_0\tau}=0, \quad (23)$$

$$\frac{\partial u_i^{(1)}}{\partial\tau}-v_0\frac{\partial u_i^{(3)}}{\partial\xi}+u_i^{(1)}\frac{\partial u_i^{(2)}}{\partial\xi}+u_i^{(2)}\frac{\partial u_i^{(1)}}{\partial\xi}+\sigma\frac{\partial p_i^{(3)}}{\partial\xi}-\sigma n_i^{(1)}\frac{\partial p_i^{(2)}}{\partial\xi}-\sigma n_i^{(2)}\frac{\partial p_i^{(1)}}{\partial\xi}+\sigma(n_i^{(1)})^2\frac{\partial p_i^{(1)}}{\partial\xi}=-\frac{\partial\phi^{(3)}}{\partial\xi}, \quad (24)$$

$$\frac{\partial p_i^{(1)}}{\partial \tau} - v_0 \frac{\partial p_i^{(3)}}{\partial \xi} + u_i^{(1)} \frac{\partial p_i^{(2)}}{\partial \xi} + u_i^{(2)} \frac{\partial p_i^{(1)}}{\partial \xi} + 3 \frac{\partial u_i^{(3)}}{\partial \xi} + 3 p_i^{(1)} \frac{\partial u_i^{(2)}}{\partial \xi} + 3 p_i^{(2)} \frac{\partial u_i^{(1)}}{\partial \xi} + \frac{3 \nu u_i^{(1)}}{v_0 \tau} = 0, \tag{25}$$

$$\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} - (1-\beta)\phi^{(3)} - \phi^{(1)}\phi^{(2)} - \frac{1+3\beta}{2}(\phi^{(1)})^3 + n_i^{(3)} = 0. \tag{26}$$

Combining Eqs. (23)–(26), a modified KdV equation is obtained,

$$\frac{\partial \phi^{(1)}}{\partial \tau} + \frac{\nu}{2\tau}\phi^{(1)} + A_1 (\phi^{(1)})^2 \frac{\partial \phi^{(1)}}{\partial \xi} + B_1 \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} = 0, \tag{27}$$

where

$$A_1 = \frac{117\sigma^2(1-\beta)^5 + 3(1-\beta)^3 + \frac{99\sigma}{2}(1-\beta)^4 - \frac{3\sigma}{2}(1-\beta)^2 - \frac{1+3\beta}{2}}{2\sqrt{3\sigma(1-\beta)+1}(1-\beta)^{(3/2)}},$$

$$B_1 = \frac{1}{2\sqrt{3\sigma(1-\beta)+1}(1-\beta)^{(3/2)}}. \tag{28}$$

By group analysis method it can be seen that the equation is invariant under the transformation

$$\tau \to e^{3a}t, \quad \xi \to e^{a}t, \quad \phi^{(1)} \to e^{-a}\phi^{(1)}.$$

This suggests that the MKdV equation can be reduced to a separable form by the substitution

$$\zeta = \frac{\xi}{\tau^{1/3}}, \quad \eta = \tau, \quad \phi^{(1)} = \frac{1}{\zeta \eta^{1/3}} f(\zeta).$$

Then the MKdV equation reduces to

$$\frac{B_1}{\zeta}\frac{d^3 f}{d\zeta^3} - \frac{3B_1}{\zeta^2}\frac{d^2 f}{d\zeta^2} + \left(\frac{6B_1}{\zeta^3} + \frac{A_1}{\zeta^3}[f(\zeta)]^2 - \frac{1}{3}\right)\frac{df}{d\zeta} + \left(\frac{\nu}{2\zeta} - \frac{6B_1}{\zeta^4}\right) f(\zeta) - \frac{A_1}{\zeta^4}[f(\zeta)]^3 = 0. \tag{29}$$

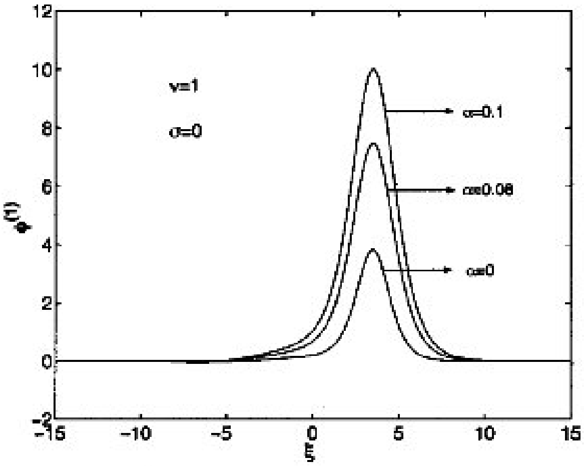

FIG. 1. Plot of $\phi^{(1)}$ vs ξ for the solution (17) for different values of α, where $\nu=1$, $\sigma=0$.

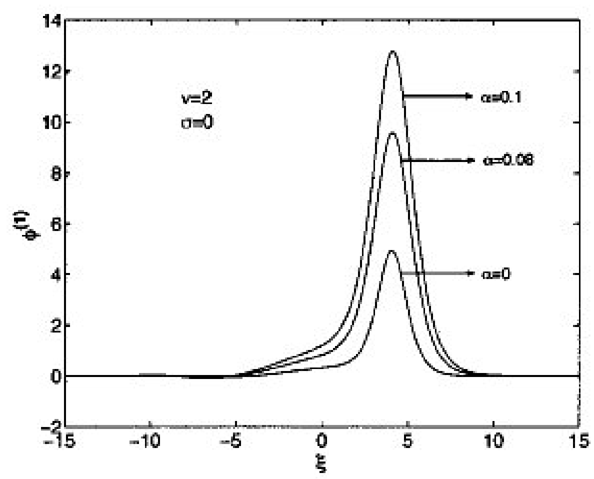

FIG. 2. Plot of $\phi^{(1)}$ vs ξ for the solution (17) for different values of α, where $\nu=2$, $\sigma=0$.

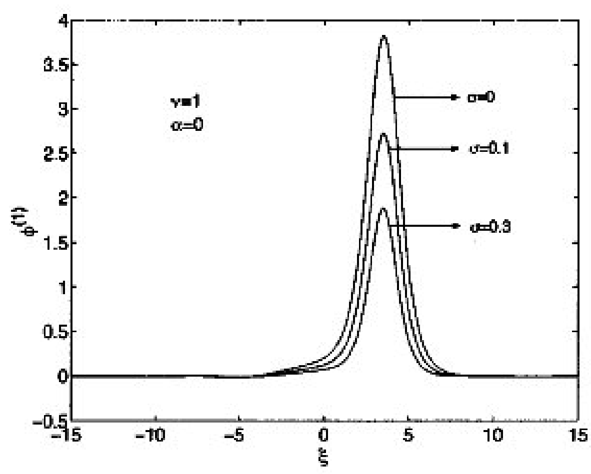

FIG. 3. Plot of $\phi^{(1)}$ vs ξ for the solution (17) for different values of σ, where $\nu=1$, $\alpha=0$.

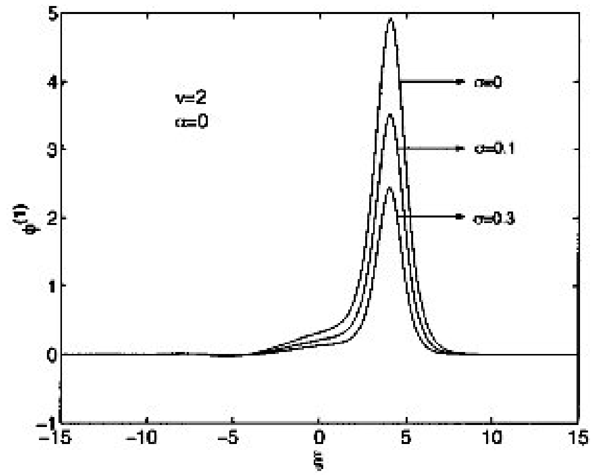

FIG. 4. Plot of $\phi^{(1)}$ vs ξ for the solution (17) for different values of σ, where $\nu=2$, $\alpha=0$.

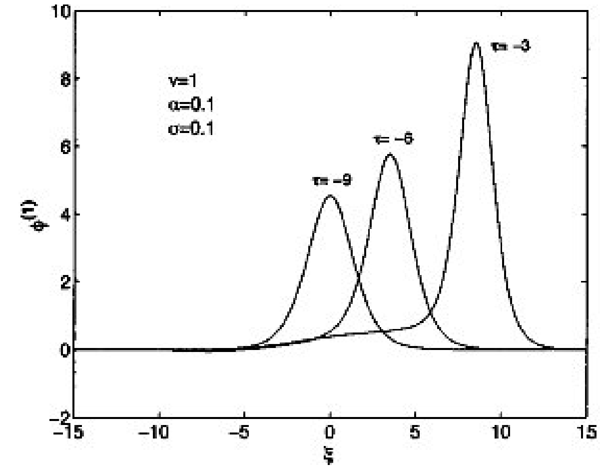

FIG. 6. Plot of $\phi^{(1)}$ vs ξ for the solution (17) for different values of τ, where $\nu=1$, $\alpha=0.1$, $\sigma=0.1$.

However a nontrivial analytical solution of $f(\zeta)$ could not be obtained.

III. NUMERICAL SOLUTION OF THE CYLINDRICAL AND SPHERICAL KdV AND MKdV EQUATIONS

Equation (17) has the following solitary wave solution for $\nu=0$ (corresponding to one dimensional geometry):

$$\phi^{(1)} = \frac{3V}{A}\operatorname{sech}^2\left[\sqrt{\frac{V}{4B}}(\xi - V\tau)\right], \tag{30}$$

where V is the soliton velocity. With this initial profile at $\tau=-9$ we solve numerically the cylindrical and spherical KdV equations. In Figs. 1 and 2 the solutions for $\nu=1$ and $\nu=2$ are plotted for several values of α, the nonthermal electron parameter for $\sigma=0$ at $\tau=-6$, respectively. It is seen that amplitude of the soliton increases with the increase of α. In Figs. 3 and 4 the solutions for $\nu=1$ and $\nu=2$ are plotted for several values of σ, the ion temperature parameter for $\alpha=0$ at $\tau=-6$, respectively. It is seen that amplitude of the soliton decreases with the increase of ion temperature parameter σ. So the amplitude of the soliton increases with the increase of α whereas amplitude decreases with the increase of σ. The dependence of the amplitude on α is very significant. This shows that the nonthermal effect is very important. Figure 5 shows the evolution of solitary wave structure at $\tau=-6$ in different geometry. It is clear that the amplitude of the solitary waves are different in different geometry. The amplitude of the cylindrical solitary wave is larger than that of the one-dimensional solitary wave but smaller than that of the spherical solitary wave. In Figs. 6 and 7 the solutions for $\nu=1$ and $\nu=2$ are plotted for several values of τ in presence of ion temperature parameter σ and nonthermal electron parameter α, respectively. It is found that as the value of τ decreases, the amplitude of the waves increases. Also the solution approaches to that of the usual KdV equation as expected.

Equation (27) has the following solitary wave solution for $\nu=0$:

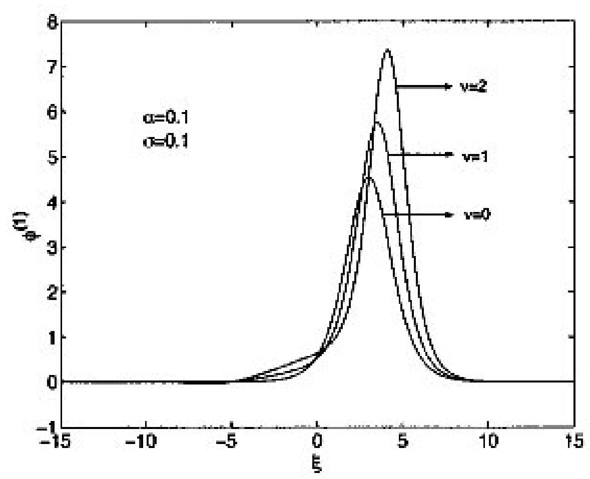

FIG. 5. Plot of $\phi^{(1)}$ vs ξ for the solution (17) in different geometries, where $\alpha=0.1$, $\sigma=0.1$.

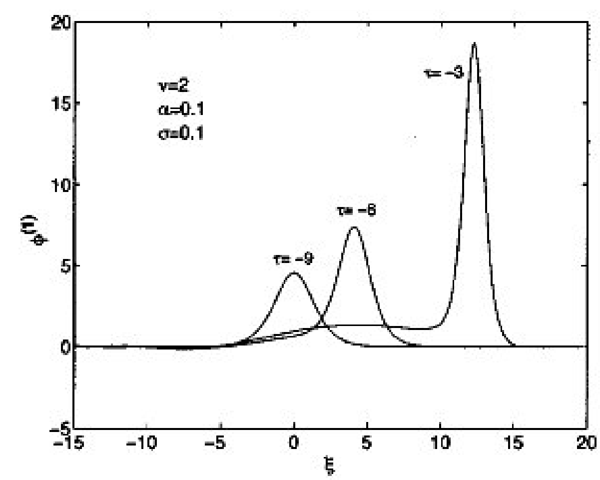

FIG. 7. Plot of $\phi^{(1)}$ vs ξ for the solution (17) for different values of τ, where $\nu=2$, $\alpha=0.1$, $\sigma=0.1$.

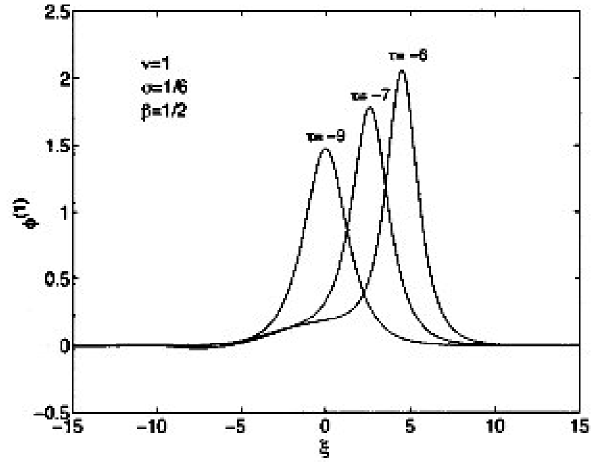

FIG. 8. Plot of $\phi^{(1)}$ vs ξ for the solution (27) for different values of τ, where $\nu=1$, $\sigma=1/6$, $\beta=1/2$ (i.e., $\alpha=1/5$).

$$\phi^{(1)} = \sqrt{\frac{6U}{A_1}}\,\mathrm{sech}\left[\sqrt{\frac{U}{B_1}}(\xi - U\tau)\right], \tag{31}$$

where U is the soliton velocity. With this initial profile at $\tau=-9$ we solve the cylindrical and spherical MKdV equations. Figures 8 and 9, respectively, show the solutions of cylindrical and spherical MKdV equations for several values of τ ranging from $\tau=-6$ to $\tau=-9$. It is seen that as magnitude of τ increases the solution looks like that for one-dimensional MKdV solitons.

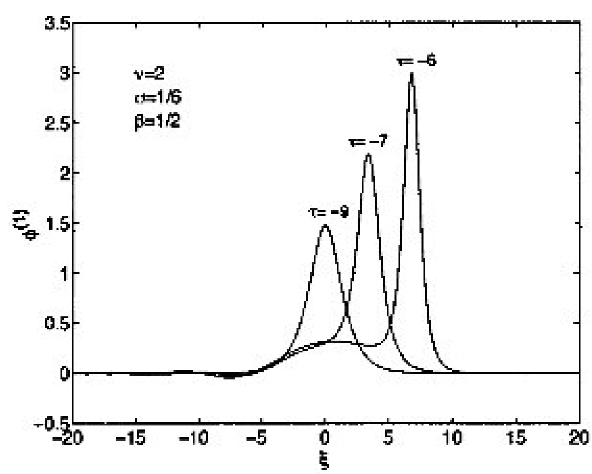

FIG. 9. Plot of $\phi^{(1)}$ vs ξ for the solution (27) for different values of τ, where $\nu=2$, $\sigma=1/6$, $\beta=1/2$ (i.e., $\alpha=1/5$).

IV. DISCUSSION AND CONCLUSION

The cylindrical and spherical KdV equations are derived for ion acoustic waves in presence of nonthermal electrons and warm adiabatic ions using reductive perturbation technique. Also exact analytical solutions for cylindrical and spherical KdV equations are found using group analysis method. The effects of nonthermal electrons and warm ions are studied in different geometry. Also here we use a numerical method assuming an initial profile similar to the one-dimensional soliton solution. It is found that nonisothermality has a very significant effect on the nature of ion acoustic solitary waves. Also, as expected, for large values of τ the solution is similar to the one-dimensional KdV solitons.

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