# By P. K. PATHAK

## Indian Statistical Institute

SUMMARY. In simple readom sampling with replacement, Besu (1968), and Dee Raj and Khamis (1958), showed that for cetimating the population mean, the average of distinct units is more officient than the overall sample mean. In this paper, a datalled treatment of the above problem is given, and the exact expression for the variance of above estimator is derived. The relative efficiency of the above estimator with other estimators is also considered. An improved estimator of the population variance is obtained. Finally, a comparison between the two simple random sampling schemes (with and without replacement) is made.

## 1. Introduction

We index the N population units as 1, 2, ..., N, and let  $Y_j$  be some real-valued characteristic (in which we are interested) of the j-th population unit.\* Here we consider the problem of estimating the population mean

$$\bar{Y} = N^{-1} \Sigma Y_4$$

and the population variance

$$\sigma^{\underline{a}} = N^{-1} \; \Sigma \; (\overline{Y}_{\underline{i}} - \overline{Y})^{\underline{a}}.$$

For simplicity we refer population units by capital letters and sample units by small letters, e.g.,  $u_i$  and  $y_i$  will denote the unit index and the variate value respectively associated with the i-th sample unit.

## 2. ESTIMATION OF Y

In simple random sampling (with replacement), Basu (1958) considered two estimators of the population mean

- (i)  $\bar{y} = 1/n \ \Sigma \ y_i = \text{average of } n \text{ sample units};$
- (ii) ȳ<sub>r</sub> = 1/ν Σ y<sub>(i)</sub> = average of ν distinct units observed in the sample.

If we record the sample of observation as

$$S = (x_1, x_2, ..., x_n),$$

where  $x_i = (y_i, u_i)$ ; and if v be the number of distinct units observed in the sample, Basu (1958) showed that the 'order-statistic' (sample units arranged in ascending order of their unit-indices)

$$T = [x_{(1)}, x_{(2)}, ..., x_{(n)}]$$

(where  $x_{(i)} = (y_{(i)}, u_{(i)})$ , and  $y_{(i)}$  is the variate value of the sample unit with unit index  $u_{(i)}$ ) forms a sufficient statistic, and therefore, for any convex (downwards) loss function

$$E(\hat{\mathbf{y}}|T) = E(\mathbf{y}_1|T) = \hat{\mathbf{y}}_{\mathbf{y}} \qquad \dots \qquad (2.1)$$

has uniformly smaller risk than  $\bar{y}$ .

An exact expression for variance of  $\bar{y}_r$ , is given below.

Variance of \( \bar{y}\_{\bar{e}} \). We have

$$V(\bar{y}_r) = E\{V(\bar{y}_r|\nu)\} = E\left(\frac{1}{\nu} - \frac{1}{N}\right) S^n$$
 ... (2.2)

where

$$S^{a} = [N/(N-1)]\sigma^{a}$$
.

<sup>\*</sup> j runs from 1 to N; i from 1 to n; and (i) from (1) to (v).

Since

$$E\left(\frac{1}{\nu}\right) = \frac{1^{n-1} + 2^{n-1} + \dots + N^{n-1}}{N^n}, \quad \text{(Pathak, 1961)}$$

$$V(g_r) = \frac{1^{n-1} + 2^{n-1} + \dots + (N-1)^{n-1}}{N^n} S^3. \qquad \dots (2.3)$$

For large samples, it is rather cumbersome to compute  $V(g_*)$ . An approximate expression for  $V(g_*)$  valid for terms up to order  $N^{-2}$  is given by

$$V(\vec{y}_r) \doteq \left[\frac{1}{n} - \frac{1}{2N} + \frac{(n-1)}{12N^2}\right] S^2.$$
 ... (2.4)

# 3. Admissibility properties of certain estimators of $\overline{Y}$

Let  $\Gamma$  denote a certain class of estimators of  $\vec{Y}$ . For a given loss function, let R(t) represent the risk (or expected loss) associated with the estimator t of  $\vec{Y}$ .

Of the two estimators  $t_1$  and  $t_2$  of  $\bar{Y}$ ,  $t_1$  will be said to be uniformly better than  $t_1$  if, for a given loss function,

$$R(t_1) \leqslant R(t_2) \qquad \dots \qquad (3.1)$$

holds for all possible values of  $(Y_1, Y_2, ..., Y_N)$  with strict sign of inequality holding for at least one  $(Y_1, Y_2, ..., Y_N)$ .

An estimator t belonging to  $\Gamma$  is said to be admissible in  $\Gamma$  if there exists no estimator in  $\Gamma$  which is better than t.

Now we consider the problem of finding admissible estimators of  $\vec{r}$ . As the 'order-statistic' T is sufficient, we have to restrict ourselves to functions of T only. Moreover, the distribution of T is not complete, therefore, many different estimators of  $\vec{r}$  can be suggested. For simplicity, we shall consider the following class of unbiased linear estimators of  $\vec{r}$ .

$$\bar{g}_{\bullet} = f_1(v) \; \bar{g}_{\nu} + f_2(v). \; ... \quad (3.2)$$

In view of the fact that

$$E[\bar{y}_{s}|v] = f_{1}(v)\bar{Y} + f_{2}(v),$$

obviously, necessary and sufficient conditions for  $\bar{y}_s$  to be an unbiased estimator of  $\bar{Y}_s$ , are

$$E[f_1(v)] = 1$$
 and  $E[f_2(v)] = 0$ . ... (3.3)

Consider now the class  $\Gamma$  of estimators  $\bar{y}_{i}$  which satisfy the conditions of (3.3).

Now

$$V(\bar{y}_s) = E\left[f_1^2(v)\left(\frac{1}{v} - \frac{1}{N}\right)S^2\right] + V[f_1(v)\bar{Y} + f_2(v)].$$
 (3.4)

In order to choose a good estimator from  $\Gamma$ , we are to minimise (3.4) by proper choices of  $f_1(v)$  and  $f_2(v)$ . The first expression on the right hand side of (3.4) is independent of  $f_3(v)$ ; so, for a proper choice of  $f_3(v)$ , we are to minimise

$$V[f_1(v)\overline{Y}+f_2(v)]$$

which is minimum if  $\bar{Y}f_1(v)+f_2(v)$  is constant for all values of v, i.e.,

$$\bar{Y}f_1(v) + f_2(v) = E[f_1(v)\bar{Y} + f_3(v)] = \bar{Y},$$
  
 $f_2(v) = \bar{Y}[1 - f_1(v)].$  ... (3.5)

Since the above solution of  $f_n(v)$  contains the unknown  $\tilde{Y}$ , the exact value of  $f_n(v)$  is not known unless  $f_1(v) = 1$ . Thus, if we choose  $f_1(v) = 1$ , the best estimator of  $\tilde{Y}$  would be  $\tilde{y}_r$ . However, in practical situations, when some a priori knowledge about  $\tilde{Y}$  is available, it seems appropriate to approximate  $f_n(v)$  by

$$f_{\mathbf{x}}(\mathbf{v}) = \bar{X}[1-f_{\mathbf{x}}(\mathbf{v})], \qquad \dots (3.6)$$

where  $\bar{X}$  is some a priori estimate of  $\bar{Y}$ . For example,  $\bar{X}$  may be taken as the estimate of the population mean of the same variate obtained from some previous survey etc. On the other hand, if no such information about  $\bar{Y}$  is available, it would be safe to take  $f_{\bar{X}}(v) = 0$ . To choose the optimum value of  $f_{\bar{X}}(v)$ , we have to minimise

$$Ef_1^2(v)\left(\frac{1}{v}-\frac{1}{N}\right)$$

subject to the condition that  $E[f_i(v)] = 1$ .

By Schwartz inequality we have

$$E\left[f_1^2(v)\left(\frac{1}{v} - \frac{1}{\overline{N}}\right)\right] \cdot E\left[\left(\frac{1}{v} - \frac{1}{\overline{N}}\right)^{-1}\right] \geqslant 1.$$
 (3.7)

The equality holds if and only if

$$f_1(v) = \left(\frac{1}{v} - \frac{1}{N}\right)^{-1} / \left[ E\left(\frac{1}{v} - \frac{1}{N}\right)^{-1} \right] = [Nv/(N-v)]/E[Nv/(N-v)]. \dots (3.8)$$

Thus, when some a priori estimate  $\overline{X}$  of  $\overline{Y}$  is available, the optimum estimate of  $\overline{Y}$  is given by

$$\bar{g}_{r(1)} = \frac{[N\nu/(N-\nu)]}{E[N\nu/(N-\nu)]} \, g_r + \bar{\chi} \, \left[ 1 - \frac{[N\nu/(N-\nu)]}{E[N\nu/(N-\nu)]} \right]. \quad ... \quad (3.9)$$

When no such information about  $\tilde{r}$  is available we may use the following estimator

$$\bar{g}_{r(2)} = \frac{[N \nu/(N-\nu)]}{E[N \nu/(N-\nu)]} \bar{g}_{r}.$$
 ... (3.10)

The two estimators are admissible in  $\Gamma$  in the sense that they minimise the first component of (3.4). Any estimator  $g_s$  different from either of them cannot be uniformly better than  $g_{r(1)}$  or  $g_{r(2)}$  because

 $V(\bar{y}_{r(1)}) < V(\bar{y}_s)$  for all populations where  $\bar{Y} = \bar{X}$ ;  $V(\bar{y}_{r(s)}) < V(\bar{y}_s)$  for all populations where  $\bar{Y} = 0$ .

Expression for  $E(N\nu/(N-\nu)]$ : Proceeding on similar lines given by the author (Pathak, 1961), it can be shown that

$$E[Nv/(N-v)] = N^{2} \sum_{m=1}^{n} \frac{\mathcal{P}_{12} \dots m}{(N-m)} \dots (3.11)$$

# SANKHYA: THE INDIAN JOURNAL OF STATISTICS: SERIES A

where

$$p_{12...m} = \begin{cases} 1 - {m \choose 1} (1 - 1/N)^n + \dots + (-)^m {m \choose m} (1 - m/N)^n \text{ for } m \leqslant n; \\ 0 \text{ otherwise.} \end{cases}$$

Thus, we see that it may be quite cumbersome to compute the estimators (3.9) and (3.10) in case of large samples owing to the difficulty of computing  $E[N\nu/(N-\nu)]$ . If, however, the sampling fraction n/N can be ignored, the estimators reduce to

$$\bar{g}_{\nu(1)}^* = \frac{\mathsf{v}}{E(\mathsf{v})} \; \bar{g}_{\nu} + \bar{X} \left[ 1 - \frac{\mathsf{v}}{E(\mathsf{v})} \right]; \qquad \dots \quad (3.12)$$

$$\bar{g}_{r(\mathbf{g})}^{\bullet} = \frac{v}{E(v)} \bar{g}_{r}. \qquad \dots (3.13)$$

It is easy to see that (3.13) is the well-known Horvitz-Thompson (1952) estimator in case of equal probability sampling. An interesting comparison between  $V(\bar{y}_{r(1)})$  and  $V(\bar{y}_{r})$  is made below.

4. Comparison between  $(\nabla \bar{y}_r)$  and  $\nabla (\hat{y}_{r(2)}^*)$ 

We have shown that

$$V(\bar{y}_r) = \frac{1^{n-1} + 2^{n-1} + \dots + (N-1)^{n-1}}{N^n} S^2;$$

and

$$\begin{split} & V(\hat{g}_{r(2)}^{*}) = E\left[V\left\{\frac{v}{E(v)}\hat{g}_{r}|v\right\}\right] + V\left[E\left\{\frac{v}{E(v)}\hat{g}_{r}|v\right\}\right] \\ & = E\left[\frac{v^{2}}{E^{2}(v)}\left(\frac{1}{v} - \frac{1}{N}\right)S^{2}\right] + \frac{\bar{Y}^{2}}{E^{2}(v)}V(v). \quad ... \quad (4.1) \end{split}$$

It can be shown that

$$\begin{split} E(\mathbf{v}) &= N \left[ 1 - \left( 1 - \frac{1}{N} \right)^n \right]; \\ E(\mathbf{v}^2) &= N \left[ 1 - \left( 1 - \frac{1}{N} \right)^n \right] + N(N-1) \left[ 1 - 2 \left( 1 - \frac{1}{N} \right)^n + \left( 1 - \frac{2}{N} \right)^n \right]; \\ \text{and} \qquad V(\mathbf{v}) &= N \left( 1 - \frac{1}{N} \right)^n - N^2 \left( 1 - \frac{1}{N} \right)^{2n} + N(N-1) \left( 1 - \frac{2}{N} \right)^n \\ & \therefore \quad V(\bar{y}^*_{r(2)}) &= \frac{S^2}{N^2 \left[ 1 - \left( 1 - \frac{1}{N} \right)^n \right]^2} \left[ N \left\{ 1 - \left( 1 - \frac{1}{N} \right)^n \right\} - \left\{ 1 - \left( 1 - \frac{1}{N} \right)^n \right\} \right] \\ & - (N-1) \left\{ 1 - 2 \left( 1 - \frac{1}{N} \right)^n + \left( 1 - \frac{2}{N} \right)^n \right\} \right] + \frac{\bar{Y}^2}{N^2 \left[ 1 - \left( 1 - \frac{1}{N} \right)^n \right]^2} \\ & \times \left[ N \left( 1 - \frac{1}{N} \right)^n - N^2 \left( 1 - \frac{1}{N} \right)^n + N(N-1) \left( 1 - \frac{2}{N} \right)^n \right] \dots (4.2) \end{split}$$

$$\begin{array}{l} \text{Now} \\ \overline{V(\mathcal{G}_r)} - \overline{V(\mathcal{G}_{r(1)}^*)} = S^2 \\ \hline \\ \frac{1^{n-1} + 2^{n-1} + \ldots + (N-1)^{n-1}}{N^n} - \frac{(N-1)}{N^2} \frac{\left[ \left(1 - \frac{1}{N}\right)^n - \left(1 - \frac{2}{N}\right)^n \right]}{\left[1 - \left(1 - \frac{1}{N}\right)^n\right]^2} \\ \hline \end{array} \right]$$

$$-\frac{p_{3}\left[\left(1-\frac{1}{N}\right)^{n}-N\left(1-\frac{1}{N}\right)^{s_{1}}+(N-1)\left(1-\frac{2}{N}\right)^{n}\right]}{\left[1-\left(1-\frac{1}{N}\right)^{n}\right]^{s_{1}}}$$

$$= C_1 S^3 - C_2 \overline{Y}^3$$
. (say) ... (4.3)

Thus  $g_r$  is better than  $g_{r(s)}^*$  if

$$\frac{S^2}{\bar{T}^2} < \frac{C_1}{\bar{C}_1}$$

and worse if

$$\frac{S^2}{\widehat{\nabla}^2} > \frac{C_2}{C_1}$$

Approximate values of  $C_1$  and  $C_2$  for large populations correct up to terms of order  $N^{-2}$ , are given by

$$\begin{aligned} & O_1 = \frac{1}{2nN} + \frac{5(n-1)}{12nN^3}; \\ & C_3 = \frac{(n-1)}{2nN} - \frac{(n-1)(n-2)}{2nN^3}. & \dots & (4.4) \end{aligned}$$

The above comparison shows that if the square of the population coefficient of variation exceeds (n-1), then  $\bar{y}_{r(1)}$  has smaller variance than  $\bar{y}_r$ . Moreover, if we have some a priori knowledge of  $\bar{Y}$ , it would be more pertinent to compare  $\bar{y}_r$  and  $\bar{y}_{r(1)}^*$ . It can be seen on similar lines that  $\bar{y}_r$  is better than  $\bar{y}_{r(2)}^*$  if

$$\frac{S^{\mathbf{a}}}{(\overline{Y}-\overline{X})^{\mathbf{a}}} < \frac{C_{\mathbf{a}}}{C_{\mathbf{1}}},$$

and worse otherwise. This result shows that if  $\bar{X}$  provides a close approximation to  $\bar{Y}$ , it is always better to use  $g_{r(1)}^*$  rather than  $\bar{y}_r$ .

We now state the following admissibility property of  $\bar{y}_r$ .

Theorem 1: If squared error be the loss function,  $\tilde{y}$ , is admissible among all functions of  $\tilde{y}$ , and v.

Proof: Let 
$$t = \bar{y}_* + f(\bar{y}_*, v)$$

be a function of  $\bar{g}$ , and v. Suppose that t is uniformly better than  $g_s$ . Now by hypothesis,

$$R(t) = E(g_r - \overline{Y})^3 + E[f(g_r, v)]^3 + 2E[(g_r - \overline{Y})f(g_r, v)] \leqslant E(g_r - \overline{Y})^3 \quad \dots \quad (4.5)$$
 holds for all  $Y_1, Y_2, \dots, Y_N$ . Take in particular  $Y_1 = Y_2 = \dots = Y_N = C$  (say). Then the above relation implies that

the spove relation implies that 
$$f(C, v) = 0, \qquad \dots \tag{4.6}$$

Since the choice of C is arbitrary, it follows that  $f(g_r, v)$  is identically zero, which proves the above theorem.

## 5. ESTIMATION OF VARIANCE

We now turn to the problem of estimating the population variance from a simple random sample (with replacement). The usual estimator of the population variance

$$\sigma^{4} = N^{-1} \Sigma (Y_{i} - \overline{Y})^{2}$$

is given by the sample variance

$$s^2 = \frac{1}{(n-1)} \sum (y_i - g)^2 = \frac{1}{2n(n-1)} \sum_{i \neq i'} (y_i - y_{i'})^2.$$
 (5.1)

In this section, we derive an estimator uniformly better than sa.

Theorem 2: For any convex (downwards) loss function, an estimator uniformly better than s\* is given by

$$\boldsymbol{s}_{r}^{2} = \left[ \frac{O_{r}(n) - O_{s}(n-1)}{O_{s}(n)} \right] \boldsymbol{s}_{s}^{2}, \qquad \dots (5.2)$$

where

$$C_s(n) = v^n - {v \choose 1}(v-1)^n + \cdots + {v \choose v-1} \cdot 1^n,$$

and

$$s_{2}^{3} = \begin{cases} \frac{1}{(\nu-1)} & \Sigma & (y_{(i)} - g_{\nu})^{3} & \text{if } \nu > 1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Since the 'order-statistic', T, is sufficient, by Rao-Blackwell theorem, an estimator uniformly better than s<sup>2</sup> is given by

$$E[e^{2}|T] = E\left[\frac{1}{2n(n-1)}\sum_{i\neq i'}(y_{i}-y_{i'})^{2}|T\right] = E\left[\frac{1}{2}(y_{1}-y_{2})^{2}|T\right]. \quad ... \quad (5.3)$$

When v = 1, (5.3) is obviously zero. To derive (5.3) when v > 1, we observe that

$$P[x_{1} = x_{(i)}, x_{1} = x_{(i')}|T] = \frac{\frac{1}{N^{3}} \sum_{i'''} \frac{(n-2)!}{\alpha_{(1)}! \dots \alpha_{(\nu)}!} \left(\frac{1}{N}\right)^{\alpha_{(1)}} \frac{1}{N} \frac{1}{N}^{\alpha_{(\nu)}}}{\sum_{i'} \frac{n!}{\alpha_{(1)}! \dots \alpha_{(\nu)}!} \left(\frac{1}{N}\right)^{\alpha_{(1)}} \left(\frac{1}{N}\right)^{\alpha_{(\nu)}}} \dots (5.4)$$

$$(i \neq i' = 1, 2, ..., v)$$

where  $\Sigma'$  means summation over all integral  $\alpha$ 's such that

$$\alpha_{(1)} + \alpha_{(2)} + \cdots + \alpha_{(r)} = \pi$$
 and  $\alpha_{(i)} > 0$  for  $i = 1, 2, ..., v$ ;

and  $\Sigma''$  means summation over all integral  $\alpha$ 's such that

$$\alpha_{(1)} + \alpha_{(2)} + ... + \alpha_{(r)} = n-2, \ \alpha_{(i)} > 0, \ \alpha'_{(i)} > 0 \ \text{and} \ \alpha_{(k)} > 0 \ \text{; for } k \neq i \neq i' = 1, 2, ..., v.$$

It follows from Lemma 1 given by the author (Pathak, 1961) that

$$\Sigma' \frac{n!}{\alpha_{(1)}! \dots \alpha_{(r)}!} = C_r(n);$$

$$\Sigma'' \frac{(n-2)!}{\alpha_{(1)}! \dots \alpha_{(r)}!} = C_r(n-2) + 2C_{r-1}(n-2) + C_{r-2}(n-2)$$

$$= \frac{C_r(n) - C_r(n-1)}{v(v-1)} \qquad \dots (5.5)$$

$$P[x_1 = x_{(i)}, x_1 = x_{(i')} | T] = \frac{C_s(n) - C_s(n-1)}{\sqrt{(v-1)} C_s(n)} \dots (5.6)$$

$$(i \neq i' = 1, 2, \dots, v).$$

Thus, if v > 1

$$E\left[\frac{(y_{t1}-y_{t2})^{2}}{2} \mid T\right] = \Sigma \frac{(y_{t1}-y_{t(t)})^{3}}{2} P[x_{1} = x_{(t)}, x_{2} = x_{(t')} \mid T]$$

$$= \frac{C_{s}(n) - C_{s}(n-1)}{C_{s}(n)} \left[\frac{1}{2\sqrt{(v-1)}} \Sigma (y_{t0} - y_{ti'})^{3}\right]$$

$$= \frac{C_{s}(n) - C_{s}(n-1)}{C_{s}(n)} \cdot \frac{1}{(v-1)} \Sigma (y_{t0} - \hat{y}_{s})^{3}. \dots (5.7)$$

Therefore, for any v, 
$$E(s^2|T) = E\left[\frac{(y_1 - y_2)^3}{2}|T\right] = \frac{C_r(n) - C_r(n-1)}{C_r(n)} s_d^3, \dots$$
 (5.8)

where  $s_d^2$  has been defined earlier.

In practice the estimator  $\sigma_r^2$  requires the knowledge of the ratio  $\frac{C_m(n-1)}{C_m(n)}$ . Table 3 gives values of  $\frac{C_m(n-1)}{C_m(n)}$  correct to seven places of decimals for  $1 \le m \le n \le 50$ ; and were computed from values of  $\frac{C_m(n)}{m}$  tabulated by Gupta (1950).

The following results are direct consequences of Theorem 2.

If there are two characters Y and Z, the covariance between Y and Z is defined by

$$\sigma_{(yz)} = \frac{1}{N} \Sigma(\bar{Y}_j - \bar{Y})(Z_j - \bar{Z}). \qquad ... (5.9)$$

The usual estimator of  $\sigma_{in}$  is given by

$$s_{(yz)} = \frac{1}{(n-1)} \sum (y_i - \hat{y})(z_i - \hat{z}).$$
 (5.10)

## SANKHYA: THE INDIAN JOURNAL OF STATISTICS: SERIES A

Corollary 1: It follows from Theorem 2 that an estimator better than  $s_{(yz)}$  is given by

$$s_{r(yz)} = \begin{cases} \frac{O_r(n) - O_r(n-1)}{O_r(n)} & s_{d(yz)} & \text{if } v > 1 \\ 0 & \text{otherwise,} \end{cases}$$
 (5.11)

where  $s_{d(ys)}$  is the sample covariance based on the distinct units observed in the sample.

The above theorem can be used to derive an unbiased ratio estimator which is better than Hartley-Ross unbiased ratio estimator (1954). In the sampling scheme under consideration, Hartley-Ross estimator is given by

$$g_{\scriptscriptstyle R} = \bar{r} \bar{Z} - \frac{n}{(n-1)} (g - \bar{r} z) = f \bar{Z} - \frac{1}{(n-1)} \sum \left( \frac{y_i}{z_i} - \bar{r} \right) (z_i - \bar{z}),$$

where  $\tilde{Z}=N^{-1}\sum Z_i$ ,  $\tilde{r}=1/n\sum y_i/z_i$  and  $z_i$  is the value of the Z-characteristic, an auxiliary characteristic related to Y-characteristic, of the *i*-th sample unit.

Corollary 2: An estimator better than g<sub>R</sub> is given by

$$E[\bar{g}_R|T] = \begin{cases} \bar{r}_r \bar{Z} + \frac{C_r(n) - C_r(n-1)}{C_r(n)} \cdot \frac{\mathbf{v}}{(\mathbf{v}-1)} \left( \bar{y}_r - \bar{r}_r \bar{z}_r \right) & \text{if } \mathbf{v} > 1 \\ \bar{r}_r \bar{Z} & \text{otherwise,} \end{cases} ... \tag{5.12}$$

where

$$\bar{r}_r = \frac{1}{v} \sum \frac{y_{(i)}}{z_{(i)}}, \ \bar{z}_r = \frac{1}{v} \sum z_{(i)}.$$

Murthy (1961) has extended the idea of ratio estimators to product estimators. Similar to the well-known ratio estimator  $\frac{\bar{y}\bar{Z}}{\bar{z}}$  he has considered the product estimator

$$\frac{\bar{g}\hat{z}}{\bar{z}}$$
 ... (5.13)

for estimating  $\bar{Y}$ .

Corollary 3: It can be verified that an estimator better than  $\frac{\hat{y}z}{z}$  is given by

$$E\left[\frac{\tilde{y}z}{\tilde{z}} \mid T\right] = \frac{1}{\tilde{z}} \left[ \frac{\sum y_{(i)}z_{(i)}}{y} - \frac{(n-1)}{n} \cdot s_{r(yz)} \right] \dots (5.14)$$

where areas is given by (5.11).

Finally, the almost unbiased product estimator of Murthy (1961), is given which makes  $\frac{g_z}{Z}$  almost unbiased. This estimator is given by

$$P_{a} = \frac{n}{(n-1)} \cdot \frac{\bar{y}\bar{s}}{\bar{Z}} - \frac{1}{(n-1)} \cdot \frac{\Sigma y_{i}z_{i}}{n\bar{Z}}.$$
 ... (5.15)

Corollary 4: An estimator better than Po is given by

$$E[P_s \mid T] = \frac{\sum_{(i)} y_{(i)}}{\sqrt{Z}} - \frac{s_{r(p)}}{Z}. \qquad ... \quad (5.16)$$

6. Some estimators of  $V(\bar{y}_s)$ 

Some unbiased estimators of  $V(\bar{y}_*)$  are given by

(I) 
$$v_1(\tilde{y}_r) = \left[ \frac{1^{n-1} + 2^{n-1} \dots + (N-1)^{n-1}}{N^n} \right] \frac{N}{(N-1)} \delta^2;$$

$$(\text{II}) \qquad v_2(\tilde{y}_r) = \left[ \begin{array}{cc} \frac{1^{n-1} + 2^{n-1} + \ldots + (N-1)^{n-1}}{N^n} \end{array} \right] \frac{N}{(N-1)} \frac{[C_r(n) - C_r(n-1)]}{C_r(n)} \hat{s}_d^2 :$$

(III) 
$$v_3(\bar{y}_{\nu}) = \frac{C_{\nu-1}(n-1)}{C_{\nu}(n)} s_d^2$$
;

(IV) 
$$v_4(\bar{y}_r) = \left[ \left( \frac{1}{r} - \frac{1}{N} \right) + \frac{(N-1)}{(N^n - N)} \right] s_d^2;$$

$$(V) \qquad v_b(\vec{y}_r) = \left[ \left( \frac{1}{\nu} - \frac{1}{N} \right) + N^{1-\kappa} \left( 1 - \frac{1}{\nu} \right) \right] s_d^2 \quad \text{(to be used for } \nu > 1).$$

The estimate (II) is known to be uniformly better than (I). It appears difficult to give direct proofs of relative efficiencies of these estimators. The estimators (IV) and (V) were given by Des Raj and Khamis (1958). The estimator (V) is conditionally unbiased for v > 1. Des Raj and Khamis suggested the use of (V) for v > 1.

It is easy to see that

$$v_4 = v_5 \frac{N^n}{(N^n - N)}$$
 ... (6.1)

A little comparison will, now, show that the conditional variance of (V) is less than the variance of (IV). The amount of decrease in the variance is given by

$$V(v_4) - V(v_5/v > 1) = \frac{1}{2\sqrt{n-1}} E(v_4^2).$$
 (6.2)

In general, this leads to the conclusion that any estimator  $\hat{\sigma}^2$  of  $\sigma^2$  which is unbiased for  $\sigma^2$  and is equal to zero for  $\nu=1$ , can be reduced to give a conditionally unbiased estimate of  $\sigma^2$  for  $\nu>1$  whose conditional variance will be less than the variance of  $\hat{\sigma}^2$ . This conditionally improved estimator is related with  $\hat{\sigma}^2$  by the following equation.

$$\hat{\sigma}^{\mathbf{a}} = \hat{\sigma}_{lm}^{\mathbf{g}} \left[ \frac{N^n}{(N^n - N)} \right] \qquad \dots \quad (6.3)$$

where  $\sigma_{-}^2$  stands for the conditionally improved estimator of  $\sigma^2$ .

## SANKHYA: THE INDIAN JOURNAL OF STATISTICS: Subias A

Numerical example. To study the relative efficiency of the estimators of  $V(\bar{y}_s)$ , we consider the following three populations given by Yates and Grundy (1953).

TABLE	1,	THREE	POPULATIONS	GIVEN	BY
		YATE8	AND GRUNDY		

population	A	В	$\boldsymbol{c}$
unit	Yj	Yj	Y
1	0.5	0.8	0.2
2	1.2	1.4	0.6
3	2.1	1.8	0.9
4	3.2	2.0	0.8
ΣΥj	7.0	6.0	2.5

These populations were deliberately chosen by them as being more extreme than will be normally encountered in practice.

The table below gives variances of unbiased estimators of  $V(\bar{y}_r)$  when n=3.  $V(v_1)$  is not given as  $V(v_1) > V(v_2)$ .

TABLE 2. VARIANCES OF UNBIASED ESTIMATORS OF V(Y)

population	$V(Y_p)$	$V(v_2)$	$V(v_3)$	$V(v_4)$	V(v5   > > 1)
A	0.29823	0.04940	0.05222	0.09017	0.07897
В	0.06125	0.00220	0.00232	0.00398	0.00348
С	0.020964	0.000279	0.000293	0.000490	0.000432

The results show that for the three populations

$$V(v_s) < V(v_s) < V(v_s | v > |) < V(v_s).$$
 (6.4)

Thus  $v_2$  appears to be most efficient estimator of  $V(\hat{y}_s)$ .

For n=2,  $v_1$  and  $v_3$  are identical. The comparison thus strongly suggests the use of  $v_2$  for estimating  $V(q_r)$ .

For getting estimators of V(t), where t is any unbiased estimator of  $\overline{Y}$ , the following procedure may be adopted.

$$v(t) = t^2 - \operatorname{est}(\bar{Y}^2), \qquad \dots \qquad (6.5)$$

where est  $(\tilde{Y}^2)$  stands for an unbiased estimator of  $\tilde{Y}^2$  and can be obtained from any of the relations

est 
$$(\bar{Y}^2) = v_i(g_r) - g_r^2$$
  $(i = 1, 2, 3, 4, 5)$ . ... (6.6)

From the example considered, it is expected that

est 
$$(\bar{Y}^2) = v_i(\bar{y}_r) - \bar{y}_r^2$$
  $(i = 2, 3)$  ... (6.7)

would fare better than the remaining estimators of  $\overline{Y}^2$ .

TABLE 3. VALUES OF  $\frac{C_m(n-1)}{C_m(n)}$ 

n →	2	3	4	5	8	7	8	0
ı				1.000000 for	all n			
2	0	. 3333333	.4285714	.4666667	.4838710	.4920635	.4960630	. 49803
3		0	. 1666667	. 2400000	. 2777778	. 2990033	.3115942	.31933
4			0	. 1000000	. 1538462	. 1857143	. 2057613	.21891
5				0	.0866667	. 1071429	.1333333	. 15106
в					0	.0476190	.0789474	.10052
7						0	.0357143	.08080
8							0	. 02777
9								0
n → m	10	11	12	13	14	15	16	17
t			1	1.000000 for	م للم			
2	.4990215	.4995112	4997557	.4998779	.4999390	.4999695	.4999847	. 49999
3	. 3242229	.3273569	. 3293923	. 3307253	.3316032	. 3321838	. 3325687	. 33282
4	. 2278258	. 2339966	. 2383479	.2414585	. 2437059	. 2453432	. 2465438	. 24742
5	. 1634568	. 1723544	. 1788676	. 1837! 18	. 1873611	.1901391	.1922722	. 19392
6	.1159154	. [27179]	.1355998	.1420028	. 1489395	. 1507901	.1538225	. 15623
7	.0785714	.0918937	.1019882	.1097724	.1158627	.1208858	.1245448	. 12765
8	.0480000	.0631313	.0747043	.0837155	.0908370	.0965357	.1011446	. 10490
9	.0222222	.0389610	.0518519	.0819607	.0700084	.0764970	.0817858	.08613
10	0	.0181818	. 0322581	.0433566	.0522416	.0594466	.0653539	.07024
11		0	.0151515	.0271493	. 0367965	.0446549	.0511277	.05651
12			.0	.0128205	.0231660	.0316239	.0386164	.04445
13				0	.0109890	.0200000	.0274725	.03372
14					0	.0095238	.0174419	.02408
16						0	.0083333	.01534
16							0	.00735
17								0

SANKHYA: THE INDIAN JOURNAL OF STATISTICS: SERIES A

TABLE 3. VALUES OF  $\frac{C_m(n-1)}{C_m(n)}$  (Contd.)

n → m	18	19	20	21	22	23	24	25
1				1.000000 for	all n			
ż	.4000062	. 4999981	.4999990	.4999995	. 4999998	. 4999999	. 4999999	.50000
3	.3329943	.3331075	.3331828	. 3332330	. 3332665	. 3332888	. 3333036	. 33331
ĭ	. 2480833	. 2485693	2489308	. 2492003	.2494015	.2495518	. 2498643	. 24974
5	. 1952045	.1062073	1989942	. 1976139	.1981031	. 1984904	.1987974	. 19904
6	.1581551	. 1597031	.1609542	. 1619699	.1627974	.1634737	.1640281	. 16448
7	1301924	. 1322655	.1339720	. 1353836	.1365562	. 1375340	.1383520	. 13903
ġ	1080005	. 1105635	.1126991	.1144881	.1159936	.1172659	,1183450	.11926
9	.0897460	.0027608	.0952950	.0974369	.0992581	.1008081	.1021373	.10327
10	.0743243	.0777549	.0806574	.0831272	.0852394	.0870540	.0880193	.08997
11	.0610250	.0848389	.0680820	.0708560	.0732409	.0753009	.0770878	.07864
12	.0493679	.0535361	.0570949	.0801513	.0627902	.0650794	.0670738	18880.
13	.0390147	.0435116	.0473637	.0506832	.0535590	.0560624	.0582511	.08017
14	0207189	.0345220	.0386476	.0422127	.0453100	.0480141	.0503862	.05247
15	.0212963	.0263854	.0307669	.0345620	.0378671	.0407598	.0433020	.04554
16	.0136054	.0189618	.0235845	.0275957	.0310960	.0341857	.0368697	.03926
17	.0065359	.0121457	.0169935	.0212082	.0248926	.0281295	.0309861	.03351
18	0	.0058480	.0109091	.0153161	.0191746	.0225696	.0255705	.02823
19	-	0	.0052632	.0098522	.0138756	.0174208	.0205584	.02334
20		•	0	.0047619	.0089419	.0126294	.0158973	.01880
21				0	.0043290	.0081522	.0115440	.01456
22					0	.0039526	.0074627	.01059
23					-	0	.0038232	.000%5
24							0	.00333
25								0
n →	26	27	28	29	30	31	32	33
278	26	27				31	32	33
1	26	27		1.0000000	for all n	31	32	33
1 2				1.0000000	for all n for n > 24			
1 2 3	.3333201	. 3333245	. 3333275	1.0000000 .5000000 .3333294	for all n for n > 24 .3333307	. 3333316	.3333322	. 33333
1 2 3 4	.3333201	. 3333245	.3333275	1.0000000 .5000000 .3333294 .2499208	for all n for n > 24 .3333307 .2490404	. 3333316 . 2499553	.3333322	. 33333
1 2 3 4 5	.3333201	.3333245 .2498587 .1993895	.3333275 .2498940 .1995125	1.0000000 .5000000 .3333294 .2499206 .1996106	for all n for n > 24 .3333307 .2490404 .1996889	.3333316 .2499553 .1097514	.3333322 .2499665 .1998012	. 33333 . 24997 . 19984
1 2 3 4 5	.3333201 .2498115 .1992352 .1648585	. 3333245 . 2498587 . 1993895 . 1651676	.3333275 .2498940 .1995125 .1654230	1.0000000 .5000000 .3333294 .2499208 .1996108 .1656342	for all n for n > 24 .3333307 .2490404 .1996889 .1658090	.3333316 .2499553 .1097514 .1659539	.3333322 .2499665 .1998012 .1660741	.33333 .24997 .19984
1 2 3 4 5 6 7	.3333201 .2498115 .1992352 .1648585 .1396157	.3333245 .2498587 .1993895 .1651676 .1401025	.3333275 .2498940 .1995125 .1654230 .1405137	1.0000000 .5000000 .3333294 .2499206 .1996106 .1656342 .1408616	for all n for n > 24 .3333307 .2490404 .1996889 .1658090 .1411565	.3333316 .249953 .1997514 .1659539 .1414068	.3333322 .2499665 .1998012 .1660741 .1410195	.33333 .24997 .19984 .16617
1 2 3 4 5 6 7	.3333201 .2498115 .1992352 .1648585 .1396157 .1200468	. 3333245 . 2498587 . 1993895 . 1651676 . 1401025 . 1207173	.3333275 .2498940 .1995125 .1654230 .1405137 .1212923	1.0000000 .5000000 .3333294 .2499206 .1996106 .1656342 .1408616 .1217864	for all n for n > 24     .3333307     .2490404     .1996889     .1658090     .1411565     .1222119	.3333316 .2499553 .1997514 .1659539 .1414068 .1225788	.3333322 .2499665 .1998012 .1660741 .1410195 .1228958	. 33333 . 24997 . 19984 . 16617 . 14180
1 2 3 4 5 6 7 8	.3333201 .2498115 .1992352 .1648585 .1396157 .1200468 .1042646	. 3333245 . 2498587 . 1993895 . 1651676 . 1401025 . 1207173 . 1051161	.3333275 .2498940 .1995125 .1654230 .1405137 .1212923 .1058542	1.0000000 .5000000 .3333294 .2499208 .1996106 .1656342 .1408616 .1217864 .1064955	for all n for n > 24     .3333307     .2490404     .1996889     .1658090     .1411565     .1222119     .1070539	.3333316 .2499553 .1997514 .1659539 .1414068 .1225788 .1075409	.3333322 .2499665 .1998012 .1660741 .1416195 .1228958	.33333 .24997 .19984 .16617 .14180 .12316
1 2 3 4 5 6 7	.3333201 .2498115 .1992352 .1648585 .1396157 .1200468	. 3333245 . 2498587 . 1993895 . 1651676 . 1401025 . 1207173	.3333275 .2498940 .1995125 .1654230 .1405137 .1212923 .1058542 .0930737	1.0000000 .5000000 .3333294 .2499208 .1996106 .1656342 .1408616 .1217864 .1064955	for all n for n > 24     .3333307     .2490404     .1996889     .1658090     .1411565     .1222119	.3333316 .2499553 .1997514 .1659539 .1414068 .1225788	.3333322 .2499665 .1998012 .1660741 .1410195 .1228958	.33333 .24997 .19984 .16617 .14180 .12316
1 2 3 4 5 6 7 8 9	.3333201 .2488115 .1992352 .1648585 .1386157 .1200468 .1042646 .0911518	.3333245 .2498687 .1993895 .1651676 .1401025 .1207173 .1051181 .0921776	.3333275 .2498940 .1995125 .1654230 .1405137 .1212923 .1058542 .0930737	1.0000000 .5000000 .3333294 .2499208 .1996106 .1656342 .1408616 .1217864 .1064955 .0938586	for all n for n > 24 .333307 .2490404 .1996889 .1658090 .1411565 .1222119 .1070539 .0945470 .0839795	.3333316 .2499553 .1997514 .1659539 .1414068 .1226788 .1075409 .0951537	.3333322 .2400665 .1998012 .1660741 .1416195 .1229058 .1079065 .0956879	. 33333 . 24997 . 19984 . 16617 . 14180 . 12316 . 10833 . 09615
1 2 3 4 5 6 7 8 9 10 11 12	.3333201 .2498115 .1992352 .1648585 .1396157 .1200468 .1042646 .0911518 .0800030	.3333245 .2498587 .1993895 .1851676 .1401025 .1207173 .1051181 .0921776 .0811943 .0716870	.3333275 .2498940 .1995125 .1654230 .1405137 .1212923 .1058542 .0930737 .0822415	1.0000000 .5000000 .3333294 .2409208 .1996108 .1656342 .1408616 .1217864 .1064955 .0938588 .0831643	for all n for n > 24 .3333307 .2499404 .1996889 .1658090 .1411565 .1222119 .1070639 .0945476 .0839796	.3333316 .2499553 .1997514 .1659539 .1414068 .1226788 .1075409 .0961637 .0847012	.3333322 .2499665 .1998012 .166071 .1410195 .1228958 .1079065 .0956879 .0853413	. 33333 . 24997 . 19984 . 16617 . 14180 . 12316 . 10833 . 09615
1 2 3 4 5 6 7 8 9 10	.3333201 .2498115 .1902352 .1648585 .1396157 .1200468 .1042646 .0911518 .0800030 .0703490 .0618648	.3333245 .2498587 .1993895 .1651678 .1401025 .1207173 .1051181 .0921776 .0811943 .0716970 .0633807	.3333276 .2498940 .1995125 .1854230 .1405137 .1212923 .1058642 .0930737 .0822415 .0728875	1.0000000 .5000000 .3333294 .2409206 .1996106 .1956342 .1408616 .1217864 .1064955 .0938586 .0831643 .0739417	for all n for n > 24	.3333316 .2499553 .2499553 .1659539 .1414088 .1225788 .1075409 .0961637 .0847012 .0757102	.3333322 .2409665 .1998012 .1660741 .1416195 .1228958 .1070065 .0056879 .0853413 .0764525 .0686950	. 33333 . 24997 . 19984 . 16617 . 14180 . 12316 . 10833 . 09615 . 08591 . 07711
1 2 3 4 5 6 7 8 9 10 11 12 13	.3333201 .2498115 .1992352 .1648585 .1396157 .1200468 .1042646 .0911518 .0800030 .0703490 .0618648	.3333245 .2498687 .1993895 .1851878 .1401025 .1207173 .1051181 .0921776 .0811943 .0716970 .0633807 .0539625	.3333275 .2498940 .1995125 .1654230 .1405137 .1212923 .1068642 .0930737 .0822415 .0728875 .0648867	1.0000000 .5000000 .3333294 .2409208 .1996106 .1656342 .1408616 .1217884 .1064955 .0938586 .0831643 .0739417 .0658656	for all n for n > 24	.3333316 .2499553 .1997514 .1850539 .1414088 .1225788 .1076409 .0981637 .0847012 .0757102 .0478054 .04809026	.3333322 .2400665 .1998012 .1660741 .1416195 .1228958 .1070665 .0056879 .0853413 .0764525 .0686950 .0618362	.33313 .24997 .19984 .16617 .14180 .12316 .10833 .09615 .08591 .07711
1 2 3 4 5 6 7 8 9 10	.3333201 .2498115 .1902352 .1648585 .1396157 .1200468 .1042646 .0911518 .0800030 .0703490 .0618648	.3333245 .2498587 .1993895 .1651678 .1401025 .1207173 .1051181 .0921776 .0811943 .0716970 .0633807	.3333276 .2498940 .1995125 .1854230 .1405137 .1212923 .1058642 .0930737 .0822415 .0728875	1.0000000 .5000000 .3333294 .2409206 .1996106 .1956342 .1408616 .1217864 .1064955 .0938586 .0831643 .0739417	for all n for n > 24	.333316 .2499553 .1997514 .1650539 .1414068 .1225768 .1076409 .0851637 .0847012 .0737102 .0078548 .0609028	.3333322 .2409665 .1998012 .1660741 .1416195 .1228958 .1070065 .0056879 .0853413 .0764525 .0686950	.33333 .24997 .19984 .16617 .14180 .12316 .10833 .09615 .08591 .07711
1 2 3 4 5 6 7 8 9 10 11 12 13	.3333201 .2498115 .1992352 .1648585 .1396157 .1200468 .1042646 .0911518 .0800030 .0703490 .0618648	.3333245 .2498687 .1993895 .1851878 .1401025 .1207173 .1051181 .0921776 .0811943 .0716970 .0633807 .0539625	.3333275 .2498940 .1995125 .1654230 .1405137 .1212923 .1068642 .0930737 .0822415 .0728875 .0648867	1.0000000 .5000000 .3333294 .2409208 .1996106 .1656342 .1408616 .1217884 .1064955 .0938586 .0831643 .0739417 .0658656	for all n for n > 24	.3333316 .2499553 .1997514 .1850539 .1414088 .1225788 .1076409 .0981637 .0847012 .0757102 .0478054 .04809026	.3333322 .2400665 .1998012 .1660741 .1416195 .1228958 .1070665 .0056879 .0853413 .0764525 .0686950 .0618362	.33333 .24997 .19984 .16617 .14180 .12316 .10833 .09615 .08591 .07711 .06944 .06247
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	.3333201 .2498115 .1992352 .1648585 .1396157 .1200468 .1042646 .0911518 .0800030 .0703490 .0618848 .0643172 .0475339	.3333245 .2498587 .1993895 .1851676 .1401025 .1207173 .1051161 .0921776 .0811943 .0716970 .0633807 .0559625 .0493009 .0432760	.3333276 .2498940 .1995125 .1654230 .1405137 .1212923 .1068642 .0930737 .0728876 .0648867 .0574068 .0509763	1.0000000 .5000000 .3333294 .2409208 .1996108 .1656342 .1408616 .1217864 .1064955 .0938586 .0831643 .0739417 .0668656 .0587037 .0522851 .0441810	for all n for n > 24 .3333307 .2490404 .1996889 .1658090 .1411565 .1222110 .1070639 .0945470 .0839795 .0748775 .0669162 .0598633 .0535481 .0478422 .0426467	.333316 .2499553 .1997514 .1650539 .1414088 .1225788 .1075409 .0961637 .0847012 .078518 .0609026 .0648833 .049084 .0439505	.3333322 .2499665 .1998012 .1660741 .1416195 .1228958 .1070685 .0056879 .0853413 .0704525 .0688950 .0618362 .0557068	.33333 .24997 .19984 .16617 .14180 .12316 .10933 .09615 .08591 .07711 .06947 .05662
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18	.3333201 .2498115 .1992352 .1448585 .1396157 .1200468 .1042646 .0911518 .0800030 .0703490 .0618848 .0643172 .0475339 .0413846 .0367685	.3333245 .2498687 .1993895 .1851878 .1401025 .1207173 .1051181 .0921776 .0811943 .071687 .0539625 .0493009 .0432760 .0377778 .0327276	.3333275 .2498940 .1995125 .1854230 .1405137 .1212923 .1058542 .0930737 .0822415 .0728875 .0646867 .0574068 .0508763 .0449662 .0395768	1.0000000 .5000000 .3333294 .2499208 .1996108 .1656342 .1498616 .1217884 .1064955 .0938586 .0831643 .0739417 .0658656 .0587037 .0522861 .0464810 .0411922 .0363408	for all n for n > 24	.3333316 .2499553 .1997514 .1659539 .1414068 .1225768 .1075409 .0951637 .0847012 .0757102 .0478348 .0409028 .0448833 .0490884 .0439595 .0392793	.3333322 .2499665 .1998012 .1660741 .1410195 .1228958 .1079065 .0050879 .0853413 .0704525 .0684950 .0618362 .0557968 .0501758 .0451473 .0405439	.33333 .24997 .19984 .16617 .14180 .12316 .10833 .09615 .08591 .07711 .06944 .06297 .05662 .05117 .04162
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	.3333201 .2498115 .1992352 .1648585 .1396157 .1200468 .1042846 .0911518 .0810030 .0703490 .0618848 .0643172 .0475339 .0413846 .0357685	.3333245 .2498587 .1993895 .1851676 .1401025 .1207173 .1051161 .0921776 .0811943 .0716970 .0633807 .0559625 .0493009 .0432760	.3333276 .2498940 .1995125 .1654230 .1405137 .1212923 .1068642 .0930737 .0728876 .0648867 .0574068 .0509763	1.0000000 .5000000 .3333294 .2409208 .1996108 .1656342 .1408616 .1217864 .1064955 .0938586 .0831643 .0739417 .0668656 .0587037 .0522851 .0441810	for all n for n > 24 .3333307 .2490404 .1996889 .1658090 .1411565 .1222110 .1070639 .0945470 .0839795 .0748775 .0669162 .0598633 .0535481 .0478422 .0426467	.333316 .2499553 .1997514 .1650539 .1414088 .1225788 .1075409 .0961637 .0847012 .078518 .0609026 .0648833 .049084 .0439505	.3333322 2409665 1998012 1660741 1416195 1228958 1070065 0056879 0853413 0764525 0688950 0618362 0557068 0501756 0451473 0405439 0363044	.33333 .24997 .19984 .16617 .14180 .12316 .10833 .09615 .08591 .07711 .06944 .06297 .05662 .05117 .04162
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18	.3333201 .2498115 .1992352 .1448585 .1396157 .1200468 .1042646 .0911518 .0800030 .0703490 .0618848 .0643172 .0475339 .0413846 .0367685	.333245 .2498687 .1993996 .1851676 .1401026 .1207173 .1051161 .0921776 .0633607 .0530025 .0493009 .0432780 .0377778 .0327876 .0237878	.3333275 .2498940 .1995125 .1854230 .1405137 .1212923 .1058542 .0930737 .0822415 .0728875 .0646867 .0574068 .0508763 .0449662 .0395768	1.000000 .500000 .500000 .3333294 .2409208 .1996108 .1656342 .1408616 .1217884 .1044955 .093858 .0831643 .0739417 .068865 .0587037 .0522851 .044810 .0411922 .04318649 .0318649 .0318649	for all n for n > 24	.3333316 .2499553 .1997514 .1659539 .1414068 .1225768 .1075409 .0951637 .0847012 .0757102 .0478348 .0409028 .0448833 .0490884 .0439595 .0392793	.3333322 .2400665 .1998012 .1600741 .1410195 .1228958 .0050879 .0853413 .0704625 .066850 .0618362 .0657068 .0601756 .0601756 .0601756	.33333 24997 19984 16617 14189 10933 .09615 08791 06944 06267 05662 05117 04822 04169 03752
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21	.3333201 2498115 1992332 1648585 1398167 1200488 0911518 0960030 0703490 0818648 0643172 0475339 0413846 0357685 0357685 0258351 0214030	.333245 .2498587 .1993896 .151676 .101025 .120173 .051163 .0716970 .063397 .0539525 .0493009 .043776 .0377778 .0230825 .02373716	.3333275 .2498940 .199ñ125 .1854230 .1405137 .121293 .1058542 .0930737 .0822415 .0728875 .0514088 .050768 .050768 .050768 .036298 .0305768 .036298 .030528 .030528 .030528 .030528 .030528	1.0000000 .5000000 .5000000 .3333294 .2409208 .1996106 .1996106 .1408615 .121786 .104895 .0938688 .0938688 .093868 .0857037 .0852851 .0411922 .0363408 .0318649 .027142	for all n 24 333307 2489407 333007 1658090 141156 1222119 1070639 049447 0698033 0535481 049452 049467 049483 053481 0478422 0426467 0334925 0294224 0266328	.333316 .2499653 .1997514 .1650539 .1414068 .122788 .1070409 .0847012 .0737102 .067534 .0690928 .064883 .049084 .0439695 .049084 .0439695 .0392793 .0392793 .0392793	.3333322 .2409665 .1998012 .1680741 .1410195 .1228858 .0050879 .0833413 .0764525 .0618362 .0557058 .061756 .0451473 .0405139 .0405139	.33333 24997 19984 16617 14180 12316 10833 .09615 .08591 .07711 .06944 .06297 .05662 .06117 .04622 .04189 .03752 .03366
1 2 3 4 5 6 7 8 9 10 11 12 13 114 15 16 17 18 19 20 21 22	.3333201 2498115 1992332 1648585 1396167 1290468 1042646 0911618 0800030 0703490 0618848 030643172 0475339 0413846 03257685 03257	.333245 .2498587 .1993967 .1401025 .1207173 .1061181 .0921776 .0811943 .0716870 .0633807 .0493009 .0432780 .0432780 .02373778 .0237378 .0237378 .0237378	.3333275 .2498940 .1996125 .1654230 .1406137 .1212923 .1058642 .0930737 .0822416 .0728875 .0546867 .054686 .044682 .034578 .044682 .034578 .044628 .034578 .044628 .034578 .044628 .034578	1.000000 .5000000 .3333294 .2409208 .1998108 .1998108 .198108 .1098108 .1217844 .1044955 .09387037 .0688656 .0587037 .0688656 .0587037 .0628951 .044810 .0411922 .033408 .031843	for all n   24   333307   2490404   1996889   1658090   1411366   1222119   1070639   0945476   0689162   0.0748175   0.069162   0.0748176   0.034942   0.034942   0.034942   0.034942   0.034942   0.034942   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.026032   0.024942   0.	.3333316 .2499553 .1997514 .1659539 .1414068 .1227788 .1073406 .0961637 .0847012 .0737102 .047804 .0490926 .0490926 .0490926 .0490926 .0390714 .0390714	.3333322 2400665 1.998012 1.600741 1.410195 1.228958 1.077065 0.0568795 0.058795 0.0570768 0.0507768 0.051756 0.051756 0.0537078 0.0537078 0.0537078 0.0537078 0.0537078	.33333 24997 19984 16617 14180 12316 10833 .09615 .08791 .07711 .04626 .05117 .04629 .04169 .03752 .03366 .03007
1 2 3 4 4 5 6 6 7 8 9 110 111 12 13 114 115 116 117 118 119 20 21 22 22 22 22 23	.3333201 2498115 1992332 1648585 1398167 1200488 0911518 0960030 0703490 0918648 0643172 0475339 0413846 0257685 0258351 0214030 0172679 1013960 00172679	.333245 249867 1993896 1851676 1401025 1207173 1051181 0921778 0811943 0716870 0534025 0493009 043276 0437778 032778 0227378 0493009 0432702 0437778	.3333275 .2498940 .1996125 .1854230 .1405137 .121293 .1058642 .0930737 .0822415 .0728875 .0546887 .054688 .0508763 .0449862 .0395768 .0346298 .030629 .036256 .0218762 .0181807 .0147103	1.0000000 5000000 5000000 2.3333294 2.409208 1.996106 1.856342 1.108815 1.217884 1.06495 0.9381643 0.0739417 0.068656 0.0587037 0.0522561 0.0411922 0.0351869 0.027142 0.022314 0.022314	for all n   24   3333307   24   190488   190484   190488   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   1904888   14   190488   14   190488   14   190488   14   190488   14	.3333316 .2499551 .199751 .1650539 .1414068 .1225788 .1070409 .0961637 .0947012 .0737102 .0737102 .0496833 .049084 .049085	.3333322 2.409645 .1998012 .1600741 .1410195 .1070065 .0050879 .0553413 .0704625 .0646850 .0617362 .0507658 .0601736 .04	.33333 .24997 .19984 .16617 .14180 .12316 .10833 .09615 .07711 .06944 .06297 .05662 .05117 .04622 .04169 .03306 .03007 .02872 .02872
1 2 3 4 4 5 6 7 8 9 9 10 11 12 13 14 15 16 17 18 20 21 222 3224	.3333201 2498115 1092335 1396167 1200468 1042646 0911618 0800030 0703490 00418048 0643172 0475339 0413846 0257885 026835 0214030 0172679 0133950 0097747 0063234	.333245 .2498587 .199398 .1651676 .1401025 .1207173 .1061181 .0921776 .0811943 .0716870 .0633807 .0493009 .0432780 .0432780 .0432780 .02373778 .023738 .0237378 .0237378 .0237378 .023738 .023	.3333275 .2498940 .199ñ125 .1854230 .140f137 .1212923 .1058642 .0930737 .0822415 .0728875 .0646887 .0547408 .034628 .034628 .034628 .034628 .034628 .034628 .034628 .034628 .034628 .034628 .034628 .034628 .034628 .034628	1.000000 .5000000 .3333294 .2409208 .1990108 .1990108 .1090108 .1090108 .1090108 .031643 .0739417 .068865 .0887037 .0522851 .0464810 .041192 .033408 .031643 .0318649 .0318649 .0318649 .0318649 .0318649 .0318649 .0318649	for all n   24   3333307   2490404   1996889   1958090   1411506   122219   1070639   0045470   0083976   0083976   0083978   00535481   0074872   0045470   0083978   00535481   00566329   00748762	.3333316 .249953 .109751 .109751 .1141-008 .122768 .1075409 .0051837 .0847012 .00787102 .0078848 .0090028 .004082 .004083 .004083 .004083 .004083 .004083 .004083 .004083	.3333322 2400665 1.998012 1.600741 1.410195 1.228958 1.077065 0.0568795 0.686895 0.667695 0.657695 0.657768 0.6	. 33333 24997 . 19984 . 10617 . 14180 . 12316 . 10833 . 09615 . 08591 . 07711 . 06944 . 06397 . 05062 . 04169 . 03152 . 03306 . 03007 . 02872 . 02872 . 02872
1 2 3 4 4 5 6 6 7 8 9 110 111 12 13 114 115 116 117 118 119 20 21 22 22 22 22 23	.3333201 2498115 1992332 1648585 1398167 1200488 0911518 0960030 0703490 0918648 0643172 0475339 0413846 0257685 0258351 0214030 0172679 1013960 00172679	.333245 249867 1993896 1851676 1401025 1207173 1051181 0921778 0811943 0716870 0534025 0493009 043276 0437778 032778 0227378 0493009 0432702 0437778	.3333275 .2498940 .1996125 .1854230 .1405137 .121293 .1058642 .0930737 .0822415 .0728875 .0546887 .054688 .0508763 .0449862 .0395768 .0346298 .030629 .036256 .0218762 .0181807 .0147103	1.0000000 5000000 5000000 2.3333294 2.409208 1.996106 1.856342 1.108815 1.217884 1.06495 0.9381643 0.0739417 0.068656 0.0587037 0.0522561 0.0411922 0.0351869 0.027142 0.022314 0.022314	for all n   24   3333307   24   190488   190484   190488   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   190488   14   1904888   14   190488   14   190488   14   190488   14   190488   14	.3333316 .2499551 .199751 .1650539 .1414068 .1225788 .1070409 .0961637 .0947012 .0737102 .0737102 .0496833 .049084 .049085	.3333322 2.409645 .1998012 .1600741 .1410195 .1070065 .0050879 .0553413 .0704625 .0646850 .0617362 .0507058 .0601736 .04	. 33333 24997 . 19984 . 10617 . 14180 . 12316 . 10833 . 09615 . 08591 . 07711 . 06944 . 06397 . 05062 . 04169 . 03152 . 03306 . 03007 . 02872 . 02872 . 02872
1 2 3 4 5 6 7 8 9 9 10 11 1 12 13 13 14 15 16 17 18 19 20 22 22 22 22 22 22 22 22 22 22 22 22	.3333201 2498115 1092335 1396167 1200468 1042646 0911618 0800030 0703490 00418048 0643172 0475339 0413846 0257885 026835 0214030 0172679 0133950 0097747 0063234	.333245 2498687 199386 1851676 1401025 1207173 1051181 0021776 0811043 071687 0633907 0432790 0432790 0432790 0432790 0432791 0493002 04373778 0493002 04373715 0493002 04373716 0493002 0438022 0437315 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 04802 0	.3333275 .2498940 .199612 .1854230 .1406137 .1212923 .1058642 .0930737 .0822415 .0728875 .0514088 .034629 .034629 .03	1.000000 .5000000 .5000000 .333394 .199010 .199010 .189610 .121784 .104895 .0938580 .081643 .0739417 .0622851 .044810 .0411922 .0363408 .031849 .027142 .022314 .018886 .0318480 .023487	for all n   24   3333307   3433307   3433307   34333307   345333307   34533307   34533	.3333316 .249953 .1997514 .1650539 .1414068 .1226798 .1076400 .0961637 .0847012 .078702 .047804 .049092 .0492793 .049084 .0392793 .049084 .0392793 .049084 .0392793 .049084 .0392793 .049084 .0392793 .049084 .0392793 .049085 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .04	.3333322 2.249965 1.998012 1.660741 1.410195 1.228958 1.077065 0.95685 0.957058 0.95	.33333 .24997 .19984 .16617 .14180 .12316 .10833 .09015 .08791 .00944 .08247 .05662 .06117 .04682 .04189 .03368 .03007 .02358 .02063 .02358 .02063 .017844
1 2 3 4 4 5 6 6 7 8 9 9 10 11 1 12 2 13 14 15 16 16 17 18 19 20 21 22 23 224 225 227	.3333201 2498115 1992332 1648585 1396167 1290468 9911518 9800030 971547 9475339 941348 9357885 9258351 9214030 91172679	.333245 249869 199389 1851876 1401025 1207173 1051181 0921776 0811943 0716870 0433807 0439709 0437776 0230925 0493009 0437776 0230925 0493009 04971777 0498025 0499121 0499121 0499121 0499121 0499121	.3333275 2.198840 .1996125 .1654230 .1405137 .1212923 .1058642 .0930737 .0922415 .0728875 .0446867 .0506763 .0449682 .0393768 .0306783 .044968 .0306783 .041968 .030678 .03078 .0318807 .0147103 .014409 .008316 .008316 .008316	1.0000000 5000000 4.2409208 2.409208 1.996106 1.856342 1.408816 1.217864 1.06495 0.938184 0.038184 0.038184 0.0587037 0.05285 0.038184 0.041192 0.038184 0.027142 0.027142 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.0023847 0	for all n   24   3333307   2490404   1996889   1058090   141136   1222119   1070039   0945476   0689162   049043   0035481   0478472   049040   0314925   049043   0314925   049043   0456329   0456329   0456331   0456	.3333316 .2499551 .1097514 .1050539 .1141068 .1225788 .1070409 .0961637 .0787102 .0787102 .078848 .0409028 .040833 .0409029 .040884 .043959 .040893 .041086 .030971 .032723 .032773 .032779 .032773 .032779 .032773 .0327744 .032774 .032774 .032774 .032774 .032774 .032774 .032774 .032774 .032774	.3333322 2.400665 1.090012 1.090012 1.141019 1.228058 1.070065 0.050679 0.05033113 0.704025 0.060679 0.0507058 0.050	.33333 24997 .19984 .16617 .14180 .12316 .10833 .09615 .08791 .00944 .06297 .05662 .04169 .03752 .03366 .03007 .02672 .02752 .02
1 2 3 4 4 5 6 7 7 8 9 10 0 111 1213 14 14 15 116 117 118 119 20 122 22 22 22 22 22 22 22 22 22 28	.3333201 2498115 1992332 1648585 1396167 1290468 9911518 9800030 971547 9475339 941348 9357885 9258351 9214030 91172679	.333245 2498687 199386 1851676 1401025 1207173 1051181 0021776 0811043 071687 0633907 0432790 0432790 0432790 0432790 0432791 0493002 04373778 0493002 04373715 0493002 04373716 0493002 0438022 0437315 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 04802 0	.3333275 .2498940 .199612 .1854230 .1406137 .1212923 .1058642 .0930737 .0822415 .0728875 .0514088 .034629 .034629 .03	1.000000 .5000000 .5000000 .333394 .199010 .199010 .189610 .121784 .104895 .0938580 .081643 .0739417 .0622851 .044810 .0411922 .0363408 .031849 .027142 .022314 .018886 .0318480 .023487	for all n   n   24   333307   333307   333307   333307   333307   333307   333307   333307   333307   333307   333307   333307   333706	.3333316 .249953 .1997514 .1650539 .1414068 .1226798 .1076400 .0961637 .0847012 .078702 .047804 .049092 .0492793 .049084 .0392793 .049084 .0392793 .049084 .0392793 .049084 .0392793 .049084 .0392793 .049084 .0392793 .049085 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .0392793 .049085 .04	.3333322 2.249965 1.999012 1.660741 1.410195 1.228958 1.070065 0.950689 0.950708 0.957058 0.9	. 33333 24997 19984 16617 14180 10333 09015 08791 07711 06944 06247 06162 04169 03752 03067 02972 02972 02972 02063 01784 012714 012714
1 2 3 4 4 5 6 6 7 8 9 9 110 111 121 13 114 115 117 118 220 221 222 224 227 229	.3333201 2498115 1992332 1648585 1396167 1290468 9911518 9800030 971547 9475339 941348 9357885 9258351 9214030 91172679	.333245 2498687 199386 1851676 1401025 1207173 1051181 0021776 0811043 071687 0633907 0432790 0432790 0432790 0432790 0432791 0493002 04373778 0493002 04373715 0493002 04373716 0493002 0438022 0437315 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 04802 0	.3333275 2.198840 .1996125 .1654230 .1405137 .1212923 .1058642 .0930737 .0922415 .0728875 .0446867 .0506763 .0449682 .0393768 .0306783 .044968 .0306783 .041968 .030678 .03078 .0318807 .0147103 .014409 .008316 .008316 .008316	1.0000000 5000000 4.2409208 2.409208 1.996106 1.856342 1.408816 1.217864 1.06495 0.938184 0.038184 0.038184 0.0587037 0.05285 0.038184 0.041192 0.038184 0.027142 0.027142 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.023847 0.0023847 0	for all n   24   3333307   2490404   1996889   1658090   141136   1628090   141136   1628090   1	.3333316 .2499551 .1097514 .1050539 .1414068 .1225788 .1070409 .0961637 .0787102 .0787102 .078848 .0409026 .046833 .049084 .043956 .049084 .04908 .049084 .04908 .049084 .049084 .049084 .04908 .04908 .04908 .049084	.3333322 2.400645 1.090012 1.090012 1.141019 1.123853 1.070065 0.050679 0.05033413 0.704025 0.0607705 0.061736 0.050705 0.050736 0.050705	.33333 .24997 .19984 .16617 .14180 .12316 .10933 .09615 .08591 .07711 .06944 .06267 .04622 .04169 .03752 .03368 .03007 .02672 .02672 .02752 .0
1 2 3 4 4 5 6 6 7 7 8 9 9 10 11 12 13 3 14 15 16 17 18 19 20 22 22 22 22 22 22 22 22 22 22 22 22	.3333201 2498115 1992332 1648585 1396167 1290468 9911518 9800030 971547 9475339 941348 9357885 9258351 9214030 91172679	.333245 2498687 199386 1851676 1401025 1207173 1051181 0021776 0811043 071687 0633907 0432790 0432790 0432790 0432790 0432791 0493002 04373778 0493002 04373715 0493002 04373716 0493002 0438022 0437315 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 04802 0	.3333275 2.198840 .1996125 .1654230 .1405137 .1212923 .1058642 .0930737 .0922415 .0728875 .0446867 .0506763 .0449682 .0393768 .0306783 .044968 .0306783 .041968 .030678 .03078 .0318807 .0147103 .014409 .008316 .008316 .008316	1.0000000 .5000000 .333294 .2409208 .199010 .189610 .121784 .104895 .0938580 .081643 .0739417 .0622851 .044810 .0411922 .0333408 .031840 .0279417 .022314 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .0103206 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010236 .01	for all n   n   24   333307   333307   333307   333307   333307   333307   333307   333307   333307   333307   333307   333307   333706	.3333316 2.49953 1.997514 1.650539 1.414068 1.226798 1.076400 0961637 0.647012 0.0785148 0.000029 0.04883 0.040084 0.39995 0.392793 0.319066 0.309714 0.22523 0.174563 0.114618 0.011829 0.001245 0.001753	.3333322 2.249965 1.999012 1.660741 1.410195 1.228958 1.070065 0.950689 0.950708 0.957058 0.9	. 33333 24997 19984 16617 14180 10333 09015 08791 07711 06944 06247 06162 04169 03752 03067 02972 02972 02972 02063 01784 012714 012714
1 2 3 4 4 5 6 6 7 8 9 9 110 111 121 13 114 115 117 118 220 221 222 224 227 229	.3333201 2498115 1992332 1648585 1396167 1290468 9911518 9800030 971547 9475339 941348 9357885 9258351 9214030 91172679	.333245 2498687 199386 1851676 1401025 1207173 1051181 0021776 0811043 071687 0633907 0432790 0432790 0432790 0432790 0432791 0493002 04373778 0493002 04373715 0493002 04373716 0493002 0438022 0437315 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 048022 04802 0	.3333275 2.198840 .1996125 .1654230 .1405137 .1212923 .1058642 .0930737 .0922415 .0728875 .0446867 .0506763 .0449682 .0393768 .0306783 .044968 .0306783 .041968 .030678 .03078 .0318807 .0147103 .014409 .008316 .008316 .008316	1.0000000 .5000000 .333294 .2409208 .199010 .189610 .121784 .104895 .0938580 .081643 .0739417 .0622851 .044810 .0411922 .0333408 .031840 .0279417 .022314 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .0103206 .010234 .010234 .010234 .010234 .010234 .010234 .010234 .010236 .01	for all n   24   3333307   2490404   1996889   1658090   141136   1628090   141136   1628090   1	.3333316 .2499551 .1097514 .1050539 .1414068 .1225788 .1070409 .0961637 .0787102 .0787102 .078848 .0409026 .046833 .049084 .043956 .049084 .04908 .049084 .04908 .049084 .049084 .049084 .04908 .04908 .04908 .049084	.3333322 2.400645 1.090012 1.090012 1.141019 1.123853 1.070065 0.050679 0.05033413 0.704025 0.0607705 0.061736 0.050705 0.050736 0.050705	.33333 .24997 .19984 .16617 .14180 .12316 .10933 .09615 .08591 .07711 .06944 .06267 .04622 .04169 .03752 .03368 .03007 .02672 .02672 .02752 .0

TABLE 3. VALUES OF  $\frac{C_m(n-1)}{C_m(n)}$  (Contd.)

n → m	34	35	36	37	38	39	40	41	42
1 2	_			1.00000 .80000 .33333	00 for	all n n>24 n>38			
3	. 3333328	.3333330	. 3333331	. 3333332	. 3333332	. 3333333	. 3333333	for n>38	
4	2499812	. 2499859	. 2499894	. 2499921	. 2400940	. 2499955	. 2499966	.2499975	. 249908
5	. 1998730	. 1998984	8810001.	.1999350	. 1999480	. 1999584	. 1999667	. 1999734	. 199978
ß	.1662566	. 1663255	. 1663827	. 1664302	. 1664698	.1665027	. 1665301	.16615530	. 166572
7	.1419544	.1420856	. 1421976	. 1422931	.1423746	.1424442	. 1425037	. 1425546	.142598
8	. 1234073	. 1236131	.1237916	. 1239467	.1240815	.1241988	.1243008	.1243897	. 124467
9	.1086654	.1089519	.1092036	. 1094250	.1096199	. 1097916	. 1099431	. 1100767	.110194
10	.0965764	.0969451	.0972731	.0975837	.0978221	.0980519	.0982566	.0984390	.098001
11	.0804167	.0868683	.0872715	.0876321	.0879548	.0882441	.0885036	.0887367	.08894
12	.0777093	.0782413	.0787190	.0791485	.0795352	.0798837	.0801982	.0804824	.08073
13	.0701262	.0707361	.0712861	.0717828	.0722320	.0726388	.0730075	.0733423	.07364
14	.0634346	.0641193	.0647390	.0653007	.0658105	.0662739	.0866957	.0670799	.06743
15	.0574637	.0582202	.0589068	.0595311	.0600994	.0606176	.0010907	.0615232	.08191
	.0520858	.0529109	.0536617	.0543460	.0549707	.0555417	.0560643	.0565434	.05698
17	.0472028	.0480935	.0489058	.0496476	.0503263	.0509481	.0515186	.0520426	.05252
	.0427382	.0436916	.0445626	.0453597	.0460903	.0467608	.0473773	.0479446	.04846
9	.0386311	. 0396445	.0408718	.0414218	.0422021	.0429196	.0435802	.0441893	.04475
20	.0348327	.0359036	.0368848	.0377855	.0386136	.0393761	.0400792	.0407284	.04132
21	.0313031	.0324290	.0334620	.0344113	.0352853	.0360910	.0368350	.0375228	.03815
22	0280093	.0291880	.0302707	.0312668	.0321848	.0330321	.0338154	.0345404	.03521
23 .	0249241	.0261535	.0272838	.0283248	.0292852	.0301725	.0309937	.0317545	.03246
24 .	0220242	.0233023	.0244785	.0255627	.0265639	.0274898	.0283474	.0291428	.02988
5 .	0192902	.0206152	.0218354	.0229613	.0240017	.0249648	.0258575	.0260863	.02748
6.	0167052	.0180754	.0193382	.0205041	.0215825	.0225813	.0235080	.0243689	.02516
7.	0142550	.0156687	.0169725	.0181771	.0192920	.0203255	.0212849	.0221769	.02300
8.	0119270	.0133827	.0147261	.0159681	.0171183	.0181852	.0191763	.0200983	.0209
9.	0097103	.0112066	.0125883	.0138664	.0150508	.01810.	.0171717	.0181228	.01900
0.	0075954	.0091310	.0105497	.0118628	.0130802	.0142107	.0152620	.0162412	.0171
1.	0055740	.0071475	.0086021	.0099490	.0111984	.0123592	.0134393	.0144458	.0153
2.	0036386	.0052489	.0067382	.0081179	.0093983	.0105885	.0116964	.0127294	.0136
3.	0017825	.0034286	.0049515	.0003030	.0076735	.0088922	.0100272	.0110858	.0120
4	0	.0016807	.0032362	.0046786	.0080183	.0072647	.0084259	.0095094	.01055
δ		0	.0015873	.0030597	.0044277	.0057010	.0068877	.0079955	.00902
6			0	.0015015	.0028972	.0041965	.0054081	.0065394	.00759
7				0	.0014225	.0027473	.0039829	.0051372	.00621
8					0	.0013495	.0026087	.0037853	.00488
9						0	.0012821	.0024804	.00360
0							0	.0012195	.00236
1 2								0	.00116

# SANKHYA: THE INDIAN JOURNAL OF STATISTICS: SMALES A

TABLE 3. VALUES OF  $\frac{C_m(n-1)}{C_m(n)}$  (Contd.)

5 6 7 8 9 10 11 12 13 14 15	.2499988 .199830 .199830 .1665878 .1426364 .1245346 .1102991 .0987470 .0891346 .0739230 .0677505 .0622814	.2499989 .1999864 .1666009 .1426672 .1246935 .1103913 .0988767 .0893043 .0818426 .0741749 .0880431 .0826139	.2499992 .1990891 .1666119 .1426945 .1246897 .1104729 .0989927 .0813730 .0744043 .0683107 .0029191	1.0000 .5000 .33333 .2499994 .1999913 .1666210 .1427178 .1246897 .1106451 .0990985	.249998 .199930 .1666287 .1427378 .1247288 .1106090 .0991893	n >24 >38 .249997 .1999044 .1666350 .1427540 .1247629 .1106558 .0992724 .0898316 .0818471	.2499997 .1999955 .1608403 .1427695 .1247928 .1107158 .0993488	.24099 .19999 .16664 .14278 .12481 .11076 .09941
5 6 7 8 9 10 11 12 13 14 15	.1909830 .1665878 .1426354 .1245346 .1102991 .0987470 .0891346 .0809719 .0739230 .0677505 .0622814 .0573868 .0529683	.1999864 .1666009 .1426072 .1245935 .1103913 .0988767 .0893043 .0811826 .0741749 .0880431 .0826139	.1999891 .1866119 .1426945 .1246897 .1104729 .0989927 .0894572 .0813736 .0744043 .0683107	.33333 .2499994 .199913 .1666210 .1427178 .1246897 .1106451 .0999985 .0896950 .0815469 .0746136	.2499996 .1999930 .1666287 .1427378 .1247288 .1106090 .0991893	>38 .249997 .1999944 .1866350 .1427549 .1247629 .1108658 .0992724 .0898316	.1666403 .1427695 .1247928 .1107158 .0903468	.19999 .18684 .14278 .12481 .11076 .09941
4 5 6 7 8 9 10 11 12 13 14 15 16 17 18	.1909830 .1665878 .1426354 .1245346 .1102991 .0987470 .0891346 .0809719 .0739230 .0677505 .0622814 .0573868 .0529683	.1999864 .1666009 .1426072 .1245935 .1103913 .0988767 .0893043 .0811826 .0741749 .0880431 .0826139	.1999891 .1866119 .1426945 .1246897 .1104729 .0989927 .0894572 .0813736 .0744043 .0683107	.249994 .199913 .1666210 .1427178 .1240897 .1106451 .0990965 .0896950 .0816469 .0746136	.249998 .1999930 .1666287 .1427378 .1247288 .1106090 .0991893	.2499987 .1999944 .1866350 .1427549 .1247629 .1106656 .0992724	.1666403 .1427695 .1247928 .1107158 .0903468	.19999 .18684 .14278 .12481 .11076 .09941
5 6 7 8 9 10 11 12 13 14 15 16 17 18	.1909830 .1665878 .1426354 .1245346 .1102991 .0987470 .0891346 .0809719 .0739230 .0677505 .0622814 .0573868 .0529683	.1999864 .1666009 .1426072 .1245935 .1103913 .0988767 .0893043 .0811826 .0741749 .0880431 .0826139	.1999891 .1866119 .1426945 .1246897 .1104729 .0989927 .0894572 .0813736 .0744043 .0683107	. 1999913 . 1666210 . 1427178 . 1240897 . 1106451 . 0990985 . 0896950 . 0816469 . 0746136	.1999930 .1666287 .1427378 .1247288 .1108090 .0991893 .0897194	.1899944 .1868350 .1427540 .1247629 .1108658 .0992724	.1666403 .1427695 .1247928 .1107158 .0903468	.19999 .18684 .14278 .12481 .11076 .09941
6 7 8 9 10 11 12 13 14 15 16 17 18	.1665878 .1426354 .1245346 .1102991 .0987470 .0891346 .0809719 .0737230 .0677505 .0622814 .0573868 .0529683	.1666009 .1426672 .1245935 .1103913 .0988767 .0893043 .0811826 .0741749 .0880431 .0826139	.1866119 .1426945 .1246897 .1104729 .0989927 .0894572 .0813736 .0744043 .0683107	.1666210 .1427178 .1240897 .1106451 .0990985	.1666287 .1427378 .1247288 .1106090 .0991893	.1886350 .1427549 .1247629 .1108658 .0992724	.1606403 .1427695 .1247928 .1107158 .0993468	.16664 .14278 .12481 .11076 .09941
7 8 9 10 11 12 13 14 15 16 17 18	1426354 11245346 1102991 0987470 0891346 0809719 0739230 0677505 0622814	.1426672 .1245936 .1103913 .0988767 .0893043 .0811826 .0741749 .0880431 .0826139	.1426945 .1246897 .1104729 .0989927 .0894572 .0813736 .0744043 .0683107	.1427178 .1246897 .1106451 .0990985 .0896950 .0815469 .0746136	.1427378 .1247288 .1108090 .0991893 .0897194 .0817042	.1427540 .1247629 .1106656 .0992724	.1427695 .1247928 .1107158 .0993468	.14278 .12481 .11076 .09941
8 9 10 11 12 13 14 15 16 17 18	1245346 1102991 .0987470 .0891346 .0809719 .0739230 .0677505 .0622814 .0573868 .0529683	.1245935 .1103913 .0988767 .0893043 .0811826 .0741749 .0880431 .0826139	.1246897 .1104729 .0989927 .0894572 .0813736 .0744043 .0683107	.1240897 .1106451 .0990985 .0896950 .0815469 .0746136	.1247288 .1108090 .0991893 .0897194 .0817042	.1247629 .1106656 .0992724	.1247928 .1107158 .0993468	.12481 .11076 .09941
9 10 11 12 13 14 15 16 17	.0891346 .0891346 .0809719 .0739230 .0677505 .0622814	.1103913 .0988767 .0893043 .0811826 .0741749 .0880431 .0826139	.0989927 .0989927 .0894572 .0813736 .0744043 .0683107	.0990985 .0896950 .0816469 .0746136	.1106090 .0991893 .0897194 .0817042	.1106656 .0992724 .0898316	.1107158 .0993468 .0899330	.09941
10 11 12 13 14 15 16 17	.0987470 .0891346 .0809719 .0739230 .0677505 .0622814 .0573868 .0529683	.0988767 .0893043 .0811826 .0741749 .0880431 .0626139	.0989927 .0894572 .0813736 .0744043 .0683107	.0990985 .0896950 .0815469 .0746136	.0991893 .0897194 .0817042	.0992724	.0993468	.09941
11 12 13 14 15	.0891346 .0809719 .0739230 .0677505 .0622814 .0573868 .0529683	.0893043 .0811826 .0741749 .0880431 .0826139	.0894572 .0813736 .0744043 .0683107	.0896950 .0815469 .0746136	.0897194 .0817042	.0898316	.0899330	.09002
12 13 14 15 16 17	.0809719 .0739230 .0677505 .0622814 .0573868 .0529683	.0811826 .0741749 .0880431 .0826139	.0813736 .0744043 .0683107	.0815469	.0817042			
12 13 14 15 16 17	.0739230 .0677505 .0622814 .0573868 .0529683	.0741749 .0880431 .0826139	.0744043	.0746136	.0817042			
13 14 15 16 17 18	.0677505 .0622814 .0573868 .0529683	.0880431	.0683107		05.00.5		.0819771	. 08209
14 15 16 17 18	.0622814 .0573868 .0529683	.0828139			.0748045	.0749789	.0751383	.07528
16 17 18	.0573868 .0529683		0400101	.0685558	.0687804	.0689864	.0691754	.06934
17 18	.0529683	.0577581	.0029191	.0631995	.0634574	.0636947	.0639132	.0641
17 18	.0529683		.0581000	.0584150	.0587054	.0589736	.0592212	.05948
18	0160108	.0533774	.0537549	.0541035	.0544259	.0547241	.0550003	.0552
		.0493954	.0498074	.0501888	.0505420	.0508696	.0511736	.05143
	.0452712	.0457520	.0481974	.0466104	.0469937	.0473498	.0476809	.04798
20	.0418842	.0423991	.0428768	.0433204	.0437328	.0441165	.0444738	.04480
21	.0387498	.0392976	.0398064	.0402796	.0407202	.0411307	.0415135	.04181
22	.0358361	.0364155	.0369545	.0374563	.0379240	.0383604	.0387679	.0391
	.0331164	.0337264	.0342944	.0348239	.0353180	.0357794	.0362108	.0366
24	.0305684	.0312080	.0318041	.0323603	.0328798	.0333656	.0338201	.0342
25	.0281735	.0288416	.0294649 •	.0300469	.0305911	.0311003	.0315773	.03202
26	.0259157	.0266113	.0272608	.0278679	.0284359	.0289679	.0294666	. 02993
27	.0237812	.0245035	.0251784	.0258097	.0264008	.0289548	.0274745	.0279
	.0217583	.0225064	.0232059	.0238606	.0244740	.0250494	.0255895	.02609
29	.0198366	.0208097	.0213330	.0220104	.0226455	.0232416	.0238016	.0243
30	.0180073	.0188046	.0195510	.0202504	.0209065	.0215227	.0221019	.0226
31	.0182624	.0170832	.0178520	.0185728	.0192493	.0198850	.0204828	.0210
32	.0145950	.0154386	.0162291	.0169707	.0178870	.0183216	.0189376	.0195
33	.0129991	.0138648	.0146784	.0154381	.0161537	.0168267	.0174603	.0180
	.0114690	.0123563	.0131884	.0139696	.0147040	.0153949	.0180458	.0166
35	.0100000	.0109082	.0117602	.0125605	.0133131	.0140214	.0146887	.0153
	.0085877	.0095162	.0103876	.0112085	.0119767	.0127020	.0133856	.01403
	.0072281	.0081762	.0090867	.0099036	.0108911	.0114329	.0121323	.01279
	.0059176	.0068852	.0077939	.0086484	.0094527	.0102108	.0109254	.01160
	.0046531	.0056395	.0085662	.0074378	.0082585	.0090321	.0097619	.01044
40	.0034317	.0044364	.0053806	.0062689	.0071056	.0078944	.0086390	.0093-
	.0022506	.0032731	.0042344	.0051391	.0059913	.0067951	.0075540	.00827
	.0011074	.0021475	.0031254	.0040460	.0049135	.0057319	.0065047	.00723
43	0	.0010571	.0020513	.0029874	.0038698	.0047024	.0054889	.00623
44		0	.0010101	.0019614	.0028584	.0037049	.0045047	.00526
45			0	.0009662	.0018773	.0027375	.0035504	.00431
46				0	.0009251	.0017988	.0026242	.00340
47					0	.0008865	.0017246	.00251
48						0	.0008503	.00165
49 50							0	.00081

<sup>\*</sup> Note: Values of  $\frac{C_{m-1}(n-1)}{C_m(n)}$  can also be obtained from  $\frac{C_m(m-1)}{C_m(n)}$  by using following relation  $\frac{1}{m} = \frac{O_m(n-1)}{O_m(n)} + \frac{C_{m-1}(n-1)}{O_m(n)} + \frac{C_{m-1}(n-1)}{O_m(n)}$ 

# 7. COMPARISON BETWEEN WITH AND WITHOUT REPLACEMENT SIMPLE RANDOM SAMPLING SCHEMES

In conclusion let us compare the two simple random sampling schemes for the purpose of estimation of  $\overline{Y}$ . If we draw a simple random sample with replacement of size n, then the variance of the sample mean is  $\sigma^2/n$ . Further, in a simple random sample without replacement of size n, the variance of the sample mean is  $\frac{\sigma^2}{n} \binom{N-n}{N-1}$ . Since

$$\frac{\sigma^3}{n} \cdot \frac{(N-n)}{(N-1)} < \frac{\sigma^3}{n}$$

it is usually claimed that sampling without replacement is better than sampling with replacement. Basy (1958) has pointed out that this comparison is not fair because the cost of selecting a sample of size n in sampling without replacement is greater than the cost of selecting a sample in sampling with replacement. For comparing the two sampling schemes, it would be appropriate to take into account the cost involved in the selection of two different samples. The comparison, thus, mainly depends on the choice of the cost function, and no sampling scheme can be said to be superior to the other unless the cost function is known in advance. Let us, for illustration, consider the case where the cost of sampling is proportional to the number of distinct units drawn. Thus the expected cost of selecting a sample with replacement of size n is equivalent to the cost of selecting a sample without replacement of size  $E(v) = N \left[1 - \left(\frac{N-1}{N}\right)^n\right]$ . Basu has shown that in this situation the sample mean of the sample with replacement is worse than the sample mean of the equivalent sample without replacement. We now compare the sample mean  $\bar{y}$  of the equivalent sample without replacement with the following estimator of with replacement sample:

$$\tilde{g}_{r(\mathbf{S})} = \frac{[N \nu / (N - \nu)]}{E[N \nu / (N - \nu)]} \tilde{g}_{r}.$$

It has been shown that

$$V(\bar{g}_{*(9)}) = \frac{S^2}{E[N\nu/(N-\nu)]} + \bar{Y}^2V \left[ \frac{N\nu/(N-\nu)}{E[N\nu/(N-\nu)]} \right], \quad ... \quad (7.1)$$

and

$$V(g) = \left[\frac{1}{E(\gamma)} - \frac{1}{N}\right] S^2$$
. ... (7.2)

Since  $N_{\nu}/(N-\nu)$  is a convex function of  $\nu(1 \le \nu \le n < N)$ ,

$$\mathbb{E}\left[N_{V}/(N-v)\right] \geqslant N\mathbb{E}_{V}/[N-\mathbb{E}_{V}] = \left[\frac{1}{\mathbb{E}_{V}} - \frac{1}{N}\right]^{-1} \qquad \dots (7.3)$$

# SANKHYA: THE INDIAN JOURNAL OF STATISTICS: SERIES A

From (7.3), it is evident that the first component of  $V(g_{r(2)})$  is smaller than V(g). Thus for a population whose coefficient of variation is sufficiently large  $V(g_{r(2)})$  would be smaller than V(g). This comparison shows that the sample mean of without replacement sample cannot be uniformly better than all estimators of with replacement sampling.

However, the comparison made above is not very satisfactory. First, because of the linearity of the cost function and secondly, because E(v) is not necessarily an integer. We hope that for some other cost functions also, similar situations may be found out where with replacement sampling would fare better than without replacement sampling.

## ACKNOWLEDGEMENT

I am grateful to Dr. D. Basu for his valuable guidance in the preparation of the paper.

#### RHYBRANCES

BASU, D. (1958): On sampling with and without replacement. Sankhya, 20, 287-294.

DES RAJ and KHANGS, S. H. (1958): Some remarks on sampling with replacement. Ann. Math. Stat., 88, 560-557.

FRASER, D. A. S. (1957): Non-parametric Methods in Statistics, John Wiley and Sons, New York.

GUPTA, H. (1950): Tables of distributions. Research Bulletin of the East Punjab University, No. 2, 3, 13-44.

HARTLEY, H. O. and Ross, A. (1954): Unbiased ratio estimators. Nature, 174, 270-271.

HOEVITZ, D. G. and TROMISSON, D. J. (1952): A generalization of sampling without replacement from finite universes. J. Amer. Stat. Ass., 47, 663-885.

MUNITARY, M. N. (1961): Some problems of estimation in sampling from finite populations. Thesis submitted for the Ph.D degree of the Indian Statistical Institute, 105-112.

PATHAE, P. K. (1961): On the evaluation of moments of distinct units in a sample, Sankhyā, Series A, 33, 415-420

YATES, F. and GRUNDY, P. M. (1953): Selection without replacement from within strata with probability proportional to size. J. Roy. Stat. Soc., Series B, 15, 253-261.

Paper received: September, 1961.