Some geometrical aspects of semidefinite linear complementarity problems

MADHUR MALIK* and S. R. MOHAN†

Indian Statistical Institute, New Delhi 110016, India

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This article studies some geometrical aspects of the semidefinite linear complementarity problem (SDLCP), which can be viewed as a generalization of the well-known linear complementarity problem (LCP). SDLCP is a special case of a complementarity problem over a closed convex cone, where the cone considered is the closed convex cone of positive semidefinite matrices. It arises naturally in the unified formulation of a pair of primal-dual semidefinite programming problems. In this article, we introduce the notion of complementary cones in the semidefinite setting using the faces of the cone of positive semidefinite matrices and show that unlike complementary cones induced by an LCP, semidefinite complementary cones need not be closed. However, under $\mathbf{R}_0$-property of the linear transformation, closedness of all the semidefinite complementary cones induced by L is ensured. We also introduce the notion of a principal subtransformation with respect to a face of the cone of positive semidefinite matrices and show that for a self-adjoint linear transformation, strict copositivity is equivalent to strict semimonotonicity of each principal subtransformation. Besides the above, various other solution properties of SDLCP will be interpreted and studied geometrically.

Keywords: Semidefinite linear complementarity problem; Face; Semidefinite complementary cone; Principal Subtransformation; $\mathbf{R}_0$-property; $\mathbf{Q}$-property; Semimonotonocity and copositivity

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1. Introduction

Consider the space S^n of all symmetric real $n \times n$ matrices and the self-dual closed convex cone S^n_+ of symmetric and real $n \times n$ positive semidefinite matrices. Given a linear transformation $L: S^n \to S^n$ and $Q \in S^n$, the semidefinite linear complementarity problem SDLCP(L, Q) is the problem of finding a matrix $X \in S^n$ such that

$$X \in S^n_+, \quad Y = L(X) + Q \in S^n_+, \quad \text{and} \quad \langle X, Y\rangle = \text{trace}(XY) = 0.$$

It is easy to observe that the above problem is a special case of a more general cone complementarity problem, where the cone is specialized to be the cone of symmetric real positive semidefinite matrices. SDLCP can also be regarded as a generalization of the well-known linear complementarity problem. But strikingly, the properties of LCP do not carry over to SDLCP trivially as the semidefinite cone is nonpolyhedral.

Various aspects of the above problem have been studied in recent years. The above form of the problem is due to Gowda and Song [4] who initiated the study of solution properties of SDLCP, particularly for Lyapunov and Stein transformations. Following this article, the solution properties (existence and uniqueness of solutions) for a general SDLCP as well as with special transformations like Lyapunov and Stein transformations have been studied in [5,7,8,18]. Some stability related issues of SDLCP as well as Lorentz cone complementarity problem have been studied in [17]. Algorithmic aspects and an interior point algorithm with a more general form of SDLCP are discussed in [14].

In this article, we introduce the notion of *semidefinite complementary cones* in connection with the SDLCP, using the faces of the positive semidefinite cone, generalizing the notion of a complementary cone studied in linear complementarity theory, see [3]. Unlike complementary cones in the LCP [17], we show that semidefinite complementary cones need not be closed. However, $\mathbf{R_0}$-property of the linear transformation L provides a sufficient condition for the closedness of the semidefinite complementary cones induced by L. In section 3, we study Murty's result [16] ($\mathbf{Q}$-property is equivalent to $\mathbf{R_0}$-property in LCP for nonnegative matrices) in the semidefinite setting and provide a necessary condition for the transformations of the type $L(S^n_+) \subseteq S^n_+$ to have the $\mathbf{Q}$-property. Continuing with the geometrical concepts, in section 4, using the notion of a projection onto the subspace generated by a face, we introduce the notion of *principal subtransformations* with respect to a face F of S^n_+ and show that for self-adjoint linear transformations, strict copositivity is equivalent to strict semimonotonicity of the principal subtransformations. Finally, we end this article with a discussion on the question as to whether the matrix representation of a transformation L with the $\mathbf{P}$-property, with respect to the canonical basis in S^n, is a P-matrix.

We use the symbol $X \succeq 0$ ($\succ 0$) to say that X is symmetric and positive semidefinite (positive definite); the symbol $X \preceq 0$ means that $-X \succeq 0$. We write $\langle X, Y\rangle$ for trace (XY). The orthogonal projection onto the subspace S is denoted by Proj_S, and $\mathrm{span}\, E$ represents the linear span of a subset E of a linear space. A nonempty subset F of a closed convex pointed cone K in S^n is a *face* of K if F is a convex cone and

$$X \in K, \quad Y - X \in K \quad \text{and} \quad Y \in F \Rightarrow X \in F.$$

A *complementary face* of F is defined as

$$F^{\triangle} := \{X \in S^n_+ \colon \langle X, Y\rangle = 0 \; \forall Y \in F\}.$$

Also the smallest face of S^n_+ containing X is the face containing X in its relative interior, see [2].

2. Semidefinite complementary cones and nondegenerate transformations

The following theorem is a restatement of a theorem proved by Hill and Waters (1987) in [10]. See also Pataki [19], wherein the notions of faces and complementary faces are used to study geometrical concepts in cone linear programming problems.

THEOREM 1 *Let $X \in S^n_+$ be a matrix of rank r. Then*

(i) *there exists an orthogonal U, such that the smallest face of S^n_+ containing X is*

$$F = \left\{ U \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} U^T \colon Y \in S^r_+ \right\}$$

(ii) *the face complementary to F is*

$$F^{\triangle} = \left\{ U \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix} U^T \colon Z \in S^{n-r}_+ \right\}$$

(iii) *the dimension of the face F is $(r(r+1))/2$ and the dimension of the complementary face $F^{\triangle}$* is $((n-r)(n-r+1))/2$.

At this point we notice that due to the nonpolyhedral nature, for any face F of S^n_+, $\operatorname{span} F + \operatorname{span} F^{\triangle}$ does not generate the whole space S^n. This leaves a nonzero residual subspace $(\operatorname{span} F + \operatorname{span} F^{\triangle})^{\perp}$ of S^n, which makes the geometry of SDLCP different from that of LCP. Note that in case of R^n_+ the residual subspace is $\{0\}$.

Definition 1

(a) Given a linear transformation $L\colon S^n \to S^n$, a *semidefinite complementary cone* of L corresponding to the face F of S^n_+ is defined as

$$K_F = \{Y - L(X) \colon X \in F,\ Y \in F^{\triangle}\}.$$

(b) A linear transformation L is said to have the $\mathbf{R_0}$-property, if $X = 0$ is the only solution to SDLCP$(L, 0)$.

(c) A linear transformation $L\colon S^n \to S^n$ has the **Q**-property, if SDLCP(L, Q) has a solution for every $Q \in S^n$.

Note that corresponding to every face F of the cone S^n_+, there is a semidefinite complementary cone K_F. Thus there are an infinite number of semidefinite complementary cones. Note that the given SDLCP(L, Q) has a solution X, if and only if there exists a semidefinite complementary cone K_F, such that $X \in F$, $Y = L(X) + Q \in F^{\triangle}$ and $Q = Y - L(X) \in K_F$. Thus the union of all semidefinite complementary cones is the set of all symmetric matrices Q, for which the SDLCP (L, Q) has a solution. But unlike complementary cones in LCP over R^n_+, complementary cones in SDLCP need not be closed.

Example 1 Let $L\colon S^n \to S^n$ be a linear transformation defined as

$$L\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & 0 \end{pmatrix}.$$

Note that

$$L\left(\begin{pmatrix} \epsilon & -1 \\ -1 & \frac{1}{\epsilon} \end{pmatrix}\right) \to \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \text{as } \epsilon \to 0.$$

However, there exist no $X \succeq 0$ such that

$$L(X) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Thus the complementary cone of L corresponding to the face S^2_+ is not closed.

In Theorem 2 we give a sufficient condition for the closedness of semidefinite complementary cones induced by a given linear transformation L in terms of its $\mathbf{R}_0$-property. For this we appeal to a result which can be deduced from Theorem 9.1 in Rockafeller [20], but an independent proof has been supplied here.

LEMMA 1 *Let C be a closed convex cone in R^n and $A: R^n \to R^m$.* If $Az = 0$, $z \in C$ *implies $z = 0$, then $A(C)$ is closed.*

Proof Let $\{x_n\}$ be a sequence in C such that $A(x_n) \to y$. We shall show that $y \in A(C)$. When $y = 0$ the result is trivial, so consider the case $y \neq 0$. Then $x_n \neq 0$ for large n and hence there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m/\|x_m\|$ converges to some $s \in C$. Thus $A(x_m)/\|x_m\| \to A(s)$. Note that the sequence $\{x_m\}$ is bounded, otherwise we have $A(s) = 0$, which by the hypothesis gives $s = 0$, a contradiction. Since $\{x_m\}$ is bounded, it has a subsequence converging to some $x \in C$. Thus we have $y = A(x) \in C$, which completes the proof. ∎

THEOREM 2 *If $L: S^n \to S^n$ has the $\mathbf{R_0}$-property, then all the complementary cones of L are closed.*

Proof By the definition of the $\mathbf{R_0}$-property of L and Lemma 1 above, it is apparent that $L(F)$ is closed for every face F of the cone S^n_+. Consider a linear transformation $M: S^n \times S^n \to S^n$ defined as $M(X, Y) = X + Y$. Let $K_F = \{Y - L(X): X \in F, \ Y \in F^\triangle\}$ be a complementary cone corresponding to the face F of S^n_+. Let $K_1 := \{Y: Y \in F^\triangle\}$ and $K_2 := \{-L(X): X \in F\}$. Then $M(K_1, K_2) = K_1 + K_2 = K_F$. Now, $M(Y, -L(X)) = 0$ for some $X \in F$ and $Y \in F^\triangle$ implies that $Y - L(X) = 0$. Hence $L(X) \in F^\triangle$, which by the $\mathbf{R_0}$-property yields $X = 0$. Thus we have $Y = L(X) = 0$. Appealing to Lemma 1 again, we get K_F is closed. ∎

Nondegenerate matrices [3] are studied to characterize the finiteness of the set of solutions to a linear complementarity problem LCP(M, q), which is geometrically equivalent to the assertion that for a given $q \in R^n$ there can be at the most one solution to LCP(M, q) in every complementary cone induced by M, i.e. each complementary cone is nondegenerate, see [3]. Since there are finitely many complementary cones (finitely many faces of R^n_+), for a nondegenerate M, LCP(M, q) has finitely many solutions, for each q. Motivated by this we define the notion of a nondegenerate semidefinite complementary cone and study its connection with the notion of nondegenerate transformations introduced and studied by Gowda and Song [7] in the semidefinite setting.

Definition 2

(a) A semidefinite complementary cone K_F above is called *nondegenerate*, if the *complementary transformation*

$$T_F: \text{span}\, F + \text{span}\, F^\triangle \to L(\text{span}\, F) + \text{span}\, F^\triangle$$

defined as $T_F(X + Y) = Y - L(X)$ is invertible.

(b) A linear transformation is *nondegenerate* (see [7]) if

$$XL(X) = 0 \Rightarrow X = 0.$$

PROPOSITION 1 *A linear transformation $L: S^n \to S^n$ is nondegenerate if and only if each of the semidefinite complementary cones of L is nondegenerate.*

Proof If part

Let all the semidefinite complementary cones of L be nondegenerate in the above sense. Let $XL(X) = 0$ for some $0 \neq X \in S^n$ of rank r. Then there exists an orthogonal matrix U and diagonal matrices D and E, such that

$$X = U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}U^T \quad \text{and} \quad L(X) = U\begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}U^T.$$

Consider the semidefinite complementary cone K_F corresponding to the face

$$F = \left\{ U\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}U^T \colon Y \in S^r_+ \right\}.$$

Since $X \in \operatorname{span} F$ and $L(X) \in \operatorname{span} F^{\triangle}$, the complementary transformation $T_F\colon \operatorname{span} F + \operatorname{span} F^{\triangle} \to L(\operatorname{span} F) + \operatorname{span} F^{\triangle}$, defined above is not invertible because $T_F(X + L(X)) = 0$ with $X + L(X) \neq 0$, which contradicts our hypothesis that K_F is nondegenerate. It follows from here that L is nondegenerate.

Only if part

Let K_F be a semidefinite complementary cone corresponding to the face F. Let there exist $X_1, X_2 \in F$ and $Y_1, Y_2 \in F^{\triangle}$, such that $Y_1 - L(X_1) = Y_2 - L(X_2)$. Then $(X_1 - X_2)(Y_1 - Y_2) = (X_1 - X_2)L(X_1 - X_2) = 0$. From the nondegeneracy of the transformation L, we have $X_1 = X_2$. Again, for $L(X) + Y \in L(\operatorname{span} F) + \operatorname{span} F^{\triangle}$, $T_F(-X + Y) = L(X) + Y$. Thus T_F is one-one and onto and hence invertible. ■

The following example due to Gowda and Song [7] shows that even if L is nondegenerate, there may be a $\overline{Q}$ for which there are infinitely many solutions to SDLCP$(L, \overline{Q})$. In fact, the same example can be used to illustrate that all the semidefinite complementary cones are nondegenerate in the sense defined above, and hence each of the infinitely many solutions of SDLCP$(L, \overline{Q})$ comes from a distinct semidefinite complementary cone.

Example 2 Let $L: S^2 \to S^2$ be defined by $L(X) = -X$. Let $Q = I$. Consider any one-dimensional face F. Note that the semidefinite complementary cone corresponding to F is $K_F = \{Y + X \colon X \in F, \ Y \in F^{\triangle}\}$. Consider now the complementary transformation $T_F\colon \operatorname{span} F + \operatorname{span} F^{\triangle} \to L(\operatorname{span} F) + \operatorname{span} F^{\triangle}$, which is seen to be $T_F(X + Y) = X + Y$. This is obviously invertible.

3. Q-property of positive semidefiniteness preserving transformations

In this section, we study the transformations $L: S^n \to S^n$ for which $L(S^n_+) \subseteq S^n_+$. We call such transformations *semidefiniteness preserving*. Special cases of such transformations have been studied earlier by [5,8,18]. These transformations generalize a nonnegative matrix in the context of the linear complementarity problem.

We first note that transformations satisfying $L(S^n_+) = S^n_+$ can be represented as $L(X) = AXA^T$ for some invertible matrix A of order n, see [21]. However, there are semidefiniteness preserving transformations which cannot be represented as AXA^T. The following is an example.

Example 3 Consider the transformation L: $S^2 \to S^2$ given by

$$L(X) = \begin{pmatrix} x_{11} + x_{22} & 0 \\ 0 & x_{11} + x_{22} \end{pmatrix}, \text{ for all symmetric } X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix}.$$

If we try to represent it in the form AXA^T we get inconsistent equations in the elements of the matrix A.

For a general L the semidefiniteness preserving property and in addition the $\mathbf{R_0}$-property, have the following interpretation in terms of the faces of S^n_+.

PROPOSITION 2

(i) *A transformation L has the property* $L(S^n_+) \subseteq S^n_+$ *if and only if for every pair of one dimensional faces F and G of the semidefinite cone* S^n_+, $\langle Y, L(X)\rangle \geq 0$, $\forall X \in F$ and $Y \in G$.
(ii) *Let* $L(S^n_+) \subseteq S^n_+$. *Then L has the* $\mathbf{R_0}$*-property if and only if* $\langle X, L(X)\rangle > 0$ *for every nonzero* $X \in F$, *where F is any 1-dimensional face of* S^n_+.

Proof

(i) The 'only if part' is trivial. For the 'if part', note that any $Y \succeq 0$ can be written as the sum of matrices on one dimensional faces, i.e. $Y = \sum_{i=1}^n VE_iV^T$ where V is orthogonal and E_i is a nonnegative matrix with all entries 0 other than the ith diagonal entry. Let $X \succeq 0$ be given. Consider an arbitrary $Y \succeq 0$. Writing $X = \sum_{i=1}^n UD_iU^T$ for U orthogonal and Y as above, we get $\langle Y, L(X)\rangle \geq 0$, since each term is nonnegative by hypothesis. Since $Y \succeq 0$ is arbitrary, from the self duality of S^n_+ we get $L(X)$ is positive semidefinite.
(ii) The proof of this part also follows easily from the definitions and the argument used above in part (i). ■

Remark 1 Note that the defining condition $\langle X, L(Y)\rangle \geq 0$, for all $X \in F$ and $Y \in G$, is a generalization of the condition that $e_iMe_j = m_{ij} \geq 0$, where M is a square matrix and e_i is a vector whose ith entry is 1 and others 0.

We say that a square matrix M of order n is a Q-matrix (see [3]), if LCP(M, q) has a solution for each $q \in R^n$. In the context of the linear complementarity problem, Murty [16] shows that a nonnegative square matrix M is a Q-matrix if and only if the diagonal entries of M are positive ($e_iMe_i > 0$), which is equivalent to saying that M is an R_0-matrix. This motivates us to introduce the following definition.

Definition 3 L has the *positive diagonal property*, if for every one-dimensional face F of S^n_+, $\langle X, L(X)\rangle > 0$ for every nonzero $X \in F$.

The above proposition shows that a semidefiniteness preserving transformation has the $\mathbf{R_0}$-property if and only if it has the positive diagonal property. At present we are unable to settle the question whether for such transformations $\mathbf{Q}$-property implies $\mathbf{R_0}$-property. However, we have the following result.

THEOREM 3 *Let $L: S^n \to S^n$ satisfy $L(S^n_+) \subseteq S^n_+$. If L has the $\mathbf{Q}$-property then for every one-dimensional face G there exists a one-dimensional face F of S^n_+, such that $\langle Y, L(X)\rangle > 0$ for all nonzero $X \in F$ and $Y \in G$.*

Proof Suppose the result is not true. Then without loss of generality we assume that $\langle E_{11}, U^T L(X) U\rangle = 0$ for all rank 1 matrices $X \succeq 0$ and some fixed orthogonal matrix U. Consider the matrix

$$Q = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $R = UQU^T$. Since $L(X)$ is positive semidefinite for all $X \succeq 0$, it follows that for all $X \succeq 0$ at least $(n-1)$ eigenvalues of $U^T(L(X)+R)U = U^T L(X) U + Q$ are positive. Since $X = 0$ cannot be a solution to SDLCP(L, R), it follows that if $\widetilde{X}$ is a solution to it, then the rank of $L(\widetilde{X}) + R$ must be $(n-1)$ and hence $\widetilde{X}$ must have rank 1. Now for any rank 1 matrix $X \succeq 0$,

$$\langle E_{11}, U^T(L(X)+R)U\rangle = \langle E_{11}, U^T L(X) U\rangle + \langle E_{11}, Q\rangle = -1,$$

which shows that any $X \succeq 0$ of rank 1 cannot be a solution to SDLCP(L, R). Since X is an arbitrary rank 1 matrix, it follows that there is no solution to SDLCP(L, R). This concludes the proof. ■

The necessary condition for $\mathbf{Q}$-property proved in Theorem 3 is seen to be equivalent to

$$L^*(X) = 0, \quad X \succeq 0 \Rightarrow X = 0,$$

where L^* is the adjoint of L, which immediately leads to a following corollary.

COROLLARY 1 *If $L: S^n \to S^n$ satisfying $L(S^n_+) \subseteq S^n_+$ has the $\mathbf{Q}$-property then $L^*(F)$ is closed for every face F* of S^n_+.

In the last result of this section we observe that a nonnegative matrix is a Q-matrix if and only if a related linear transformation on S^n, to be defined below, has the $\mathbf{Q}$-property.

THEOREM 4 *Let M be a given nonnegative matrix and define the $n \times n$ diagonal matrix A_i by taking its jth diagonal entry as m_{ij}, the (i, j)th entry of M. Let the transformation L*

be defined by

$$L(X) = \begin{pmatrix} \langle A_1, X\rangle & 0 & \dots & 0 \\ 0 & \langle A_2, X\rangle & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \langle A_n X\rangle \end{pmatrix}.$$

Then L has the **Q**-*property if and only if M is a Q-matrix.*

Proof Suppose L has the **Q**-property. Given any $q \in R^n$ let $\overline{Q}$ denote the diagonal matrix whose ith diagonal entry is q_i. Note that SDLCP($L, \overline{Q}$) has a solution X, since L has the **Q**-property. Define $x \in R^n$ as $x_i = x_{ii}$. Then it is easy to note that $x \geq 0$ and that x is a solution to the LCP(M, q) proving that M is a Q-matrix. Conversely, note that M is a Q-matrix if and only if all the diagonal entries of M are positive, see Murty [16]. We shall show that L is $\mathbf{R_0}$ when M is a Q-matrix. Let $X \succeq 0$ solve SDLCP(L, 0). Then $XL(X) = 0$ implies $x_{ii}(\sum_{i \neq j} m_{ij}x_{jj} + m_{ii}x_{ii}) = 0\ \forall i.$ Since $m_{ii} > 0\ \forall i$ and $m_{ij} \geq 0\ \forall i \neq j$ we get $x_{ii} = 0\ \forall i,$ which in turn gives $X = 0$. Also on observing the fact that for any $Q \succ 0$, SDLCP(L, Q) has the unique solution $X = 0$, it follows from Karamardian's theorem [13] that L has the **Q**-property. ■

4. Relationship between strict copositivity and strict semimonotonicity

For a linear transformation $L\colon S^n \to S^n$, we say that

(a) L is copositive (strictly copositive) if $\langle X, L(X)\rangle \geq 0\ (> 0)$ for all $X \succeq 0$ (nonzero $X \succeq 0$.)

(b) L has strict semimonotone (**SSM** or **E**)-property (see [4]) if

$$X \succeq 0, \quad XL(X) = L(X)X \preceq 0 \Rightarrow X = 0.$$

(c) L has the semimonotone property or the $\mathbf{E_0}$-property (see [4]) if $L + \varepsilon I$ has the **SSM**-property for every $\varepsilon > 0$.

The above definitions are motivated by the notions of copositivity and semimonotonicity in linear complementarity problems and the next two theorems study the relationship between the two properties in the semidefinite setting.

THEOREM 5 *For any linear transformation* $L\colon S^n \to S^n$, *we have the following implications*

(i) *L is copositive $\Rightarrow$ L has $\mathbf{E_0}$-property.*
(ii) *L is strictly copositive $\Rightarrow$ L has* **SSM**-*property.*
(iii) *L is copositive $\Rightarrow$ SDLCP(L, Q) has a unique solution for $Q \succ 0$.*
(iv) *L is strictly copositive $\Rightarrow$ SDLCP(L, Q) has a unique solution for $Q \succeq 0$.*

Proof We shall present the proof of (i) and (iii). The proofs of the other two parts are similar.

(i) Fix an $\varepsilon > 0$. Let $X \succeq 0$ and $X(L+\varepsilon I)(X) = (L+\varepsilon I)(X)X \preceq 0$. If $X \neq 0$, then by copositivity of L it follows that

$$\langle X, L(X) + \varepsilon X\rangle = \langle X, L(X)\rangle + \varepsilon\langle X, X\rangle > 0.$$

On the other hand $X(L+\varepsilon I)(X) \preceq 0 \Rightarrow \langle X, L(X) + \varepsilon X\rangle \leq 0$, a contradiction!

(iii) Let $X \succeq 0$ be a solution to the SDLCP(L, Q) for some $Q \succ 0$. Then

$$X \succeq 0, \quad L(X) + Q \succeq 0 \quad \text{and} \quad \langle X, L(X) + Q\rangle = 0.$$

Now, $\langle X, L(X) + Q\rangle = 0$ implies $\langle X, Q\rangle \leq 0$, which in turn gives $\langle X, Q\rangle = XQ = 0$ resulting in $X = 0$. ■

In the context of a LCP, strict copositivity and strict semimonotonicity are known to be equivalent for real symmetric matrices, see [3]. To study a similar kind of relationship in a SDLCP, we introduce the notion of a principal subtransformation of a linear transformation corresponding to a face F of S^n_+. Observe that this is a generalization of the concept of a principal submatrix of a matrix in $R^{n\times n}$.

Definition 4

(a) Let $L: S^n \to S^n$ be a linear transformation and F be a face of S^n_+. Then a principal subtransformation of L, with respect to a face F is a linear transformation L_{FF}: span $F \to$ span F such that $L_{FF}(X) = \text{Proj}_{\text{span}\,F} L(X)$ for $X \in$ span F.

(b) L_{FF} *is strictly semimonotone* if

$$X \in F, \quad \text{X and } L_{FF}(X) \text{ commute}, \quad \text{and} \quad XL_{FF}(X) \in -F \Rightarrow X = 0.$$

Remark 2 The notion of a principal subtransformation in a semidefinite setting was also introduced independently by Gowda et al. [8]. Though the connection of their notion with our notion has been described in detail in [15], our notion of the principal subtransformations corresponding to faces of S^n_+ seems to be more general and geometric in nature.

In order to study the connection between strict copositivity and the **SSM**-property, we shall generalize Theorem 1 in [12] to the semidefinite setting.

LEMMA 2 *Let $L: S^n \to S^n$ be a self-adjoint linear transformation. Then, L is strictly copositive if every principal subtransformation L_{FF} of L has no eigenvector V in the relative interior of F with associated eigenvalue $\lambda \leq 0$.*

Proof Suppose there exists a nonzero $X_0 \succeq 0$ with $\langle X_0, L(X_0)\rangle \leq 0$. Define $\mathcal{S} := \{X \succeq 0: X \neq 0, \langle X, L(X)\rangle \leq 0\}$. Let $m(X)$ denote the number of positive eigenvalues of X. Since we can choose an $X \in \mathcal{S}$ that has the least number of positive eigenvalues among all $X \in \mathcal{S}$, we can assume, without loss of generality, that $r = m(X_0) \leq m(X)$, $\forall X \in \mathcal{S}$. We consider the case $r > 1$. (For $r = 1$ the proof follows easily.) Let F be the smallest face containing X_0 and $\Im := \{Y: Y \in F, \|Y\| = 1\}$. We can also assume without loss of generality that $\|X_0\| = 1$. Consider the function

$Q(Y) = \langle Y, L(Y)\rangle$ restricted to the set $\Im$. Note that X_0 is in the relative interior of $\Im$ and $\langle X_0, L(X_0)\rangle \leq 0$. Moreover, any matrix X on the relative boundary of $\Im$ will have less than r positive eigenvalues and hence for such a matrix X, $\langle X, L(X)\rangle > 0$. It follows that $Q(Y)$ restricted to $\Im$ will attain its minimum at some point V in the relative interior of $\Im$ and $Q(V) \leq 0$. But then V would be an eigenvector of L_{FF} with a negative or zero eigenvalue, contradicting our hypothesis. Thus, L is strictly copositive on S^n_+. ■

THEOREM 6 *Suppose L is self-adjoint. Then L is strictly copositive if and only if every principal subtransformation L_{FF} of L is strictly semimonotone.*

Proof For the 'only if' part note that if L is strictly copositive over S^n_+, then every principal subtransformation L_{FF} of L is stricly copositive on F. Thus it is sufficient to show that L is stricly semimonotone, which follows from Theorem 5 proved above. Now assume that L is not strictly copositive. By Lemma 2 above, there exists a nonzero face F of the cone S^n_+ such that $L_{FF}(X) = \lambda X$ for some X in the relative interior of F and $\lambda \leq 0$. Thus we have

$$X \in F, \quad XL_{FF}(X) = L_{FF}(X)X \in -F,$$

which contradicts the fact that L_{FF} has the **SSM**-property. ■

5. Relationship between P-property and P-matrix property

Definition 5 For a linear transformation $L: S^n \to S^n$, we say that

(a) L has the **P**-*property* (see [4]) if

$$X \in S^n, \quad XL(X) = L(X)X \preceq 0 \Rightarrow X = 0.$$

(b) L_{FF} has the **P**-*property* if

$$X \in \operatorname{span} F, \quad X \text{ and } L_{FF}(X) \text{ commute}, \quad \text{and} \quad XL_{FF}(X) \in -F \Rightarrow X = 0.$$

(c) a matrix $M \in R^{n\times n}$ is a *P-matrix* (see [3]) if all principal minors of M are positive.

A matrix $A \in R^{n\times n}$ is *positive stable* if the real part of every eigenvalue of A is positive. Given a matrix A, the *Lyapunov and Stein transformations* are defined by $L_A(X) := AX + XA^T$ and $S_A(X) := X - AXA^T$, respectively.

Given a linear transformation $L: S^n \to S^n$, we denote by $\mathcal{N}(L)$ the matrix of L of order $n(n+1)/2$ corresponding to the basis $\{E_{ij}\}$ where E_{ij}, for $i \neq j$, is the symmetric matrix whose ijth and jith elements are $1/\sqrt{2}$ and other elements are 0, and E_{ii} is the symmetric matrix whose ith diagonal entry is 1 and all other entries are equal to 0. The elements in a column of this matrix represent the matrix $L(E_{rs})$ as a linear combination of the basis elements E_{ij} taken in the order $\{E_{11}, E_{12}, E_{22}, E_{13}, E_{23}, E_{33}, E_{14}, \ldots, E_{nn}\}$. Note that each column will have $n(n+1)/2$ entries.

We say that L has the *P-matrix property* if $\mathcal{N}(L)$ is a P-matrix. The motivation for asking whether a L with **P**-property has the P-matrix property is partly the issue studied in Theorem 8 of [4] (also see [6]). Also, when L is self-adjoint, we have the following equivalence:

L_{FF} has the **P**-property for all $F \Leftrightarrow L$ has **P**-property $\Leftrightarrow L$ is strictly monotone $\Leftrightarrow \mathcal{N}(L)$ is symmetric positive definite (see Theorem 1 in [8]). In this section we shall study the relationship between the **P**-property of L and the P-matrix property of L.

The following example shows that $\mathcal{N}(L)$ may be a P-matrix, but L_{FF} does not have the **P**-property for all F.

Example 4 For

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$

the matrix of the Lyapunov transformation L_A is

$$\mathcal{N}(L_A) = \begin{pmatrix} 2 & -2\sqrt{2} & 0 \\ 0 & 2 & -2\sqrt{2} \\ 0 & 0 & 2 \end{pmatrix}.$$

Note that $\mathcal{N}(L_A)$ is a P-matrix. However A is not positive definite, which implies from Theorem 12 in [9], that not every principal subtransformation of L_A has the **P**-property.

The next example shows that **P**-property of L need not imply that $\mathcal{N}(L)$ is a P-matrix.

Example 5 Consider a Lyapunov transformation $L_A: S^2 \to S^2$. Take

$$A = \begin{pmatrix} -1 & 2 \\ -2 & 2 \end{pmatrix}.$$

Note that A is positive stable and hence from Theoerm 5 in [4] L_A has the **P**-property. The matrix of L_A is

$$\mathcal{N}(L_A) = \begin{pmatrix} -2 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 4 \end{pmatrix}$$

which is not a P-matrix.

Given a set of indices $= \{i_1 < i_2 < \cdots < i_k\}$ where $1 \leq i_1;\ i_k \leq n$, the canonical face F of S^n_+ corresponding to α is the face defined as

$$F_\alpha := P \begin{pmatrix} S_+^{|\alpha|} & 0 \\ 0 & 0 \end{pmatrix} P^T,$$

where P is a permutation matrix such that for any $X \in S^n$, $(P^TXP)_{\beta\beta} = X_{\alpha\alpha}$, where $\beta = \{1, 2, \ldots, |\alpha|\}$. As has been discussed earlier, due to the nonpolyhedral nature of S^n_+, span F + span $F^\triangle$ does not generate the whole space S^n. This motivates us to study a class of linear transformations for which

$$L(\text{span}\, F_\alpha) \subseteq \text{span}\, F_\alpha + \text{span}\, F_\alpha^\triangle,$$

where F_α is a canonical face. In what follows we shall characterize these transformations and study the **P**-property and the P-matrix property for these transformations. We shall assume, without loss of generality, the following form of a linear transformation L.

$$L(X) = \begin{pmatrix} \langle A_{11}, X\rangle & \cdots & \langle A_{1n}, X\rangle \\ \vdots & \ldots & \vdots \\ \langle A_{n1}, X\rangle & \cdots & \langle A_{nn}, X\rangle \end{pmatrix} \tag{1}$$

where A_{ij} and A_{ji} are $n \times n$ symmetric matrices and $A_{ij} = A_{ji}$ for all $i, j \in \{1, \ldots, n\}$. We will use the notation a^{rs}_{ij} for the (i,j)th entry in the matrix A_{rs}.

Theorem 7 *A linear transformation $L : S^n \to S^n$ written in the form* (1) *satisfies $L(\text{span}\, F_\alpha) \subseteq \text{span}\, F_\alpha + \text{span}\, F_\alpha^\triangle$ for all $\alpha \subseteq \{1,2\ldots, n\}$…' iff every entry other than the (i,j)th entry of the $n \times n$ symmetric matrix A_{ij} in* (1) *is zero for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$.*

Proof If part

Let $\alpha \subseteq \{1, 2, \ldots, n\}$ and $L_{F_\alpha F_\alpha}$ be a principal subtransformation of L corresponding to F_α. We shall show that $L(\text{span}\, F_\alpha) \subseteq \text{span}\, F_\alpha + \text{span}\, F_\alpha^\triangle$. Without loss of generality assume that $\alpha = \{1, \ldots, k\}$, where $1 \leq k \leq n$. Consider an arbitrary $X \in \text{span}\, F_\alpha$ and a matrix A_{ij} for any $(i,j) \in \alpha \times \beta$, where β is the complement of α in $\{1, 2, \ldots, n\}$. Then by our hypothesis $\langle A_{ij}, X\rangle = 0$, which immediately proves our claim.

Only if part

Consider an $n \times n$ symmetric matrix A_{ij} for $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$. We shall show that every (k,l)th entry of A_{ij} is zero where $(k,l) \neq (i,j)$. Let F_{α_1} be a canonical face of S^n_+ corresponding to $\alpha_1 := \{1, 2, \ldots, n\} \setminus \{i\}$. Since $L(\text{span}\, F_{\alpha_1}) \subseteq \text{span}\, F_{\alpha_1} + \text{span}\, F_{\alpha_1}^\triangle$, we have $\langle A_{ij}, X\rangle = 0 \;\forall X \in \text{span}\, F_{\alpha_1}$, which gives $(A_{ij})_{\alpha_1\alpha_1} = 0$. Thus all the (k,l)th entries of A_{ij} other than $k = i$ or $l = i$ are 0. Similarly $(A_{ij})_{\alpha_2\alpha_2} = 0$ for $\alpha_2 := \{1, 2, \ldots, n\} \setminus \{j\}$, which shows that every (k,l)th entry of A_{ij} other than $k = j$ or $l = j$ is 0. Thus every entry other than (i,j)th entry of A_{ij} is 0. ■

Theorem 8 *Suppose $L : S^n \to S^n$ has the property that $L(\text{span}\, F_\alpha) \subseteq \text{span}\, F_\alpha + \text{span}\, F_\alpha^\triangle$ for all $\alpha \subseteq \{1, \ldots, n\}$ with $L(\text{span}\, F_\alpha) \subseteq \text{span}\, F_\alpha$ for $\alpha = \{1,2, \ldots, r\}$, $1 \leq r \leq n$. Then (i)$\Rightarrow$(ii) $\Longleftrightarrow$ (iii) in the following statements*

(i) *L has the* **P**-*property.*
(ii) *All the real eigenvalues of L and those of its canonical principal subtransformations are positive.*
(iii) *$\mathcal{N}(L)$ is a P-matrix.*

Proof (i) $\Rightarrow$ (ii)
Let F be a face of S^n_+ for which $L(\operatorname{span} F) \subseteq \operatorname{span} F + \operatorname{span} F^{\triangle}$. Then for any $X \in \operatorname{span} F$, $XL(X) = XL_{FF}(X)$ and $L(X)X = L_{FF}(X)X$. This immediately gives us that L_{FF} has the **P**-property when L has the **P**-property. Now by using Theorem 1 in [8], the above implication follows.

(ii)$\Leftrightarrow$(iii)

We assume w.l.g that the given L is represented in the form (1). The proof is by induction on n. We first verify the theorem for $n = 2$. For $n = 2$, $\mathcal{N}(L)$ is given by:

$$\mathcal{N}(L) = \begin{pmatrix} a^{11}_{11} & \sqrt{2}a^{11}_{12} & a^{11}_{22} \\ 0 & 2a^{12}_{12} & 0 \\ 0 & \sqrt{2}a^{22}_{12} & a^{22}_{22} \end{pmatrix}.$$

The hypothesis that the real eigenvalues of the canonical principal subtransformations are positive shows that the diagonal entries a^{11}_{11} and a^{22}_{22}, and the determinant of the above matrix are positive. From the structure of the matrix it follows that $2a^{12}_{12}$ is also positive. Further note that any principal minor of the above matrix is a product of a subset of the diagonal entries and hence is positive. Thus the theorem holds for $n = 2$.

Induction hypothesis: The theorem is true when $n \leq k$.

We shall now show that the theorem holds when $n = k+1$. When $n = k+1$ the matrix (L) of order $((k+1)(k+2))/2$ can be partitioned as follows: Let $\alpha = \{1, 2 \ldots k\}$. Now

$$\mathcal{N}(L) = \begin{pmatrix} A_{\alpha\alpha} & C \\ B & G \end{pmatrix},$$

where $A_{\alpha\alpha}$ is of order $k(k+1)/2$, B is of order $(k+1) \times k(k+1)/2$, C is of order $(k(k+1)/2) \times (k+1)$ and G is of order $(k+1) \times (k+1)$. The matrix A is the same as $\mathcal{N}(L_{F_\alpha F_\alpha})$ where F_α is the canonical face of S^n_+ corresponding to α. Since L and all its canonical principal subtransformations have the property that all their real eigenvalues are positive, it follows by the induction hypothesis that $A_{\alpha\alpha}$ is a P-matrix. We now note that $B = 0$. This is so since the column entries in the block B are the coefficients of $E_{l(k+1)}$ for $1 \leq l \leq (k+1)$ in the representation of $L(E_{ij})$, $1 \leq i \leq k$, $1 \leq j \leq k$, $i \leq j$, and by our hypothesis $L(E_{ij}) \in L(\operatorname{span} F_\alpha) \subseteq \operatorname{span} F_\alpha$. Since B is zero, any principal minor of $\mathcal{N}(L)$ will either be a principal minor of $A_{\alpha\alpha}$ or a product of a principal minors of $A_{\alpha\alpha}$ and a principal minor of G. Note that G is given by

$$G = \begin{pmatrix} 2a^{1(k+1)}_{1(k+1)} & 0 & \ldots & 0 \\ 0 & 2a^{2(k+1)}_{2(k+1)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \sqrt{2}a^{(k+1)(k+1)}_{1(k+1)} & \sqrt{2}a^{(k+1)(k+1)}_{2(k+1)} & \ldots & a^{(k+1)(k+1)}_{(k+1)(k+1)} \end{pmatrix}.$$

Since G is lower triangular, it follows that any principal minor of G is a product of some of its diagonal entries. That these diagonal entries are positive follows by considering the canonical principal subtransformations $L_{F_{[i,k+1]}}$ of L and using our hypothesis about the eigenvalues of such canonical principal subtransformations. From these observations it follows that $\mathcal{N}(L)$ is a P-matrix. ■

Below we give an example to illustrate the above proposition.

Example 6 Consider a Stein transformation $S_A(X) = X - AXA^T$ corresponding to

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & 0 \end{pmatrix}.$$

It is easy to check that S_A satisfies the assumption made in Theorem 8. The matrix of S_A with respect to the basis $\{E_{11}, E_{12}, E_{22}\}$ is

$$\mathcal{N}(S_A) = \begin{pmatrix} 1 - a_{11}^2 & -\sqrt{2}a_{11}a_{12} & -a_{12}^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is not a self-adjoint matrix. The eigenvalues of A are 0 and a_{11} and from Theorem 11 in [5] we have S_A has the **P**-property iff $|a_{11}| < 1$. Thus choosing $|a_{11}| < 1$ it is immediate that $\mathcal{N}(S_A)$ is a P-matrix.

Proposition 3 *For $A \in R^{2\times 2}$ we have the following implications:*

(i) *$(L_A)_{FF}$ has the **P**-property for all $F \Rightarrow (L_A)$ is a P-matrix.*
(ii) *$\mathcal{N}(L_A)$ is a P-matrix $\Rightarrow L_A$ has the **P**-property.*

Proof
(i) For

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

note that

$$\mathcal{N}(L_A) = \begin{pmatrix} 2a_{11} & \sqrt{2}a_{12} & 0 \\ \sqrt{2}a_{21} & a_{11} + a_{22} & \sqrt{2}a_{12} \\ 0 & \sqrt{2}a_{21} & 2a_{22} \end{pmatrix}.$$

Let $(L_A)_{FF}$ have the **P**-property for all F. Then from Theorem 12 of [9], A is positive definite. From simple calculations we can easily see that all the principal minors of $\mathcal{N}(L_A)$ are positive. Hence $\mathcal{N}(L_A)$ is a P-matrix.

(ii) If (L_A) is a P-matrix, then $\det A > 0$. The eigenvalues of A are given by

$$\lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{(\mathrm{Tr}(A))^2 - 4\det(A)}}{2}.$$

Since $\det(A) > 0$, the real parts of the eigenvalues of A are positive. Hence A is positive stable and L_A has the **P**-property. ■

We do not know if the above proposition can be proved for any n.

We conclude the article by presenting an example, which shows that **P**-property of L_{FF}, for all F need not imply the P-matrix property of L.

Example 7 Consider $L(X) = AXA^T$ with

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}.$$

Note that

$$\mathcal{N}(L) = \begin{pmatrix} 1 & -2\sqrt{2} & 4 \\ 2\sqrt{2} & -1 & -6\sqrt{2} \\ 4 & 6\sqrt{2} & 9 \end{pmatrix}.$$

Since A is positive definite we can check easily that L_{FF} has the **P**-property for all F (see also Corollary 6 in [8]) but $\mathcal{N}(L)$ is not a P-matrix.

6. Concluding remarks

(i) Since $\mathbf{R_0}$-property is implied by the **P**-property, strict semimonotonicity and the nondegeneracy of a linear transformation L on S^n, complementary cones are closed under any of the above stated properties.

(ii) Another question of interest, as it is relevant to the solvability of a SDLCP(L, Q), is: are the complementary cones corresponding to a transformation L with the **Q**-property closed? Except for an affirmative answer in some special cases like Lyapunov [4] and Stein transformations [5], where **Q**-property is equivalent to **P**-property, this question remains open.

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